# Simultaneous reduction to a block triangular form by a unitary congruence transformation

H. Faßbender* and Kh. D. Ikramov†

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## 1 Introduction

A classical theorem on commuting matrices says the following(e.g., see [4, Section I.4.21]:

**Theorem 1** *If $A, B \in \mathbb{C}^{n\times n}$ commute, then they can be brought to triangular form by the same unitary similarity transformation; i.e., there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that both $U^*AU$ and $U^*BU$ are triangular matrices.*

For definitiness, we always assume that the reduced matrices are upper triangular.

The standard proof of Theorem 1 is based on the following proposition:

**Theorem 2** *If $A, B \in \mathbb{C}^{n\times n}$ commute, then they have a common eigenvector.*

Our intention in this paper is to prove two assertions that are similar to Theorems 1 and 2 but concern unitary congruence transformations rather than unitary similarity ones. These assertions are stated and proved in Section 3. The necessary definitions and facts are given in Section 2. The implications of our main theorems for complex symmetric and, more generally, conjugate-normal matrices are discussed in Section 5.

Actually, Theorems 1 and 2 hold in the following stronger form:

**Theorem 3** *If $\{A_i\} \subset \mathbb{C}^{n\times n}$ is an arbitrary family (finite or infinite) of commuting matrices, then this family can be brought to triangular form by the same unitary similarity transformation.*

*Institut *Computational Mathematics*, TU Braunschweig, Pockelsstr. 14, D-38023 Braunschweig, Germany

†Faculty of Computational Mathematics and Cybernetics, Moscow State University, 119992 Moscow, Russia. The work of the second author has been supported by the Deutsche Forschungsgemeinschaft.

**Theorem 4** *If $\{A_i\} \subset \mathbb{C}^{n\times n}$ is an arbitrary family (finite or infinite) of commuting matrices, then there exists a common eigenvector for all matrices in $\{A_i\}$.*

In Section 4, we give the analogs of Theorems 3 and 4 concerning unitary congruence transformations.

# 2 Preliminaries

The concept of a coninvariant subspace of a matrix (first given in [2]) is of primary importance throughout the paper. For a subspace $\mathcal{L}$ in $\mathbb{C}^{n\times n}$, define the subspace $\overline{\mathcal{L}}$ by the relation

$$\overline{\mathcal{L}} = \{\overline{x} \mid x \in \mathcal{L}\}.$$

We say that $\mathcal{L}$ is a coninvariant subspace of $A \in \mathbb{C}^{n\times n}$ if

$$A\mathcal{L} \subset \overline{\mathcal{L}}.$$

If $\dim \mathcal{L} = 1$, then any nonzero vector $x \in \mathcal{L}$ is said to be a coneigenvector of $A$.

Not every matrix in $\mathbb{C}^{n\times n}$ has coneigenvectors; in fact, most matrices do not have them. However, the following proposition is true:

**Theorem 5** *Let $A \in \mathbb{C}^{n\times n}$. Then $A$ has a one- or two-dimensional coninvariant subspace.*

For the proof, see [1].

With each matrix $A \in \mathbb{C}^{n\times n}$, we associate the set of $n$ complex scalars called the coneigenvalues of $A$. Our definition of the coneigenvalues is different from that given in [3, Section 4.6] and relies on the following remarkable properties of the spectrum of the matrix $A_L = \overline{A}A$:

1. It is symmetric with respect to the real axis. Moreover, the eigenvalues $\lambda$ and $\overline{\lambda}$ are of the same multiplicity.
2. The negative real eigenvalues of $A_L$ (if any) are necessarily of even algebraic multiplicity.

For the proofs of these properties, we refer the reader to [3, pp. 252–253].

Let $\lambda(A_L) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of $A_L$.

**Definition 1** *The coneigenvalues of $A$ are the $n$ scalars $\mu_1, \ldots, \mu_n$ obtained as follows:*

- *If $\lambda_i \in \lambda(A_L)$ does not lie on the negative real half axis, then the corresponding coneigenvalue $\mu_i$ is defined as the square root of $\lambda_i$ with nonnegative real part and the multiplicity of $\mu_i$ is set to that of $\lambda_i$:*

$$\mu_i = \lambda_i^{\frac{1}{2}}, \quad \mathrm{Re}\mu_i \geq 0.$$

- *With a real negative $\lambda_i \in \lambda(A_L)$, we associate two conjugate purely imaginary coneigenvalues*
$$\mu_i = \pm\lambda_i^{\frac{1}{2}},$$
*the multiplicity of each being half the multiplicity of $\lambda_i$.*

*The set*
$$c\lambda(A) = \{\mu_1, \ldots, \mu_n\}$$
*is called the conspectrum of $A$.*

In particular, the coneigenvalues of a complex symmetric matrix $A$ are real nonnegative scalars that are equal to the singular values of $A$.

The following theorem plays the same role in the theory of unitary congruence as the Schur triangularization theorem does in the theory of unitary similarity (see [5]).

**Theorem 6 (Youla Theorem)** *Any matrix $A \in \mathbb{C}^{n\times n}$ can be brought by a unitary congruence transformation to a block triangular form with the diagonal blocks of orders 1 and 2. The $1\times 1$ blocks correspond to real nonnegative coneigenvalues of $A$, while each $2\times 2$ block corresponds to a pair of complex conjugate coneigenvalues. This block triangular matrix is called the Youla normal form of $A$. It can be upper or lower block triangular.*

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be conjugate-normal if
$$AA^* = \overline{A^*A}. \tag{1}$$

The Youla normal form of a conjugate-normal matrix $A$ is a block diagonal matrix with the diagonal blocks of orders 1 and 2. Again, the $1 \times 1$ blocks correspond to real nonnegative coneigenvalues of $A$, while each $2 \times 2$ block corresponds to a pair of complex conjugate coneigenvalues.

Complex symmetric matrices are conjugate-normal, and their Youla forms are real nonnegative diagonal matrices. This is the content of the classical Takagi's theorem (see [3, Section 4.4]).

**Theorem 7 (Takagi's factorization)** *Let $S \in \mathbb{C}^{n\times n}$ be a symmetric matrix. Then, there exist a unitary matrix $U$ and a real nonnegative diagonal matrix*
$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$$
*such that*
$$S = U\Sigma U^T.$$
*The scalars $\sigma_1, \ldots, \sigma_n$ are the singular values or (which is the same) the coneigenvalues of $S$. Moreover, $U$ can be chosen so that the coneigenvalues appear in any prescribed order along the diagonal of $\Sigma$.*

## 3 Main Results

We begin with the following lemma.

**Lemma 1** *Let $A, B \in \mathbb{C}^{n\times n}$ be concommutative; i.e.,*

$$A\overline{B} = B\overline{A}. \tag{2}$$

*Then $\{\overline{A}A, \overline{B}B, \overline{A}B\}$ is a commuting family of matrices.*

**Proof.** Using (2), we have

$$\begin{aligned}
(\overline{A}A)(\overline{B}B) &= \overline{A}(A\overline{B})B = \overline{A}(B\overline{A})B = (\overline{A}B)(\overline{A}B) = (\overline{B}A)(\overline{B}A) \\
&= \overline{B}(A\overline{B})A = \overline{B}(B\overline{A})A = (\overline{B}B)(\overline{A}A), \\
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\end{aligned}$$

□

Now, we prove an analog of Theorem 2.

**Theorem 8** *If $A, B \in \mathbb{C}^{n\times n}$ concommute, then they have a common coninvariant subspace of dimension one or two.*

**Proof.** Lemma 1 gives us an idea on which way to pursue in proving the required assertion. By Theorem 4, the commuting family $\{\overline{A}A, \overline{B}B, \overline{A}B\}$ has a common eigenvector. Let $x$ be this vector. Thus,

$$\overline{A}Ax = \lambda x \tag{3}$$

$$\overline{B}Bx = \nu x \tag{4}$$

$$\overline{A}Bx = \delta x \tag{5}$$

for some scalars $\lambda, \nu$, and $\delta$. Define the vectors $y$ and $z$ by the relations

$$\overline{y} = Ax \tag{6}$$

and

$$\overline{z} = Bx. \tag{7}$$

Then, using (3) - (5), we have

$$Ay = A\overline{(Ax)} = \overline{\overline{A}Ax} = \overline{\lambda}\,\overline{x}, \tag{8}$$

$$Az = A\overline{(Bx)} = \overline{\overline{A}Bx} = \overline{\delta}\,\overline{x}, \tag{9}$$

$$By = B\overline{(Ax)} = \overline{\overline{B}Ax} = \overline{\overline{A}Bx} = \overline{\delta}\,\overline{x} \tag{10}$$

$$Bz = B\overline{(Bx)} = \overline{\overline{B}Bx} = \overline{\nu}\,\overline{x}. \tag{11}$$

Relations (6) - (11) can be combined to yield

$$\begin{aligned} A[x\ y\ z] &= \overline{[x\ y\ z]}\Lambda_A, &(12)\\ B[x\ y\ z] &= \overline{[x\ y\ z]}\Lambda_B, &(13)\end{aligned}$$

where

$$\Lambda_A = \begin{bmatrix} 0 & \overline{\lambda} & \overline{\delta} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

and

$$\Lambda_B = \begin{bmatrix} 0 & \overline{\delta} & \overline{\nu} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{15}$$

Equalities (12) and (13) say that $\mathcal{L} = \text{span}\{x, y, z\}$ is a common coninvariant subspace of $A$ and $B$. If $\dim \mathcal{L} \leq 2$, then we are done. Therefore, we assume that $x, y$, and $z$ are linearly independent.

From (12) and (13), we derive the relations

$$\overline{A}B[x\ y\ z] = [x\ y\ z]\overline{\Lambda_A}\Lambda_B$$

and

$$\overline{B}A[x\ y\ z] = [x\ y\ z]\overline{\Lambda_B}\Lambda_A.$$

Since $\overline{A}B = \overline{B}A$ and $\text{rank}(x\ y\ z) = 3$, we have

$$\overline{\Lambda}_A\Lambda_B = \overline{\Lambda}_B\Lambda_A. \tag{16}$$

However,

$$\overline{\Lambda}_A\Lambda_B = \begin{bmatrix} \delta & 0 & 0 \\ 0 & \overline{\delta} & \overline{\nu} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\overline{\Lambda}_B\Lambda_A = \begin{bmatrix} \delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \overline{\lambda} & \overline{\delta} \end{bmatrix}.$$

Thus, equality (16) is possible if and only if

$$\lambda = \delta = \nu = 0.$$

This says that

$$Ay = 0, \qquad Az = 0$$

and

$$By = 0, \qquad Bz = 0.$$

It follows that each of the vectors $y$ and $z$ spans a common one-dimensional coninvariant subspace for $A$ and $B$, while $\mathrm{span}\{y, z\}$ is a common two-dimensional coninvariant subspace. This proves the theorem. $\square$

Now, we can state an analog of Theorem 1.

**Theorem 9** *If $A, B \in \mathbb{C}^{n\times n}$ concommute, then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^T AU$ and $U^T BU$ have the same block triangular form with the diagonal blocks of orders* 1 *and* 2*.*

**Proof.** We will follow the same line as in the standard proof of Theorem 1; that is, we will construct explicitly a unitary matrix that reduces $A$ and $B$ simultaneously to the desired form. Let $\mathcal{L}$ be a common coninvariant subspace of dimension one or two for $A$ and $B$. Its existence is guaranteed by Theorem 8. Choose an orthonormal basis in $\mathcal{L}$. This is a single normalized vector $u_1$ if $\dim \mathcal{L} = 1$ (case 1) or orthonormal vectors $u_1, u_2$ if $\dim \mathcal{L} = 2$ (case 2). Build a unitary matrix $U_1$ with $u_1$ as its first column (case 1) or $u_1, u_2$ as the first two columns (case 2). Then, perform the unitary congruence transformation

$$A \to A_1 = U_1^T AU_1, \qquad B \to B_1 = U_1^T BU_1. \tag{17}$$

Since

$$Au_1 = \mu_A \overline{u}_1, \qquad Bu_1 = \mu_B \overline{u}_1$$

(case 1) and

$$A[u_1\ u_2] = \overline{[u_1\ u_2]} M_A, \qquad B[u_1\ u_2] = \overline{[u_1\ u_2]} M_B$$

(case 2), the matrices $A_1$ and $B_1$ in (17) must be block triangular

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad B_1 = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}. \tag{18}$$

The transition from $A$ and $B$ to the matrices in (18) is the first step in the reduction process.

Observe that the concommutation relation (2) is preserved by unitary congruence transformations. This implies that we have

$$A_{22}\overline{B}_{22} = B_{22}\overline{A}_{22}$$

for the diagonal blocks in (18). At the second step of the reduction, we process $A_{22}$ and $B_{22}$ in the same way as the matrices $A$ and $B$ were processed at the first step. Let $V$ be the resulting unitary matrix of order $n-1$ (case 1) or $n-2$ (case 2). Define

$$U_2 = 1 \oplus V$$

(case 1) or

$$U_2 = I_2 \oplus V$$

(case 2), where $I_2$ denotes the $2 \times 2$ identity matrix. Then,

$$A_2 = U_2^T A_1 U_2 = (U_1 U_2)^T A (U_1 U_2)$$

and

$$B_2 = U_2^T B_1 U_2 = (U_1 U_2)^T B (U_1 U_2)$$

are block triangular matrices with an identical block structure:

$$A_2 = \begin{bmatrix} A_{11} & \widetilde{A}_{12} & \widetilde{A}_{13} \\ 0 & \widetilde{A}_{22} & \widetilde{A}_{23} \\ 0 & 0 & \widetilde{A}_{33} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} B_{11} & \widetilde{B}_{12} & \widetilde{B}_{13} \\ 0 & \widetilde{B}_{22} & \widetilde{B}_{23} \\ 0 & 0 & \widetilde{B}_{33} \end{bmatrix}.$$

The order (1 or 2) of the blocks $\widetilde{A}_{22}$ and $\widetilde{B}_{22}$ depends on the dimension of a common coninvariant subspace chosen for the concommuting matrices $A_{22}$ and $B_{22}$.

Continuing in this way, we finally reduce $A$ and $B$ to a block triangular form with the diagonal blocks of orders 1 and 2. The product of the unitary matrices $U_i$ used at individual steps yields the desired unitary matrix $U$. □

# 4 Simultaneous reduction of more than two matrices

In this section, we refine Theorems 8 and 9 to obtain unitary congruence analogs of Theorems 3 and 4.

We begin with an obvious extension of Lemma 1.

**Lemma 2** *Let $\mathcal{A} = \{A_i\} \subset \mathbb{C}^{n\times n}$ be such that*

$$A_i \overline{A}_j = A_j \overline{A}_i \tag{19}$$

*for any two matrices $A_i$ and $A_j$ in $\mathcal{A}$ (i not necessarily different from j). Then $\{\overline{A}_i A_j\}$ is a family of commuting matrices.*

Now, we extend Theorem 8.

**Theorem 10** *Let $\mathcal{A} = \{A_i\} \subset \mathbb{C}^{n\times n}$ be such that any two matrices $A_i, A_j \in \mathcal{A}$ concommute. Then, there exits a common coninvariant subspace of dimension one or two for all the matrices in $\mathcal{A}$.*

**Proof.** Observe that we can restrict ourselves to examining finite families $\mathcal{A}$. Indeed, assume that the original family $\mathcal{A}$ is infinite and $A_{i_1}, \ldots, A_{i_m}$ is its maximal linearly independent subset. Then, any common coninvariant subspace for $A_{i_1}, \ldots, A_{i_m}$ is actually a common coninvariant subspace for the entire family

$\mathcal{A}$. This observation also shows that the matrices in a finite family $\mathcal{A} = \{A_i\}$ $(i = 1, \ldots, m)$ can be assumed to be linearly independent.

By Lemma 2, the family $\{\overline{A_i} A_j\}$ $(i, j = 1, \ldots, m)$ consists of commuting matrices. Let $x$ be a common eigenvector for this family (see Theorem 4); thus,

$$\overline{A}_i A_j x = \delta_{ij} x, \qquad i, j = 1, \ldots, m. \tag{20}$$

Note that $\delta_{ij} = \delta_{ji}$ in view of the concommutation condition. Define the vectors $y_1, \ldots, y_m$ by the relations

$$\overline{y}_i = A_i x, \qquad i = 1, \ldots, m. \tag{21}$$

Then,

$$A_i y_j = A_i \overline{A_j x} = \overline{\overline{A}_i A_j x} = \overline{\delta}_{ij} \overline{x}, \qquad i, j = 1, \ldots, m. \tag{22}$$

Relations (21)-(22) can be combined to yield

$$A_i [x \; y_1 \; \cdots \; y_m] = \overline{[x \; y_1 \; \cdots \; y_m]} \Lambda_i, \qquad i = 1, \ldots, m, \tag{23}$$

where

$$\Lambda_1 = \begin{bmatrix} 0 & \overline{\delta}_{11} & \overline{\delta}_{12} & \cdots & \overline{\delta}_{1m} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{24}$$

$$\Lambda_2 = \begin{bmatrix} 0 & \overline{\delta}_{21} & \overline{\delta}_{22} & \cdots & \overline{\delta}_{2m} \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{25}$$

and so on.

Equalities (23) say that $\mathcal{L} = \operatorname{span}\{x, y_1, \ldots, y_m\}$ is a common coninvariant subspace for the entire family $\mathcal{A}$. If $\dim \mathcal{L} \leq 2$, then we are done. Therefore, we assume that

$$d = \dim \mathcal{L} = \operatorname{rank}(x \; y_1 \; \ldots \; y_m) \geq 3. \tag{26}$$

Consider two possible cases.

<u>Case 1:</u> $x$ is a linear combination of $y_1, \ldots, y_m$; i.e.,

$$x = \alpha_1 y_1 + \ldots + \alpha_m y_m \tag{27}$$

for some scalars $\alpha_1, \ldots, \alpha_m$. Then, applying $A_i$ $(i = 1, \ldots, m)$ to both sides of (27), we have (see (22))

$$A_i x = (\sum_{j=1}^{m} \alpha_j \overline{\delta}_{ij}) \overline{x}, \qquad i = 1, \ldots, m.$$

It follows that $x$ is a coneigenvector for each matrix $A_i$ and $\mathrm{span}\{x\}$ is a common one-dimensional coninvariant subspace of the family $\mathcal{A}$.

Case 2: $x$ is *not* a linear combination of $y_1, \dots, y_m$. Let $r$ be the maximal number of linearly independent vectors among $y_1, \dots, y_m$. To simplify the notation, we assume that $y_1, \dots, y_r$ are linearly independent. According to (26),

$$r = d - 1 \geq 2.$$

Any triple

$$x, y_i, y_j,$$

where

$$1 \leq i, j \leq r \quad \text{and} \quad i \neq j,$$

is linearly independent, and the argument in Theorem 8 can be applied. For instance, for $i = 1, j = 2$, we have

$$\begin{aligned} A_1[x\ y_1\ y_2] &= \overline{[x\ y_1\ y_2]}M_1, \\ A_2[x\ y_1\ y_2] &= \overline{[x\ y_1\ y_2]}M_2, \end{aligned}$$

where

$$M_1 = \begin{bmatrix} 0 & \overline{\delta}_{11} & \overline{\delta}_{12} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$M_2 = \begin{bmatrix} 0 & \overline{\delta}_{21} & \overline{\delta}_{22} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The concommutation condition $\overline{A}_1 A_2 = \overline{A}_2 A_1$ implies that $\overline{M}_1 M_2 = \overline{M}_2 M_1$, which yields

$$\delta_{11} = \delta_{12} = \delta_{21} = \delta_{22} = 0.$$

In a similar way, we show that

$$\delta_{ij} = 0, \qquad i, j = 1, \dots, r.$$

Thus (see (22)),

$$A_i y_j = 0, \qquad i, j = 1, \dots, r. \tag{28}$$

If $r = m$, then we are already done. Indeed, each of the vectors $y_i$ $(i = 1, \dots, m)$ spans a common one-dimensional coninvariant subspace for $\mathcal{A}$, while $\mathrm{span}\{y_i, y_j\}$ $(i \neq j)$ is a common two-dimensional coninvariant subspace.

Let $r < m$ and

$$r < j \leq m.$$

Since $y_j$ is a linear combination of $y_1, \dots, y_r$, equalities (28) imply that

$$A_i y_j = 0, \qquad i = 1, \dots, r; j = r+1, \dots, m.$$

However,

$$A_j y_i = A_j \overline{A_i} \overline{x} = A_i \overline{A_j} \overline{x} = A_i y_j = 0,$$

for $i = 1, \dots, r; j = r+1, \dots, m$. This means that each of the vectors $y_i$ ($i = 1, \dots, r$) belongs to a common null space of the entire family $\mathcal{A}$. The theorem is proved. □

An immediate implication of Theorem 10 is the following counterpart of Theorem 3.

**Theorem 11** *Let $\mathcal{A} = \{A_i\} \subset \mathbb{C}^{n\times n}$ be such that any two matrices $A_i, A_j \in \mathcal{A}$ concommute. Then, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that all the matrices $U^T A_i U$, $A_i \in \mathcal{A}$, have the same block triangular form with the diagonal blocks of orders* 1 *and* 2.

The proof of Theorem 11 is an almost word-for-word repetition of the proof of Theorem 9. The only alterations are: at each step, $\mathcal{L}$ is chosen as a common coninvariant subspace of the entire current concommuting family (rather than a common coninvariant subspace of two matrices as in Theorem 9) and Theorem 10 rather than Theorem 8 is used.

# 5 Simultaneous reduction of conjugate-normal matrices

Let $A$ be a conjugate-normal matrix. Equating the diagonal entries of the two matrices in (1), we conclude that the 2-norm of row $i$ in $A$ is equal to the 2-norm of column $i$ ($1 \le i \le n$). It easily follows from this observation that, if a conjugate-normal matrix $A$ is block triangular, then, actually, $A$ is block diagonal with the same diagonal blocks. This leads to the following corollary of Theorem 9:

**Theorem 12** *If conjugate-normal matrices $A, B \in \mathbb{C}^{n\times n}$ concommute, then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^T A U$ and $U^T B U$ have the same block diagonal form with the diagonal blocks of orders* 1 *and* 2.

An analogous corollary can be stated for Theorem 11.

If $A$ and $B$ are symmetric, then, certainly, Theorem 12 holds true. However, in this case, we can do better.

**Theorem 13** *If $A, B \in \mathbb{C}^{n\times n}$ are concommuting symmetric matrices, then there exists a unitary matrix $V \in \mathbb{C}^{n\times n}$ such that both $V^T A V$ and $V^T B V$ are real diagonal matrices.*

**Proof.** Since $A = A^T$ and $B = B^T$, we deduce from (2) that

$$A\overline{B} = B\overline{A} = B^T\overline{A^T} = (\overline{B})^*A^* = (A\overline{B})^*.$$

Thus, $A\overline{B}$ is a Hermitian matrix.

According to [3, Corollary 4.5.18(b)], symmetric matrices $A$ and $B$ can be simultaneously reduced to diagonal form by a unitary congruence transformation if and only if $A\overline{B}$ is normal. Let $W$ be a unitary matrix that accomplishes such a reduction for our matrices $A$ and $B$. Thus,

$$\Lambda = W^TAW = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

and

$$M = W^TBW = \mathrm{diag}(\mu_1, \ldots, \mu_n)$$

are diagonal matrices. If both $\Lambda$ and $M$ are real, we are done; hence, we assume that they are not real.

Since concommutation is preserved by unitary congruence transformations, we have

$$\Lambda\overline{M} = M\overline{\Lambda}$$

or

$$\lambda_i\overline{\mu}_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$

Now, we can make all $\mu$'s real by a congruence transformation with an appropriate diagonal unitary matrix $D_1$. This transformation, in general, changes $\Lambda$; however,

$$\widetilde{\Lambda} = D_1\Lambda D_1 \tag{29}$$

remains a diagonal matrix. Moreover, since the modified $\lambda$'s and $\mu$'s satisfy the relations

$$\widetilde{\lambda}_i\widetilde{\mu}_i \in \mathbb{R}, \quad i = 1, \ldots, n,$$

matrix (29) is real in all positions where $\mu_i \neq 0$. If there remain nonreal $\widetilde{\lambda}_j$ in $\widetilde{\Lambda}$, we can make them real by another congruence transformation with a diagonal unitary matrix $D_2$. This matrix has ones in the diagonal positions for which $\widetilde{\lambda}_i$ are already real; hence, it does not change $\widetilde{M} = D_1MD_1$. We conclude that

$$\widehat{\Lambda} = D_2\widetilde{\Lambda}D_2 \qquad \text{and} \qquad \widehat{M} = D_2\widetilde{M}D_2$$

are the desired real diagonal matrices and

$$V = WD_1D_2$$

is the desired unitary matrix. □

It is easy to see that the following converse of Theorem (13) is true:

**Theorem 14** *If symmetric matrices $A, B \in \mathbb{C}^{n\times n}$ can be reduced to real diagonal matrices by the same unitary congruence transformation, then $A$ and $B$ concommute.*

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