On Numerical Methods For Discrete Least-Squares Approximation By Trigonometric Polynomials

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Abstract

Fast, efficient and reliable algorithms for discrete least-squares approximation of a real-valued function given at arbitrary distinct nodes in $[0, 2\pi)$ by trigonometric polynomials are presented. The algorithms are based on schemes for the solution of inverse unitary eigenproblems and require only $O(mn)$ arithmetic operations as compared to $O(mn^2)$ operations needed for algorithms that ignore the structure of the problem. An algorithm which solves this problem with real-valued data and real-valued solution using only real arithmetic is given. Numerical examples are given that show that the proposed algorithms produces consistently accurate results that are often better than those obtained by general QR decomposition methods for the least-squares problem.

1 Introduction

A number of signal processing problems can be seen to require numerical methods for different unitary eigenvalue problems. One of these problems is the discrete least-squares approximation of a real-valued function f given at arbitrary distinct nodes $\{\theta_k\}_{k=1}^m$ in $[0, 2\pi)$ by trigonometric polynomials t in the discrete norm $||f - t|| = (\sum_{k=1}^m |f(\theta_k) - t(\theta_k)|^2 \omega_k^2)^{\frac{1}{2}}$, where the $\{\omega_k^2\}_{k=1}^m$ are positive weights. The problem can easily be reformulated as the standard least-squares problem of minimizing $DAc - Dg$ over all coefficient vectors c in the Euclidian norm, where $D = diag(\omega_1, ..., \omega_m)$ and A is the transposed $m \times n$ Vandermonde matrix

$$A = \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_m & z_m^2 & \cdots & z_m^{n-1} \end{pmatrix}$$

with $z_k = exp(i\theta_k)$.

The usual way to solve this least-squares problem is to compute the QR decomposition of DA. But DA is just the Krylov matrix

$$K(\Lambda, q_0, n) = [q_0, \Lambda q_0, ..., \Lambda^{n-1} q_0]$$

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(where $\Lambda = diag(z_1, ..., z_m)$ and $q_0 = (\omega_1, ..., \omega_m)^T$). We may therefore use the following consequence of the Implicit Q Theorem to compute the desired QR decomposition. If there exists a unitary matrix U such that $U^H \Lambda U = H$ is a unitary upper Hessenberg matrix with positive subdiagonal elements, then the unique QR decomposition of $K(\Lambda, q_0, m)$ is given by UR with $R = K(H, e_1, m)$. The construction of such a unitary Hessenberg matrix from spectral data, here contained in Λ and q_o, is an inverse eigenproblem. Thus the best trigonometric approximation to f can be computed via solving this inverse eigenproblem. Reichel, Ammar and Gragg observe in [1] that solving an inverse eigenproblem for unitary Hessenberg matrices is equivalent to computing Szegö polynomials, that is to computing polynomials that are orthogonal with respect to an inner product on the unit circle. In order to compute the least-squares solution $R^{-1}Q^H Dg$ observe that $R = K(H, e_1, m)$ is the Cholesky factor of a Toeplitz-matrix. Its inverse can therefore be computed by the Levinson algorithm. The algorithms require only $O(mn)$ arithmetic operations as compared with $O(mn^2)$ operations needed for algorithms that ignore the special structure of DA.

2 A real-valued approach

New, fast algorithms to solve the discrete least-squares approximation are developed, particularly algorithms, which solve this problem with real-valued data und real-valued solution in $O(mn)$ arithmetic operations using only real arithmetic. Our approach is to reformulate the approximation problem as the standard least-squares problem of minimizing $D\tilde{A}\tilde{t} - D\tilde{f}$ over all coefficient vectors $\tilde{t}$ in the Euclidian norm, where

$$\tilde{A} = \begin{pmatrix} 1 & \sin\theta_1 & \cos\theta_1 & \cdots & \sin l\theta_1 & \cos l\theta_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \sin\theta_m & \cos\theta_m & \cdots & \sin l\theta_m & \cos l\theta_m \end{pmatrix}$$

and l is the degree of the desired trigonometric polynomial. $D\tilde{A}$ is the product of the modified Krylov matrix

$$\kappa(\Lambda, q_0, l) = [q_0, \Lambda q_0, \Lambda^H q_0, \Lambda^2 q_0, {\Lambda^H}^2 q_0, ..., \Lambda^l q_0, {\Lambda^H}^l q_0]$$

and a block diagonal matrix $F = diag(1, B, B, ..., B)$ with $B = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$. If there exists a unitary matrix $\tilde{Q}$ such that $\tilde{Q}^H(\Lambda - \lambda I)\tilde{Q}G_e = G_o - \lambda G_e$ is a unitary matrix pencil in parametrized form, where G_o, G_e are unitary block diagonal matrices with block size at most two, then the unique QR decomposition of $\kappa(\Lambda, q_o, l)$ is given by $\tilde{Q}\tilde{R}$ with $\tilde{R} = \kappa(G_o G_e^H, e_1, l)$. From this a unique (real-valued) QR factorization of $D\tilde{A}$ is easily obtained. The construction of such a unitary matrix pencil in parametrized form from spectral data is a generalized inverse eigenproblem. The scheme for solving an inverse eigenproblem for unitary matrix pencils in parametrized form is developed from a backward stable algorithm given by Bunse-Gerstner and Elsner in [2] which reduces a unitary matrix pencil to parametrized form. It can be shown that this is equivalent to computing Laurent polynomials (rational functions) that are orthonormal with respect to an inner product on the unit circle. A relationship of these orthogonal Laurent polynomials to the Szegö polynomials is noted and a two-term recurrence is developed. In order to compute the least-squares solution $\tilde{R}^{-1}\tilde{Q}^H D\tilde{f}$ observe that $\tilde{R} = \kappa(G_o G_e^H, e_1, l)$. Using the fact that $G_o G_e^H$ is similar to a unitary upper Hessenberg matrix, it is shown that the inverse of $\tilde{R}$ can be computed by a modification of the Levinson algorithm.

As $D\tilde{A}$ is a real $m \times n$ matrix, there exists a unique, real-valued QR decomposition $\hat{Q}\hat{R}$ of $D\tilde{A}$. In order to develop an algorithm to compute $\hat{Q}$ using only real arithmetic the effect of the transformation $\tilde{Q}$ on the real- and imaginary-part of $\Lambda = diag(z_1, ..., z_m) = diag(\cos\theta_1, ..., \cos\theta_m) + diag(\sin\theta_1, ..., \sin\theta_m)$ is considered. It is shown that essentially only the real-part of Λ is needed for computing $\hat{Q}$ (and $\hat{R}$). As $\hat{R}$ is the Cholesky factor of a bordered block-Toeplitz-plus-block-Hankel-matrix, a new fast algorithm to compute $\hat{R}^{-1}$ is developed.

The algorithms discussed require only $O(mn)$ arithmetic operations as compared with $O(mn^2)$ operations needed for algorithms that ignore the special structure of DA or $D\tilde{A}$.

3 Numerical results

We present some numerical examples that compare accuracy and speed of the following methods :

- AGR : algorithm proposed in [1] as sketched in the introduction, the desired QR decomposition of DA is computed via an invers unitary eigenvalue problem and the Levinson algorithm (with complex arithmetic)

- ver2.1 : the desired QR decomposition of $D\tilde{A}$ is computed via a generalized inverse unitary eigenvalue problem and a modification of the Levinson algorithm (with complex arithmetic)

- ver4.1 : the desired QR decomposition of $D\tilde{A}$ is computed via a simultaneous reduction of the real- and imaginary part of Λ to a compact form and an algorithm to compute the Cholesky factor of a special bordered block-Toeplitz-plus-block-Hankel-matrix (with real arithmetic)

- linpack : the matrix $D\tilde{A}$ is explicitly formed, the solution is computed by the LINPACK routines sqrdc and sqrsl with real arithmetic

For comparison of accuracy we compute the solution $\tilde{t}_d$ of the system

$$min||D\tilde{A}\tilde{t} - D\tilde{f}||_2$$

in double precision using the NAG routine F04AMF. The figures display the relative error $||\tilde{t} - \tilde{t}_d||_2/||\tilde{t}_d||_2$ where $\tilde{t}$ is the coefficient vector computed by the method under consideration. Each graph displays the errors for $m = 50$ and increasing values of n. The arguments of the nodes are either equispaced in the interval $[0, \pi)$, $[0, 3/2\pi)$ or $[0, 2\pi)$ or the arguments are randomly generated uniformly distributed numbers in $[0, 2\pi)$. The weights are all equal to one, the elements of the real vector $\tilde{f}$ are randomly generated uniformly distributed numbers in $[-5, 5]$.

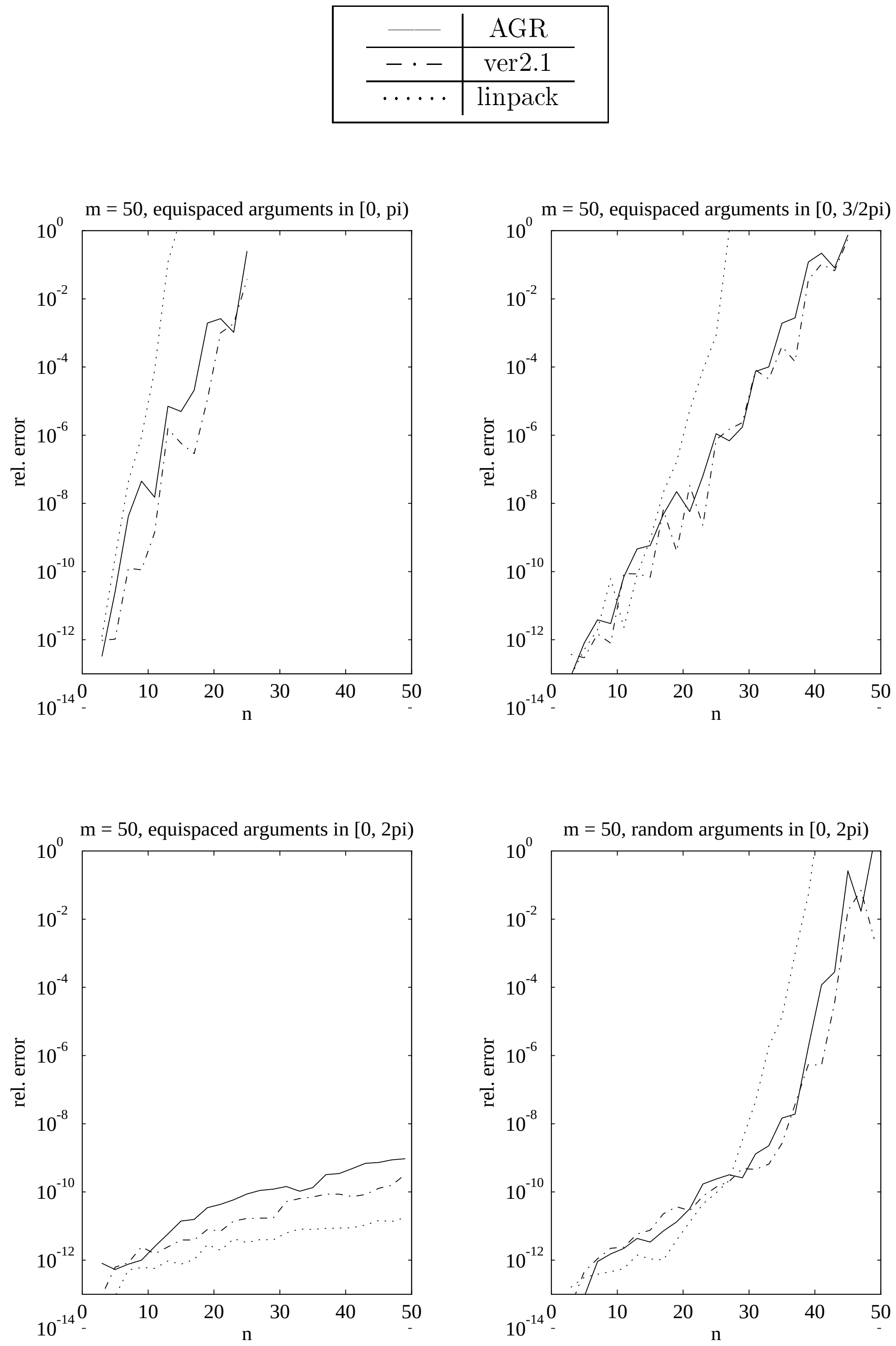
AGR
ver2.1
linpack
m = 50, equispaced arguments in [0, pi)
m = 50, equispaced arguments in [0, 3/2pi)
m = 50, equispaced arguments in [0, 2pi)
m = 50, random arguments in [0, 2pi)
rel. error
n
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
0
10
20
30
40
50

Figure 1:

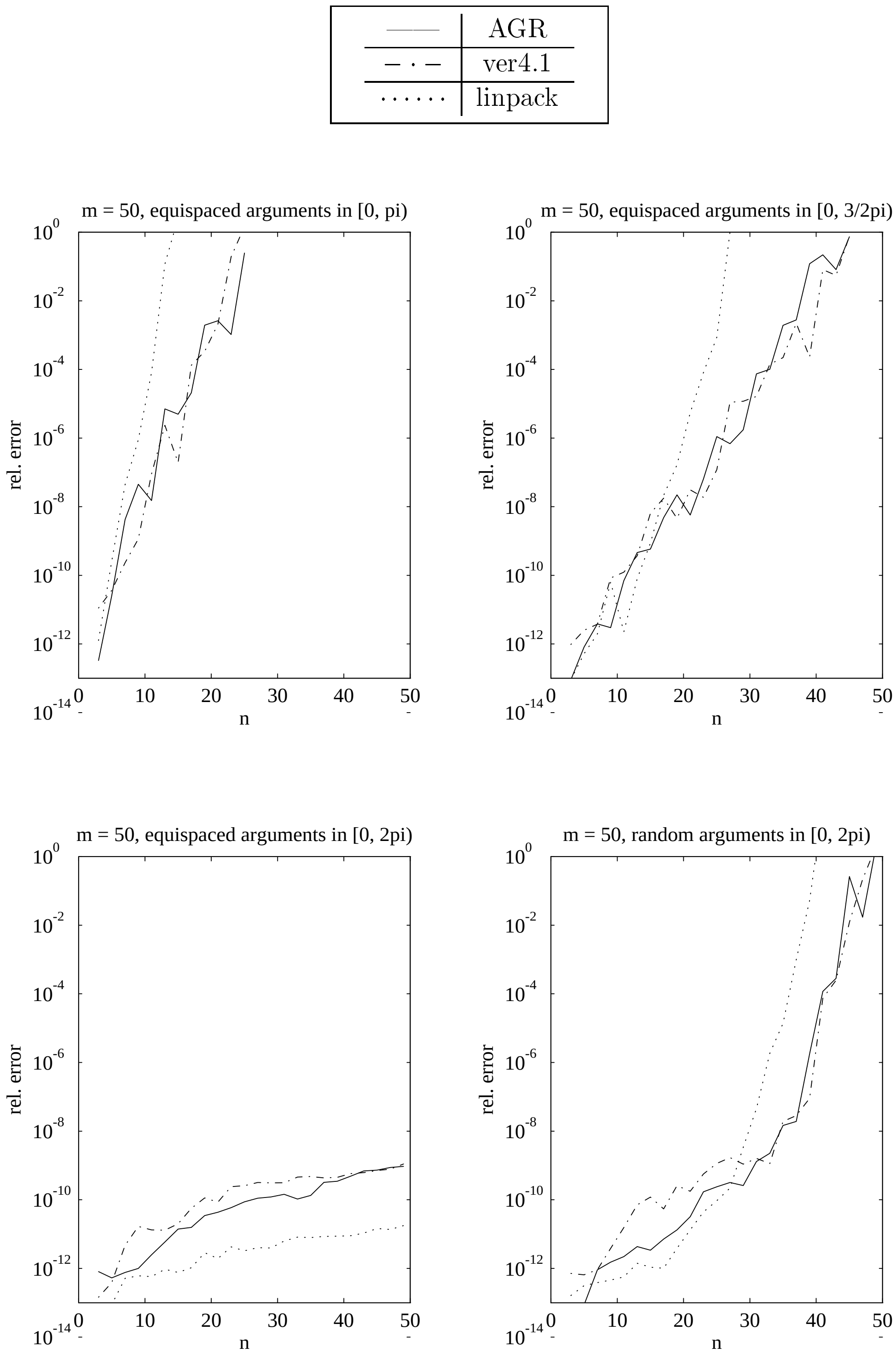
AGR
ver4.1
linpack
m = 50, equispaced arguments in [0, pi)
m = 50, equispaced arguments in [0, 3/2pi)
m = 50, equispaced arguments in [0, 2pi)
m = 50, random arguments in [0, 2pi)
rel. error
n
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
0
10
20
30
40
50

Figure 2:

The graphs at the top of Figure 1 and Figure 2 display the relative errors in the coefficient vectors for equispaced nodes in intervalls smaller than 2π. As n increases, and the problem becomes more ill conditioned, the LINPACK routines are the first to produce inaccurate results. ver2.1 produces errors that are somewhat smaller than AGR, while ver4.1 produces errors that are about the same as AGR. The graphs at the bottom of Figure 1 and Figure 2 display the relative error when the arguments are equispaced in $[0, 2\pi)$ and when the arguments are randomly generated uniformly distributed numbers in $[0, 2\pi)$. In the first case the LINPACK routines and ver2.1 produce smaller errors than AGR, while ver4.1 produces slightly larger errors. When the arguments are randomly generated uniformly distributed points in $[0, 2\pi)$ the least-squares problem is relatively well conditioned and the AGR algorithm, ver2.1 and ver4.1 yield roughly the same accuracy as n gets close to m. We obtained similar results to those in Figure 1 and Figure 2 with other choices for the nodes and the weights.

4 Final Remarks

This brief note is a partial summary of [3].

References

[1] L. Reichel, G. S. Ammar and W. B. Gragg, *Discrete Least Squares Approximation by Trigonometric Polynomials*, Math. Comp., 57, pp 273 - 289, 1991.

[2] A. Bunse-Gerstner and L. Elsner, *Schur Parameter Pencils for the Solution of the Unitary Eigenproblem*, Lin. Alg. and its Appl., 154 - 156, pp 741 - 778, 1991.

[3] H. Faßbender, *Numerische Verfahren zur diskreten trigonometrischen Polynomaproximation*, Dissertation Universität Bremen, Germany, Oktober 1993.