Some observations on the Youla form and conjugate-normal matrices

H. Faßbender* and Kh. D. Ikramov†

August 31, 2006

Abstract: Two square complex matrices A, B are said to be unitarily congruent if there is a unitary matrix U such that $A = UBU^T$. The Youla form is a canonical form under unitary congruence. We give a simple derivation of this form using coninvariant subspaces. For the special class of conjugate-normal matrices the associated Youla form is discussed.

Keywords: consimilarity, unitary congruence, coneigenvalue, coninvariant subspace, conjugate-normal matrix, Youla theorem
MSC Classification: 15A21, 65F99

1 Introduction

Matrices $A, B \in \mathbb{C}^{n\times n}$ are said to be consimilar if $A = SB\overline{S}^{-1}$ for a nonsingular matrix $S \in \mathbb{C}^{n\times n}$, where, as usual, $\overline{S}$ is the component-wise conjugate of S. Consimilarity is an equivalence transformation and expresses the change of basis formula for a basis representation of an antilinear transformation. Unitary congruence is an important particular case of consimilarity obtained when $S = U$ is a unitary matrix: $A = UBU^T$. Here U^T denotes the usual matrix transposition $C = U^T, c_{ij} = u_{ji}$, while U^* denotes the Hermitian adjoint of U, $C = U^*, c_{ij} = \overline{u}_{ji}$. If $S = Q$ is complex orthogonal, then $A = QBQ^*$; if $S = R$ is a real nonsingular matrix, then $A = RBR^{-1}$. Thus, special cases of consimilarity include ^{T}congruence, *congruence and ordinary similarity. There exists an extensive literature on consimilarity and unitary congruence, which provides a rather complete theory for these matrix relations.

*Institut *Computational Mathematics*, TU Braunschweig, Pockelsstr. 14, D-38023 Braunschweig, Germany

†corresponding author, Faculty of Computational Mathematics and Cybernetics, Moscow State University, 119992 Moscow, Russia. The work of the second author has been supported by the Deutsche Forschungsgemeinschaft.

Consimilarity has a very long history, going back to [20] and perhaps earlier. Early work on this subject can be found under keywords like antilinear transformation [6], semilinear transformations [7, 1], pseudolinear transformations [16, 17]. Canonical forms under consimilarity have been considered by a number of authors, see [12] for a summary. The consimilarity analog of the Jordan canonical form, the concanonical form, has been considered, e.g., in [1, 6, 8]. As the concanonical form is essentially unique, it follows that two complex matrices are consimilar if and only if they have the same concanonical form. Moreover, every complex matrix A is consimilar to a real matrix; its square is similar to $\overline{A}A$ (see [1, Satz 20] or [12, Theorem 4.9]). It is further shown in [12] that every matrix is consimilar to its own conjugate, transpose and adjoint, and to a Hermitian matrix. Just as one can derive a symmetric Jordan canonical form from the Jordan canonical form of a matrix under similarity, a Hermitian canonical form under consimilarity is derived in [9] from canonical forms in [12] and [4]. The reduction of a matrix to triangular or diagonal form by consimilarity has been discussed in [11].

The n eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ are its simplest (and most important) similarity invariants. We want to define analogous invariants with respect to consimilarity transformations. To this end, we introduce the matrices

$$\widehat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}, \tag{1}$$

$$A_L = \bar{A}A, \qquad \text{and} \qquad A_R = A\bar{A}.$$

Although the products AB and BA need not be similar in general, A_L is always similar to A_R (see [14, p. 246, Problem 9 in Section 4.6]). Therefore, in the subsequent discussion of their spectral properties, it will be sufficient to refer to one of them, say, A_L.

The spectrum of A_L has two remarkable properties:

1. It is symmetric with respect to the real axis. Moreover, the eigenvalues λ and $\bar{\lambda}$ are of the same multiplicity.

2. The negative eigenvalues of A_L (if any) are necessarily of even algebraic multiplicity.

For the proofs of these properties, we refer the reader to [14, p. 252–253,].

Let

$$\lambda(A_L) = \{\lambda_1, \ldots, \lambda_n\}$$

be the spectrum of A_L. The *coneigenvalues* of A are the n scalars $\mu_1, \ldots, \mu_n$ defined as follows:

If $\lambda_i \in \lambda(A_L)$ does not lie on the negative real axis, then the corresponding coneigenvalue μ_i is defined as a square root of λ_i with nonnegative real part and the multiplicity of μ_i is set to that of λ_i :

$$\mu_i = \lambda_i^{1/2}, \qquad \mathrm{Re}\mu_i \geq 0.$$

With a real negative $\lambda_i \in \lambda(A_L)$, we associate two conjugate purely imaginary coneigenvalues

$$\mu_i = \pm\lambda_i^{1/2}.$$

It can be easily shown that, if $\mu_1, \dots, \mu_n$ are the coneigenvalues of A, then

$$\lambda(\widehat{A}) = \{\mu_1, \dots, \mu_n, -\mu_1, \dots, -\mu_n\}. \tag{2}$$

Note that the definition of the coneigenvalues given above is similar or identical to the definitions in [4, 5] and is different from the definition in [14, p. 245]. In particular, the coneigenvalues as defined in [14] can exist only if A_L has real nonnegative eigenvalues. The coneigenvalues as defined above exist for any $n \times n$ complex matrix A.

In this paper, we are mostly concerned with unitary congruence transformations. Two matrices A, B are unitarily congruent if there exist a unitary U such that $A = UBU^T$. This is the same as $A = UB\overline{U}^{-1}$ since $U^{-1} = U^*$, so unitary congruence is the same as unitary consimilarity. A theorem characterizing unitary congruence of two square matrices is given in [13]. It says that two matrices A, B are unitarily congruent if and only if the pairs $(AA^*, BB^*), (A\overline{A}, B\overline{B})$ and $(A^T\overline{A}, B^T\overline{B})$ are simultaneously unitarily similar. An important theorem in the theory of unitary congruence is the Youla theorem (see, e.g., [24]). This is a unitary congruence analog of the Schur triangularization theorem. In a sense, it is even closer to the real version of the Schur theorem. The theorem says that, given a complex square matrix A, there is a unitary matrix U such that UAU^T is a block triangular matrix with diagonal blocks of order 1×1 and 2×2. The 1×1 blocks correspond to the nonnegative eigenvalues of $A\overline{A}$ (if any); the 2×2 blocks correspond to the negative and/or nonreal eigenvalues of $A\overline{A}$.

It is well known that the Schur triangular form becomes a diagonal matrix for a normal matrix A. A similar fact in the theory of unitary congruence was observed in [23]. For any conjugate-normal matrix A (that is, $AA^* = \overline{A^*A}$), there is a unitary matrix U such that UAU^T is a block diagonal matrix with diagonal blocks of order 1×1 and 2×2. The 1×1 blocks correspond to the real nonnegative eigenvalues of $A\overline{A}$; the 2×2 blocks correspond either to pairs of equal negative eigenvalues of $A\overline{A}$ or to conjugate pairs of nonreal eigenvalues of $A\overline{A}$.

The paper is organized as follows. In Section 2, the concept of a coninvariant subspace is introduced. We show that any matrix $A \in \mathbb{C}^{n\times n}$ has a coninvariant subspace of dimension one or two. Using this, we give a simple derivation for the Youla normal form of a matrix under unitary congruences (see [24]). In Section 3, the special class of conjugate-normal matrices is examined. The canonical form for these matrices with respect to unitary congruences is known [23]; however, the derivation we give for this form allows us to characterize conjugate-normal matrices as unitarily congruent ones to ordinary real normal matrices.

2 The Youla Theorem

Let X be an $n \times s$ matrix. The symbol $\mathcal{L}_X$ will denote the subspace in $\mathbb{C}^n$ spanned by the columns in X.

Definition 1 *A subspace $\mathcal{L}_X$ is said to be a coninvariant subspace of A (or A-coninvariant subspace) if*

$$AX = \overline{X}M \tag{3}$$

for some matrix M [10].

Every matrix $A \in \mathbb{C}^{n \times n}$ has at least two coninvariant subspaces: the zero subspace (set $X = 0 \in \mathbb{C}^n$ in (3)) and the entire space $\mathbb{C}^n$ (take any nonsingular $X \in \mathbb{C}^{n \times n}$ and set $M = \overline{X}^{-1}AX$ in (3)). Are there any nontrivial A-coninvariant subspaces? The following theorem is a "con"-version of the well-known fact that any complex $n \times n$ matrix has an eigenvector.

Theorem 1 *Let $A \in \mathbb{C}^{n \times n}$. Then A has a one- or two-dimensional coninvariant subspace.*

Proof. The observation preceding the theorem resolves the cases $n = 1$ and $n = 2$. Assume that $n \geq 3$. Let x be an eigenvector of $\overline{A}A$; i.e., $\overline{A}Ax = \lambda x$ for some $\lambda \in \mathbb{C}$. Then either (a) $\{Ax, \overline{x}\}$ is linearly dependent or (b) $\{Ax, \overline{x}\}$ is linearly independent. In case (a), we have $Ax = \mu\overline{x}$ for some scalar μ, and x spans a one-dimensional coninvariant subspace. In case (b), we have $A(\overline{Ax}) = \overline{\lambda}\overline{x}$; thus, x and $\overline{Ax}$ span a two-dimensional coninvariant subspace of A. $\square$

This observation is not new; see, e.g., [6]. In that paper, invariant subspaces of $A\overline{A}$ were considered, and a complete canonical form for consimilarity was given, from which the presence of the low-dimensional subspaces asserted above can be read off.

As already said in the Introduction, unitary congruence is the most important particular case of consimilarity. There is an important theorem in the theory of unitary congruence called the Youla theorem (see, e.g., [24]). We give a proof of the Youla theorem in order to demonstrate how the concept of a coninvariant subspace simplifies the argument.

Theorem 2 (Youla Theorem) *Any matrix $A \in \mathbb{C}^{n \times n}$ can be brought by a unitary congruence transformation to a block triangular form with the diagonal blocks of orders 1 and 2. The 1×1 blocks correspond to real nonnegative coneigenvalues of A, while each 2×2 block corresponds to a pair of complex conjugate coneigenvalues. This block triangular matrix is called the Youla normal form of A. It can be upper or lower block triangular.*

Proof. We outline the proof that essentially mimics the standard proof of the Schur theorem. For definiteness, we consider the reduction to the upper Youla form.

The theorem obviously holds for $n = 1$. Suppose it holds for all matrices of order $n-1$ or less. Choose a coneigenvalue μ of A. The actions to be taken depend on whether we deal with a real nonnegative coneigenvalue μ (case 1) or with a pair of conjugate coneigenvalues $\mu, \overline{\mu}$ (case 2). In any case, an orthonormal basis of the corresponding coninvariant subspace has to be chosen. This is a single normalized vector u_1 (case 1) or orthonormal vectors u_1, u_2 (case 2). Next, a unitary matrix U_1 is built up with u_1 as its first column (case 1) or u_1, u_2 as the first two columns (case 2). Finally, the unitary congruence transformation is performed

$$A \rightarrow A_1 = U_1^T A U_1. \tag{4}$$

Since

$$A u_1 = \mu \overline{u_1}$$

(case 1) and

$$A[u_1 \ u_2] = [\overline{u_1} \ \overline{u_2}] M$$

(case 2), the matrix A_1 in (4) must be block triangular

$$A_1 = \left[\begin{array}{cc} A_{11} & A_{12} \\ 0 & A_{22} \end{array} \right]. \tag{5}$$

Here, $A_{11} = \mu$ is a 1×1 block in case 1 and $A_{11} = M$ is a 2×2 block in case 2.

By induction there is a unitary matrix V of order $n-1$ (case 1) or $n-2$ (case 2) such that $V^T A_{22} V$ is in the Youla normal form. Define

$$U_2 = 1 \oplus V$$

(case 1) or

$$U_2 = I_2 \oplus V$$

(case 2). Then

$$A_2 = U_2^T A_1 U_2 = (U_1 U_2)^T A (U_1 U_2) \tag{6}$$

is in the Youla normal form. □

Remark. Different Youla forms can be constructed for the same matrix A. Moreover, for any given ordering of coneigenvalues (with the only limitation that complex conjugate coneigenvalues go by pairs), it is possible to construct the Youla form Y with that ordering of the coneigenvalues on the main diagonal of Y.

It is well known that the Schur triangular form becomes a diagonal matrix for a normal A. We have a similar fact in the theory of unitary congruence.

Definition 2 *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be conjugate-normal if*

$$AA^* = \overline{A^*A}. \tag{7}$$

It seems that conjugate-normal matrices were first introduced in [23]. An easy implication of (7) is the following property which conjugate-normal matrices share with ordinary normal matrices.

Proposition 1 *Let $A \in \mathbb{C}^{n\times n}$ be a conjugate-normal matrix. Then the 2-norm of row i is equal to the 2-norm of column i ($1 \le i \le n$).*

To verify Proposition 1, it suffices to equate the diagonal entries of the matrices in (7).

Suppose that Y is an upper Youla form of a conjugate-normal matrix A. If A_{11} is a 1×1 block, then an application of Proposition 1 with $i = 1$ yields

$$|a_{11}|^2 + \sum_{j=2}^{n} |a_{1j}|^2 = |a_{11}|^2$$

and

$$a_{12} = a_{13} = \ldots = a_{1n} = 0.$$

If A_{11} is a 2×2 block, then Proposition 1 yields for $i = 1$ and $i = 2$

$$|a_{11}|^2+|a_{12}|^2+|a_{21}|^2+|a_{22}|^2+\sum_{j=3}^{n}|a_{1j}|^2+\sum_{j=3}^{n}|a_{2j}|^2 = |a_{11}|^2+|a_{12}|^2+|a_{21}|^2+|a_{22}|^2$$

and

$$a_{13} = \ldots = a_{1n} = a_{23} = \ldots = a_{2n} = 0.$$

Thus, all the entries in the first block row outside of the diagonal block A_{11} are zero. Performing similar considerations for the blocks $A_{22}, A_{33}, \ldots$ in succession, we finally arrive at the following result (see also [23] and [13, Theorem 3.7]).

Theorem 3 *Any Youla form of a conjugate-normal matrix $A \in \mathbb{C}^{n\times n}$ is a block diagonal matrix with the diagonal blocks of orders 1 and 2.*

In Section 3, a different proof of this theorem is given after some facts about conjugate-normal matrices have been derived.

3 Conjugate-normal matrices

The special classes of matrices with respect to unitary similarities are Hermitian, skew-Hermitian, unitary, and, most generally, normal matrices. In order to determine which classes of matrices are special with respect to unitary congruences, consider matrix (1).

Proposition 2 *Let $A \in \mathbb{C}^{n\times n}$. The matrix $\widehat{A}$ in (1) is normal (respectively, Hermitian, skew-Hermitian, unitary) if and only if A is conjugate-normal (respectively, symmetric, skew-symmetric, unitary).*

This assertion is verified by simple calculations using (1) and the formula

$$\widehat{A}^* = \begin{bmatrix} 0 & A^T \\ A^* & 0 \end{bmatrix}. \tag{8}$$

A useful tool in the theory of unitary similarities is the Toeplitz (also called Cartesian) decomposition of a matrix A:

$$A = B + C, \qquad B = B^*, \quad C = -C^*. \tag{9}$$

The matrices B and C are called the real and imaginary parts, respectively, of A and are determined uniquely:

$$B = \frac{1}{2}(A + A^*), \qquad C = \frac{1}{2}(A - A^*). \tag{10}$$

The convenience of the Toeplitz decomposition is related to the fact that it is respected by unitary similarity in the following sense: for a unitary matrix U, the matrices U^*BU and U^*CU are the real and imaginary parts, respectively, of U^*AU and all the three matrices preserve their eigenvalues. This is generally not true for a nonunitary U.

An analog of the Toeplitz decomposition for unitary congruences can be found by considering again matrix (1). According to (8), its real and imaginary parts are

$$\widehat{B} = \begin{bmatrix} 0 & \frac{1}{2}(A + A^T) \\ \frac{1}{2}(\overline{A} + A^*) & 0 \end{bmatrix} \text{ and } \widehat{C} = \begin{bmatrix} 0 & \frac{1}{2}(A - A^T) \\ \frac{1}{2}(\overline{A} - A^*) & 0 \end{bmatrix}.$$

Thus, the equality

$$\widehat{A} = \widehat{B} + \widehat{C}$$

induces the decomposition

$$A = S + K, \tag{11}$$

where the matrices

$$S = \frac{1}{2}(A + A^T) \text{ and } K = \frac{1}{2}(A - A^T) \tag{12}$$

are symmetric and skew-symmetric, resp.. They are called the symmetric and skew-symmetric parts, respectively, of A.

Decomposition (11) is respected by unitary congruences in the sense that, for a unitary U, the matrices U^TSU and U^TKU are the symmetric and skew-symmetric parts, respectively, of U^TAU. What is especially important to us is that all the three matrices preserve their coneigenvalues.

Note that the coneigenvalues of S, being the square roots with nonnegative real parts of the eigenvalues of $S_L = \overline{S}S = S^*S$, are just the singular values $\sigma_1(S), \ldots, \sigma_n(S)$. The coneigenvalues of K are purely imaginary, because they are square roots of the eigenvalues of the negative semidefinite matrix $K_L = \overline{K}K = -K^*K$. The coneigenvalues of a unitary U, being the square roots of the eigenvalues of the unitary matrix $\overline{U}U$, have the modulus 1.

The fact that a normal matrix A can be brought to diagonal form by a unitary similarity transformation can be proved in many different ways. In particular, one can reason as follows: A is normal if and only if the matrices B and C in its Toeplitz decomposition (9) commute. However, commuting Hermitian matrices (and C is Hermitian up to the factor i) can always be brought to diagonal form by the same similarity transformation.

We now want to give a proof along the same lines for Theorem 3. Recall that, in Section 2, this theorem was derived as a corollary to Theorem 2 on the Youla normal form of an arbitrary square matrix.

Proposition 3 *A matrix $A \in \mathbb{C}^{n\times n}$ is conjugate-normal if and only if the matrices S and K in its decomposition (11), (12) satisfy the relation*

$$S\overline{K} = K\overline{S}. \tag{13}$$

Proof. Relation (13) is obtained by substituting (11) into definition (7). $\square$

Remark. Note that the concommutativity expressed by (13) is preserved by unitary congruences: if $\widetilde{S} = U^T S U$ and $\widetilde{K} = U^T K U$ for a unitary U, then

$$\widetilde{S}\overline{\widetilde{K}} = \widetilde{K}\overline{\widetilde{S}}.$$

In our proof of Theorem 3, we use the following two results. The first result, known as Takagi's factorization [22], has been rediscovered repeatedly (see [12, p. 144]: [18] in 1939, [21] in 1943, [15] in 1944, [19] in 1945, and [3] in 1984. Historical priority must be given to Autonne [2] for $\det S \neq 0$ as early as 1915.

Proposition 4 (Takagi's factorization) *Let $S \in \mathbb{C}^{n\times n}$ be a symmetric matrix. Then, there exist a unitary matrix U and a real nonnegative diagonal matrix*

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \tag{14}$$

such that

$$S = U\Sigma U^T. \tag{15}$$

The scalars $\sigma_1, \ldots, \sigma_n$ are the singular values or (which is the same) the coneigenvalues of S. Moreover, U can be chosen so that the coneigenvalues appear in any prescribed order along the diagonal of Σ.

This is easily seen from the Youla form of S which is block upper triangular and symmetric. Thus, it is block diagonal. There are no 2×2 blocks because $S\overline{S} = SS^*$ is positive semidefinite and hence has only nonnegative eigenvalues. This is essentially the proof given by Siegel in 1943 [21] (see also [14, Problem 22 in Section 4.4 and p. 218]). Other proofs of the Takagi factorization can be found in [14, Section 4.4] and [13].

Proposition 5 *Let $K \in \mathbb{C}^{n\times n}$ be a skew-symmetric matrix. Then, there exists a unitary matrix V such that*

$$V^T K V = 0 \oplus \cdots \oplus 0 \oplus K_1 \oplus \cdots \oplus K_\ell, \tag{16}$$

where each K_j $(1 \le j \le \ell)$ is a matrix of the form

$$\begin{bmatrix} 0 & z_j \\ -z_j & 0 \end{bmatrix}. \tag{17}$$

The scalars $z_1, \ldots, z_\ell$ can be chosen to be real positive. If K is a real skew-symmetric matrix, then V can be chosen to be a real orthogonal matrix.

This skew-symmetric analog of Takagi's factorization can be proved using the Youla form of K, which has to be block diagonal. Skew symmetry now ensures that all the diagonal entries are zero; any 2×2 block can be rotated to be real if necessary (see also [14, Problems 22, 25, and 26 in Section 4.4]).

Now, we embark on our second proof of Theorem 3.

Proof of Theorem 3. Let A be a given conjugate-normal matrix. Consider its decomposition (11), (12). Let $\sigma_1 > \sigma_2 > \ldots > \sigma_k$ be the distinct coneigenvalues of S. Choose a unitary matrix U so that

$$D = U^T S U = \sigma_1 I_{m_1} \oplus \sigma_2 I_{m_2} \oplus \cdots \oplus \sigma_k I_{m_k}, \quad m_1 + m_2 + \cdots + m_k = n. \tag{18}$$

Partition the matrix

$$L = U^T K U$$

conformably with (18):

$$L = (L_{ij})_{i,j=1}^k.$$

According to Proposition 3, it holds that

$$D\overline{L} = L\overline{D}$$

or

$$\sigma_i \overline{L}_{ij} = \sigma_j L_{ij}, \quad i, j = 1, \ldots, k.$$

It follows that

$$L_{ij} = 0, \quad i \neq j;$$

i.e., L is a block diagonal matrix. Also,

$$L_{ii} = \overline{L}_{ii}, \quad i = 1, \ldots, k-1;$$

i.e., all the diagonal blocks in L (with the possible exclusion of the block L_{kk}) are real skew-symmetric matrices. The block L_{kk} is also real if $\sigma_k > 0$; otherwise, it may be complex.

By Proposition 5, there exists a unitary matrix V_i $(1 \leq i \leq k)$ that brings the skew-symmetric matrix L_{ii} to block diagonal form with the diagonal blocks of orders 1 and 2. Moreover, V_i can be chosen to be real orthogonal for $i = 1, 2, \ldots, k-1$ if $\sigma_k = 0$ and for all i if $\sigma_k > 0$. Setting

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_k,$$

we conclude that

$$M = V^T L V$$

is a block diagonal matrix with diagonal blocks of order 1 or 2, while

$$V^T D V = D.$$

It follows that the conjugate-normal matrix

$$J = V^T U^T A U V = (UV)^T S (UV) + (UV)^T K (UV) = D + M \tag{19}$$

is also block diagonal. Moreover, its 2×2 blocks have the form

$$\begin{bmatrix} \sigma & z \\ -z & \sigma \end{bmatrix}, \tag{20}$$

where z is (or can be made) real. Theorem 3 is proved. □

The normal form J of a conjugate-normal matrix A is a real normal matrix. This allows us to make an important conclusion.

Theorem 4 *Every conjugate-normal matrix is unitarily congruent to a real normal matrix.*

The reverse statement is obvious; any unitary congruence transformation of a real normal matrix yields a conjugate-normal matrix.

Acknowledgments The authors are very grateful to one of the referees of our paper who invested a huge amount of his/her time in our manuscript.

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