DETAILED VERSION OF "STRUCTURED POLYNOMIAL EIGENPROBLEMS RELATED TO TIME-DELAY SYSTEMS"

H. FASSBENDER*, D. S. MACKEY†, N. MACKEY† , AND C. SCHRÖDER‡

Abstract. A new class of structured polynomial eigenproblems arising in the stability analysis of time-delay systems is identified and analyzed together with new types of closely related structured polynomials. Relationships between these polynomials are established via the Cayley transformation. Their spectral symmetries are revealed, and structure-preserving linearizations constructed. A structured Schur decomposition for the class of structured pencils associated with time-delay systems is derived, and an algorithm for its computation that compares favorably with the QZ algorithm is presented along with numerical experiments.

Key words. polynomial eigenvalue problem, palindromic matrix polynomial, quadratic eigenvalue problem, even matrix polynomial, structure-preserving linearization, matrix pencil, structured Schur form, real QZ algorithm, spectral symmetry, Cayley transformation, involution, time-delay systems, delay-differential equations, stability analysis.

AMS subject classifications. 15A18, 15A21, 15A22, 34K06, 34K20, 47A56, 47A75, 65F15

1. Introduction. In this paper we discuss a new class of structured matrix polynomial eigenproblems $Q(\lambda)v = 0$, where

$$Q(\lambda) = \sum_{i=0}^{k} \lambda^i B_i\,, \quad B_i \in \mathbb{C}^{n\times n}\,, \quad B_k \neq 0\,, \qquad \text{and} \quad B_i = P\overline{B}_{k-i}P\,, \quad i = 0,\dots,k \tag{1.1}$$

for a real involutory matrix P (i.e. $P^2 = I$). Here $\overline{B}$ denotes the entrywise conjugation of the matrix B. With

$$\overline{Q}(\lambda) := \sum_{i=0}^{k} \lambda^i \overline{B}_i \qquad \text{and} \qquad \operatorname{rev} Q(\lambda) := \lambda^k Q\left(\frac{1}{\lambda}\right) = \sum_{i=0}^{k} \lambda^i B_{k-i}\,, \tag{1.2}$$

we see that $Q(\lambda)$ in (1.1) satisfies

$$P \cdot \operatorname{rev}\overline{Q}(\lambda) \cdot P \;=\; Q(\lambda). \tag{1.3}$$

As shown in Section 2, the stability analysis of time-delay systems is one important source of eigenproblems as in (1.1). Throughout this paper we assume that all matrix polynomials $Q(\lambda)$ are regular, i.e. that $\det Q(\lambda) \not\equiv 0$.

Matrix polynomials satisfying (1.3) are reminiscent of the various types of palindromic polynomials defined in [17]:

- palindromic: $\operatorname{rev} Q(\lambda) = Q(\lambda)$,
- anti-palindromic: $\operatorname{rev} Q(\lambda) = -Q(\lambda)$,
- $\star$-palindromic: $\operatorname{rev} Q^\star(\lambda) = Q(\lambda)$,

*TU Braunschweig, Carl-Friedrich-Gauß-Fakultät, Institut *Computational Mathematics*, AG Numerik, 38023 Braunschweig, Germany, h.fassbender@tu-bs.de

†Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA, (steve.mackey@wmich.edu, nil.mackey@wmich.edu). Supported by National Science Foundation grant DMS-0713799.

‡TU Berlin, Institut für Mathematik, MA 4-5, 10623 Berlin, Germany, schroed@math.tu-berlin.de, Supported by MATHEON, the DFG research center in Berlin.

- $\star$-anti-palindromic: $\operatorname{rev} Q^\star(\lambda) = -Q(\lambda)$,

where $\star$ denotes transpose T in the real case and either T or conjugate transpose $*$ in the complex case. These palindromic matrix polynomials have the property that reversing the order of the coefficient matrices, followed perhaps by taking their transpose or conjugate transpose, leads back to the original matrix polynomial (up to sign). Several other types of structured matrix polynomial are also defined in [17],

- even, odd: $Q(-\lambda) = \pm Q(\lambda)$,
- $\star$-even, $\star$-odd: $Q^\star(-\lambda) = \pm Q(\lambda)$,

and shown there to be closely related to palindromic polynomials via the Cayley transformation.

We will show that matrix polynomials satisfying (1.3) have properties parallel to those of the palindromic polynomials discussed in [17]. Hence we refer to polynomials with property (1.3) as P-conjugate-P-palindromic polynomials, or PCP polynomials for short. Analogous to the situation in [17], we examine four related types of PCP-like structures,

- PCP: $P \cdot \operatorname{rev} \overline{Q}(\lambda) \cdot P = Q(\lambda)$,
- anti-PCP: $P \cdot \operatorname{rev} \overline{Q}(\lambda) \cdot P = -Q(\lambda)$,
- PCP-even: $P \cdot \overline{Q}(-\lambda) \cdot P = Q(\lambda)$,
- PCP-odd: $P \cdot \overline{Q}(-\lambda) \cdot P = -Q(\lambda)$,

revealing their spectral symmetry properties, their relationships to each other via the Cayley transformation, as well as their structured linearizations. Here we continue the practice stemming from Lancaster [13] of developing theory for polynomials of degree k wherever possible in order to gain the most insight and understanding.

There are a number of ways in which palindromic matrix polynomials can be thought of as generalizations of symplectic matrices. For example, palindromic polynomials and symplectic matrices both have reciprocal pairing symmetry in their spectra. In addition, the Cayley transformation relates palindromic polynomials to even/odd matrix polynomials in the same way as it relates symplectic matrices to Hamiltonian matrices, and even/odd matrix polynomials represent generalizations of Hamiltonian matrices. Further information on the relationship between symplectic matrices and palindromic polynomials can be found in [22] and, in the context of optimal control problems, in [2].

The classical approach to investigate or numerically solve polynomial eigenvalue problems is linearization. A $kn \times kn$ pencil $L(\lambda)$ is said to be a linearization for an $n \times n$ polynomial $Q(\lambda)$ of degree k if $E(\lambda)L(\lambda)F(\lambda) = \operatorname{diag}[Q(\lambda), I_{(k-1)n}]$ for some $E(\lambda)$ and $F(\lambda)$ with nonzero *constant* determinants. The companion forms [4] provide the standard examples of linearization for a matrix polynomial $Q(\lambda)$. Let $X_1 = X_2 = \operatorname{diag}(B_k, I_n, \ldots, I_n)$,

$$Y_1 = \begin{bmatrix} B_{k-1} & B_{k-2} & \cdots & B_0 \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}, \quad \text{and} \quad Y_2 = \begin{bmatrix} B_{k-1} & -I_n & & 0 \\ B_{k-2} & 0 & \ddots & \\ \vdots & \vdots & \ddots & -I_n \\ B_0 & 0 & \cdots & 0 \end{bmatrix}. \tag{1.4}$$

Then $C_1(\lambda) = \lambda X_1 + Y_1$ and $C_2(\lambda) = \lambda X_2 + Y_2$ are the first and second companion forms for $Q(\lambda)$. These linearizations do not reflect any structure that might be present in the matrix polynomial Q, so only standard numerical methods can be applied to solve the eigenproblem $C_i(\lambda)v = 0$. In a finite precision environment this may produce physically meaningless results [23], e.g., loss of symmetries in the spectrum. Hence it is useful to construct linearizations that reflect the structure of the given polynomial, and then to develop numerical methods for the resulting linear eigenvalue problem that properly address these structures.

It is well known that for regular matrix polynomials, linearizations preserve algebraic and partial multiplicities of all finite eigenvalues [4]. In order to preserve the multiplicities of the eigenvalue ∞, one has to consider linearizations $L(\lambda)$ which have the additional property that $\operatorname{rev} L(\lambda)$ is also a linearization for $\operatorname{rev} Q(\lambda)$, see [3]. Such linearizations have been named strong linearizations in [14]. Both the first and the second companion form are strong linearizations for any regular matrix polynomial [3, Proposition 1.1].

Several recent papers have systematically addressed the tasks of broadening the menu of available linearizations, providing criteria to guide the choice of linearization, and identifying structure-preserving linearizations for various types of structured polynomial. In [18], two vector spaces of pencils generalizing the companion forms were constructed and many interesting properties were proved, including that almost all of these pencils are linearizations. The conditioning and backward error properties of some of these linearizations were analyzed in [6], [8], and [10], developing criteria for choosing a linearization best suited for numerical computation. Linearizations within these vector spaces were identified in [17], [7], and [9] that respect palindromic and odd-even structure, symmetric and Hermitian structure, and definiteness structure, respectively.

In this paper we investigate the four types of PCP-structured matrix polynomial, analyzing their spectral symmetries in Section 3, the relationships between the various PCP-structures via the Cayley transformation in Section 4, and then showing how to build structured linearizations for each type of PCP-structure in Section 5. The existence and computation of a structured Schur-type decomposition for PCP-pencils is discussed in Section 6, and Section 7 concludes with numerical results for some examples arising from physical applications. We first, though, discuss in more detail a key source of PCP-structured eigenproblems.

2. Time-delay systems. To motivate our consideration of matrix polynomials with PCP-structure, we describe how the stability analysis of time-delay systems (also known as delay-differential equations, see e.g. [5, 20]) leads to eigenproblems with this structure. A *neutral linear time-delay system* (TDS) with m constant delays $h_1, \ldots, h_m \geq 0$ and $h_0 = 0$ is given by

$$\mathcal{S} = \begin{cases} \sum_{k=0}^{m} D_k \dot{x}(t - h_k) = \sum_{k=0}^{m} A_k x(t - h_k)\,, & t \geq 0 \\ x(t) = \varphi(t)\,, & t \in [-\widehat{h}, 0) \end{cases} \tag{2.1}$$

with $\widehat{h} = \max_i\{h_i\}$, $x : [-\widehat{h}, \infty) \to \mathbb{R}^n$, $\varphi \in \mathcal{C}^1[-\widehat{h}, 0]$, and $A_k, D_k \in \mathbb{R}^{n\times n}$ for $k = 0, \ldots, m$. An important special case of (2.1) is the class of *retarded time-delay systems*, in which $D_0 = I$ and $D_k = 0$ for $k = 1, \ldots, m$.

The stability of a TDS can be determined from its characteristic equation, i.e., from the nontrivial solutions of the nonlinear eigenvalue problem

$$\begin{gathered} \mathbb{M}(s)v = 0\,, \quad \text{where} \quad \mathbb{M}(s) = -s\mathbb{D}(s) + \mathbb{A}(s) \\ \text{with} \quad \mathbb{D}(s) = \sum_{k=0}^{m} D_k e^{-h_k s} \quad \text{and} \quad \mathbb{A}(s) = \sum_{k=0}^{m} A_k e^{-h_k s}. \end{gathered} \tag{2.2}$$

As usual, $s \in \mathbb{C}$ is called an eigenvalue associated with the eigenvector $v \in \mathbb{C}^n$, and the set of all eigenvalues $\sigma(\mathcal{S})$ is called the spectrum of $\mathcal{S}$. Having an eigenvalue in the right half-plane implies that $\mathcal{S}$ is unstable; conversely, having $\sigma(\mathcal{S})$ completely contained in the left half-plane usually implies that $\mathcal{S}$ is stable, although some additional technical assumptions are required. For further details see [12], [20].

A time-delay system $\mathcal{S}$ is called *critical* if $\sigma(\mathcal{S}) \cap i\mathbb{R} \neq \emptyset$, $i = \sqrt{-1}$. The set of all points $(h_1, h_2, \ldots, h_m)$ in delay-parameter space for which $\mathcal{S}$ is critical form the critical curves ($m = 2$) or critical surfaces ($m > 2$) of the TDS. Since a TDS can change stability when an eigenvalue pair crosses the imaginary axis, the critical curves/surfaces are important in the study of the delay-parameter space stability domain. In most cases of practical interest, the boundary of the stability domain is just a subset of the critical curves/surfaces [20, Section 1.2].

Thus the computation of critical sets, for which a number of approaches exist (see [12] for a list of references), is a key step in the stability analysis of time-delay systems. Here we outline the new method for this computation developed in [12], leading ultimately to a quadratic eigenproblem with PCP-palindromic structure that will have to be solved repeatedly for many different parameter values.

To determine critical points in delay-parameter space, we need to compute purely imaginary eigenvalues of $\mathbb{M}(s)$ in (2.2), i.e., to find $s = i\omega$ with $\omega \in \mathbb{R}$ such that

$$\mathbb{M}(i\omega)v = 0\,. \tag{2.3}$$

As shown in [12], for any $\omega \in \mathbb{R}$ and $v \in \mathbb{C}^n$ such that $v^*v = 1$ and $\widehat{v} := \mathbb{D}(i\omega)v \neq 0$, equation (2.3) is equivalent to the pair of conditions

$$\mathbb{L}(vv^*, i\omega) = 0 \quad \text{and} \quad \widehat{v}^*\mathbb{M}(i\omega)v = 0\,, \tag{2.4}$$

where $\mathbb{L}$ is a Lyapunov-type operator

$$\begin{aligned} \mathbb{L}(X, s) \;&:=\; \mathbb{M}(s)X\mathbb{D}(s)^* + \mathbb{D}(s)X\mathbb{M}(s)^* \\ &=\; \mathbb{A}(s)X\mathbb{D}(s)^* + \mathbb{D}(s)X\mathbb{A}(s)^* - 2\mathbb{D}(s)X\mathbb{D}(s)^*\mathrm{Re}(s) \end{aligned} \tag{2.5}$$

for $X \in \mathbb{C}^{n\times n}$, $s \in \mathbb{C}$. That (2.3) $\Rightarrow$ (2.4) follows immediately from

$$\begin{aligned} \mathbb{L}(vv^*, i\omega) \;&=\; \mathbb{M}(i\omega)vv^*\mathbb{D}(i\omega)^* + \mathbb{D}(i\omega)vv^*\mathbb{M}(i\omega)^* \\ &=\; \mathbb{M}(i\omega)v\,\widehat{v}^* + \widehat{v}\,\big(\mathbb{M}(i\omega)v\big)^*, \end{aligned} \tag{2.6}$$

while the implication (2.4) $\Rightarrow$ (2.3) follows by pre-multiplying (2.6) with $\widehat{v}^*$ and using the assumption $\widehat{v} \neq 0$.

Note that the assumption $\widehat{v} = \mathbb{D}(i\omega)v \neq 0$ is not very restrictive, and can be regarded as a kind of genericity condition, since $\mathbb{D}(i\omega)v = 0$ in (2.3) implies that $\mathbb{A}(i\omega)v = 0$ would have to simultaneously hold. In addition $\mathbb{D}(i\omega)v = 0$ if and only if the difference equation

$$D_0x(t) + D_1x(t-h_1) + \ldots + D_mx(t-h_m) = 0 \tag{2.7}$$

has a purely imaginary eigenvalue, which happens only in very special situations.

We now see how (2.4) can be used to systematically explore delay-parameter space to find the critical set. From (2.5) we have

$$\begin{aligned} &\mathbb{L}(vv^*, i\omega) \;=\; \mathbb{A}(i\omega)vv^*\mathbb{D}(i\omega)^* + \mathbb{D}(i\omega)vv^*\mathbb{A}(i\omega)^* \\ &\text{with}\quad \mathbb{D}(i\omega) = \sum_{k=0}^{m} D_k e^{-i\omega h_k} \quad \text{and} \quad \mathbb{A}(i\omega) = \sum_{k=0}^{m} A_k e^{-i\omega h_k}. \end{aligned} \tag{2.8}$$

Because of the periodicity in the exponential terms of $\mathbb{D}(i\omega)$ and $\mathbb{A}(i\omega)$, there is an ω-dependent periodicity in the critical set; if $(h_1, h_2, \ldots, h_m)$ is a critical delay corresponding to the solution $i\omega, v$ of the equation $\mathbb{L}(vv^*, i\omega) = 0$, then

$$(h_1, h_2, \ldots, h_m) \,+\, (2\pi/\omega)(p_1, p_2, \ldots, p_m)$$

is also a critical delay for any $(p_1, p_2, \ldots, p_m) \in \mathbb{Z}^m$. Thus it suffices to consider only the angles $\varphi_k := \omega h_k$ for $k = 1, \ldots, m$ where $\varphi_k \in [-\pi, \pi]$. These can be explored by a line-search strategy: with $\varphi_0 := \omega h_0 = 0$, for each fixed choice of $\varphi_1, \ldots, \varphi_{m-1}$ view $z := e^{-i\varphi_m}$ as a variable and rewrite $\mathbb{L}(vv^*, i\omega) = 0$ as an eigenproblem in terms of z and vv^*. Defining

$$A_{\mathrm{S}} := \sum_{k=0}^{m-1} A_k e^{-i\varphi_k} \quad \text{and} \quad D_{\mathrm{S}} := \sum_{k=0}^{m-1} D_k e^{-i\varphi_k} \tag{2.9}$$

and using (2.8), we have

$$\big(A_m z + A_{\mathrm{S}}\big) vv^* \big(D_m z + D_{\mathrm{S}}\big)^* \,+\, \big(D_m z + D_{\mathrm{S}}\big) vv^* \big(A_m z + A_{\mathrm{S}}\big)^* \;=\; 0\,. \tag{2.10}$$

Expanding and vectorizing (2.10) yields

$$(zE \,+\, F \,+\, \overline{z}G)\,\mathrm{vec}\,(vv^*) \;=\; 0\,, \tag{2.11}$$

where

$$\begin{aligned} E &= \overline{D}_{\mathrm{S}} \otimes A_m \,+\, \overline{A}_{\mathrm{S}} \otimes D_m\,, \\ F &= D_m \otimes A_m \,+\, \overline{D}_{\mathrm{S}} \otimes A_{\mathrm{S}} \,+\, \overline{A}_{\mathrm{S}} \otimes D_{\mathrm{S}} \,+\, A_m \otimes D_m\,, \\ G &= D_m \otimes A_{\mathrm{S}} \,+\, A_m \otimes D_{\mathrm{S}}\,, \end{aligned} \tag{2.12}$$

and $\otimes$ denotes the usual Kronecker product [11, Chapter 4.3]. Then multiplying (2.11) by z with $|z| = 1$ results in the quadratic eigenvalue problem

$$(z^2 E + zF + G)u = 0\,. \tag{2.13}$$

A solution (z, u) of (2.13) with $|z| = 1$ and u of the form $\mathrm{vec}\,(vv^*)$ completes the determination of $(\varphi_1, \varphi_2, \ldots, \varphi_m) = \omega(h_1, h_2, \ldots, h_m)$, and hence of a critical delay up to a real scalar multiple ω. The scaling factor ω, and hence a pure imaginary eigenvalue $s = i\omega$ of (2.2), is determined by invoking the second condition in (2.4):

$$\begin{aligned} 0 &= i\widehat{v}^*\mathbb{M}(i\omega)v \\ &= i\widehat{v}^*\big(-i\omega\mathbb{D}(i\omega) \,+\, \mathbb{A}(i\omega)\big)v \\ &= \omega\widehat{v}^*\mathbb{D}(i\omega)v \,+\, i\widehat{v}^*\mathbb{A}(i\omega)v \\ &= \omega\widehat{v}^*\widehat{v} \,+\, i\widehat{v}^*\big(A_m z + A_{\mathrm{S}}\big)v\,, \end{aligned}$$

and hence

$$\omega = -i\widehat{v}^*\big(A_m z + A_{\mathrm{S}}\big)v/\big(\widehat{v}^*\widehat{v}\big)\,.$$

From (2.10) we can see that $\omega \in \mathbb{R}$. Define $\widehat{x} := \big(A_m z + A_{\mathrm{S}}\big)v$, so that (2.10) says $\widehat{x}\widehat{v}^* + \widehat{v}\widehat{x}^* = 0$. Then $\widehat{v}^*(\widehat{x}\widehat{v}^* + \widehat{v}\widehat{x}^*)\widehat{v} = 0 \Rightarrow (\widehat{v}^*\widehat{v})(\widehat{v}^*\widehat{x} + \widehat{x}^*\widehat{v}) = 0 \;\Rightarrow\; \widehat{v}^*\widehat{x} + \widehat{x}^*\widehat{v} = 0$, so $\widehat{v}^*\widehat{x} \in i\mathbb{R}$ and hence $\omega \in \mathbb{R}$.

The preceding discussion can be summarized in the following theorem.

THEOREM 2.1 ([12]). *Assume that the difference equation* (2.7) *has no purely imaginary eigenvalues. With* $\varphi_0 = 0$ *and any given combination of angles* $\varphi_k \in [-\pi, \pi]$ *for* $k = 1, \ldots, m-1$*, consider the quadratic eigenvalue problem*

$$(z^2 E + zF + G)u = 0 \tag{2.14}$$

where $E, F, G \in \mathbb{C}^{n^2 \times n^2}$ *are given by* (2.12)*. Then for any solution of* (2.14) *with* $|z| = 1$ *and* u *of the form* $u = \mathrm{vec}\,(vv^*) = \overline{v} \otimes v$ *for some* $v \in \mathbb{C}^n$ *with* $v^*v = 1$*, critical delays for the TDS* (2.1) *can be constructed as follows. Let*

$$\widehat{v} \;=\; \big(D_m z + D_{\mathrm{S}}\big)v \quad \text{and} \quad \omega \;=\; -i\widehat{v}^*\big(A_m z + A_{\mathrm{S}}\big)v/\big(\widehat{v}^*\widehat{v}\big)\,. \tag{2.15}$$

Then for any $(p_1, p_2, \ldots, p_m) \in \mathbb{Z}^m$*,*

$$(h_1, h_2, \ldots, h_m) \;=\; \left(\frac{1}{\omega}\right)\Big[(\varphi_1, \ldots, \varphi_{m-1}, -\mathrm{Arg}\ z) \;+\; 2\pi(p_1, p_2, \ldots, p_m)\Big]$$

is a critical delay for (2.1)*.*

It is now straightforward to see why the quadratic matrix polynomial

$$Q(z) \;=\; z^2 E + zF + G \tag{2.16}$$

in (2.14) has PCP-structure. By [11, Cor 4.3.10] there exists an involutory, symmetric permutation matrix $P \in \mathbb{R}^{n^2 \times n^2}$ (i.e. $P = P^{-1} = P^T$) such that

$$B \otimes C \;=\; P(C \otimes B)P \tag{2.17}$$

for all $B, C \in \mathbb{C}^{n \times n}$. Thus we have in (2.16) that $E = P\overline{G}P$ and $F = P\overline{F}P$, since

$$\begin{aligned} E \;&=\; \overline{D}_{\mathrm{S}} \otimes A_m \;+\; \overline{A}_{\mathrm{S}} \otimes D_m \\ &=\; P\Big[A_m \otimes \overline{D}_{\mathrm{S}}\Big]P \;+\; P\Big[D_m \otimes \overline{A}_{\mathrm{S}}\Big]P \\ &=\; P\Big[A_m \otimes \overline{D}_{\mathrm{S}} \;+\; D_m \otimes \overline{A}_{\mathrm{S}}\Big]P \;=\; P\overline{G}P\,. \end{aligned}$$

The fact that $F = P\overline{F}P$ follows in a similar fashion. This implies

$$Q(z) = z^2 E + zF + G = P(z^2\overline{G} + z\overline{F} + \overline{E})P = P \cdot \mathrm{rev}\,\overline{Q}(z) \cdot P\,,$$

that is, (2.16) is a matrix polynomial as in (1.1) and (1.3).

Time-delay systems arise in a variety of applications [21], including electric circuits, population dynamics, and the control of chemical processes. Several realistic problems are discussed in Section 7, and some numerical results are given.

3. Spectral symmetry. Suppose $Q(\lambda)$ has property (1.3), and let $\lambda \neq 0$ be an eigenvalue of $Q(\lambda)$ associated to the eigenvector v, that is $Q(\lambda)v = 0$. Then we have

$$0 = Q(\lambda)v = P \cdot \mathrm{rev}\,\overline{Q}(\lambda) \cdot Pv \;\;\Rightarrow\;\; \mathrm{rev}\,\overline{Q}(\lambda) \cdot (Pv) = 0\,,$$

which from definition (1.2) of rev implies that

$$Q(1/\overline{\lambda}) \cdot (P\overline{v}) \;=\; 0\,.$$

Hence if λ is an eigenvalue with eigenvector v, then $1/\overline{\lambda}$ is an eigenvalue with eigenvector $P\overline{v}$. Note that for any matrix polynomial Q, (1.2) implies that the nonzero finite eigenvalues of $\operatorname{rev} Q(\lambda)$ are the reciprocals of those of Q.

The following theorem extends this observation of reciprocal pairing for eigenvalues of PCP-palindromic polynomials to include eigenvalues at ∞, pairing of eigenvalue multiplicities, as well as to an analogous eigenvalue pairing for PCP-even/odd polynomials. As in [17], we will employ the convention that $Q(\lambda)$ has an eigenvalue at ∞ with eigenvector x if $\operatorname{rev} Q(\lambda)$ has the eigenvalue 0 with eigenvector x. The algebraic, geometric, and partial multiplicities of an eigenvalue at ∞ are defined to be the same as the corresponding multiplicities of the zero eigenvalue of $\operatorname{rev} Q(\lambda)$.

THEOREM 3.1 (Spectral Symmetry). *Let $Q(\lambda) = \sum_{i=0}^{k} \lambda^i B_i$, $B_k \neq 0$ be a regular matrix polynomial and P a real involution.*

(a) *If $Q(\lambda) = \pm P \operatorname{rev} \overline{Q}(\lambda) P$, then the spectrum of $Q(\lambda)$ has the pairing $(\lambda, 1/\overline{\lambda})$.*

(b) *If $Q(\lambda) = \pm P \overline{Q}(-\lambda) P$, then the spectrum of $Q(\lambda)$ has the pairing $(\lambda, -\overline{\lambda})$.*

Moreover, the algebraic, geometric, and partial multiplicities of the eigenvalues in each such pair are equal. (Here we allow $\lambda = 0$ and interpret $1/\overline{\lambda}$ as the eigenvalue ∞.)

Proof. We first recall some well-known facts [4] about strict equivalence of pencils and about the companion form $C_1(\lambda)$ of a matrix polynomial $Q(\lambda)$:

1. $Q(\lambda)$ and $C_1(\lambda)$ have the same eigenvalues (including ∞) with the same algebraic, geometric, and partial multiplicities.
2. Any two strictly equivalent pencils have the same eigenvalues (including ∞) with the same algebraic, geometric, and partial multiplicities.

Because of these two facts it suffices to show that $C_1(\lambda)$ is strictly equivalent to $\operatorname{rev} \overline{C}_1(\lambda) = \lambda \overline{Y}_1 + \overline{X}_1$ for part (a), and to $\overline{C}_1(-\lambda) = -\lambda \overline{X}_1 + \overline{Y}_1$ for part (b). The desired eigenvalue pairings and equality of multiplicities then follow.

For part (a), suppose that $Q(\lambda) = \varepsilon P \cdot \operatorname{rev} \overline{Q}(\lambda) \cdot P$ where $\varepsilon = \pm 1$, or equivalently, that $B_i = \varepsilon P \overline{B}_{k-i} P$ for $Q(\lambda) = \sum_{i=0}^{k} \lambda^i B_i$. Consider the nonsingular matrix

$$R_k \otimes I_n = \begin{bmatrix} 0 & & I_n \\ & \cdot^{\cdot^{\cdot}} & \\ I_n & & 0 \end{bmatrix}, \quad \text{where} \quad R_k := \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}_{k \times k}.$$

Then

$$C_1(\lambda) \cdot (R_k \otimes I_n) \;=\; \lambda \begin{bmatrix} 0 & & & B_k \\ & & I_n & \\ & \cdot^{\cdot^{\cdot}} & & \\ I_n & & & 0 \end{bmatrix} + \begin{bmatrix} B_0 & B_1 & \dots & B_{k-1} \\ 0 & 0 & & -I_n \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ 0 & -I_n & & 0 \end{bmatrix}.$$

Multiplying on the left with the nonsingular matrix

$$T := \begin{bmatrix} I_n & B_{k-1} & \dots & B_1 \\ 0 & 0 & & -I_n \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ 0 & -I_n & & 0 \end{bmatrix},$$

we then have

$$T \cdot C_1(\lambda) \cdot (R_k \otimes I_n) \;=\; \lambda \begin{bmatrix} B_1 & \dots & B_{k-1} & B_k \\ -I_n & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & -I_n & 0 \end{bmatrix} + \begin{bmatrix} B_0 & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix}.$$

Replacing each block B_i by $\varepsilon P\overline{B}_{k-i}P$ we obtain

$$T\cdot C_1(\lambda)\cdot(R_k\otimes I_n) \;=\; \lambda\begin{bmatrix} \varepsilon P\overline{B}_{k-1}P & \dots & \varepsilon P\overline{B}_1P & \varepsilon P\overline{B}_0P \\ -I_n & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & -I_n & 0 \end{bmatrix} + \begin{bmatrix} \varepsilon P\overline{B}_kP & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix},$$

which can then be factored as

$$\begin{aligned} T\cdot C_1(\lambda)\cdot(R_k\otimes I_n) &= \widehat{P}_1\left(\lambda\begin{bmatrix} \overline{B}_{k-1} & \dots & \overline{B}_1 & \overline{B}_0 \\ -I_n & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & -I_n & 0 \end{bmatrix} + \begin{bmatrix} \overline{B}_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix}\right)\widehat{P}_2 \\ &= \widehat{P}_1\cdot \operatorname{rev}\overline{C}_1(\lambda)\cdot\widehat{P}_2\,, \end{aligned}$$

where

$$\widehat{P}_1 = \begin{bmatrix} \varepsilon & \\ & I_{k-1} \end{bmatrix}\otimes P \quad \text{and} \quad \widehat{P}_2 = I_k\otimes P\,. \tag{3.1}$$

Thus $C_1(\lambda)$ is strictly equivalent to $\operatorname{rev}\overline{C}_1(\lambda)$, which completes the proof of part (a) of the theorem.

We proceed in a similar manner for part (b). Suppose $Q(\lambda) = \varepsilon P\cdot\overline{Q}(-\lambda)\cdot P$ holds for $\varepsilon = \pm 1$, that is $(-1)^iB_i = \varepsilon P\overline{B}_iP$ for $Q(\lambda) = \sum_{i=0}^{k}\lambda^iB_i$. In this situation we aim to show that $C_1(\lambda)$ is strictly equivalent to $\overline{C}_1(-\lambda)$. Consider the nonsingular "alternating signs" matrix

$$\Sigma_k := \begin{bmatrix} (-1)^{k-1} & & & \\ & (-1)^{k-2} & & \\ & & \ddots & \\ & & & (-1)^0 \end{bmatrix}_{k\times k}.$$

Then

$$\begin{aligned} C_1(\lambda)\cdot(\Sigma_k\otimes I_n) \;=\; &\lambda\begin{bmatrix} (-1)^{k-1}B_k & & & & \\ & (-1)^{k-2}I_n & & & \\ & & \ddots & & \\ & & & (-1)^1I_n & \\ & & & & (-1)^0I_n \end{bmatrix} \\ &+\begin{bmatrix} (-1)^{k-1}B_{k-1} & (-1)^{k-2}B_{k-2} & \dots & (-1)^1B_1 & (-1)^0B_0 \\ (-1)^{k-2}I_n & 0 & \dots & 0 & 0 \\ & (-1)^{k-3}I_n & \ddots & & 0 \\ & & \ddots & & \vdots \\ & & & (-1)^0I_n & 0 \end{bmatrix}. \end{aligned}$$

Multiplying on the left by the nonsingular matrix $\widetilde{T} := \begin{bmatrix} 1 & \\ & -\Sigma_{k-1} \end{bmatrix}\otimes I_n$, we then have

$$\widetilde{T} \cdot C_1(\lambda) \cdot (\Sigma_k \otimes I_n) \;=\; (-\lambda) \begin{bmatrix} (-1)^k B_k & & & & \\ & I_n & & & \\ & & I_n & & \\ & & & \ddots & \\ & & & & I_n \end{bmatrix}$$

$$+ \begin{bmatrix} (-1)^{k-1} B_{k-1} & (-1)^{k-2} B_{k-2} & \dots & (-1)^1 B_1 & (-1)^0 B_0 \\ -I_n & 0 & \dots & 0 & 0 \\ & -I_n & \ddots & & 0 \\ & & \ddots & & \vdots \\ & & & -I_n & 0 \end{bmatrix}.$$

Using the property of $Q(\lambda)$ that $(-1)^i B_i = \varepsilon P \overline{B}_i P$, we obtain

$$\widetilde{T} \cdot C_1(\lambda) \cdot (\Sigma_k \otimes I_n) \;=\; (-\lambda) \begin{bmatrix} \varepsilon P \overline{B}_k P & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} \varepsilon P \overline{B}_{k-1} P & \dots & \varepsilon P \overline{B}_1 P & \varepsilon P \overline{B}_0 P \\ -I_n & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & -I_n & 0 \end{bmatrix},$$

which can then be factored as

$$\widetilde{T} \cdot C_1(\lambda) \cdot (\Sigma_k \otimes I_n) = \widehat{P}_1 \left((-\lambda) \begin{bmatrix} \overline{B}_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} \overline{B}_{k-1} & \dots & \overline{B}_1 & \overline{B}_0 \\ -I_n & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & -I_n & 0 \end{bmatrix} \right) \widehat{P}_2$$

$$= \widehat{P}_1 \cdot \overline{C}_1(-\lambda) \cdot \widehat{P}_2 \,,$$

where $\widehat{P}_1$ and $\widehat{P}_2$ are as in (3.1). Thus $C_1(\lambda)$ is shown to be strictly equivalent to $\overline{C}_1(-\lambda)$, completing the proof of part (b). □

The same eigenvalue pairings have also been previously observed in [17] for $*$-(anti)-palindromic and $*$-even/odd matrix polynomials; these results are summarized in Table 3.1. Observe further that when the coefficient matrices of Q are all real, then

Table 3.1
Spectral symmetries

Structure of $Q(\lambda)$	eigenvalue pairing
(anti)-palindromic, T-(anti)-palindromic	$(\lambda, 1/\lambda)$
$*$-palindromic, $*$-anti-palindromic	$(\lambda, 1/\overline{\lambda})$
(anti)-PCP	$(\lambda, 1/\overline{\lambda})$
even, odd, T-even, T-odd	$(\lambda, -\lambda)$
$*$-even, $*$-odd	$(\lambda, -\overline{\lambda})$
PCP-even, PCP-odd	$(\lambda, -\overline{\lambda})$

for all the palindromic structures listed in Table 3.1 the eigenvalues occur not just in pairs but in quadruples $(\lambda, \overline{\lambda}, 1/\lambda, 1/\overline{\lambda})$. This property is sometimes referred to as "symplectic spectral symmetry", since real symplectic matrices exhibit this behavior. In the context of the time-delay problem, though, the coefficient matrices E, F, G of $Q(z)$ in (2.16) are typically not all real unless there is only a single delay h_1 in the problem.

4. Relationships between structured polynomials. It is well known that the Cayley transformation and its generalizations to matrix pencils relates Hamiltonian structure to symplectic structure for both matrices and pencils [15, 19]. By using the extensions of the classical definition of this transformation to matrix polynomials as given in [17], we develop analogous relationships between the structured matrix polynomials considered here.

The Cayley transforms of a degree k matrix polynomial $Q(\lambda)$ with pole at $+1$ or -1, respectively, are the matrix polynomials $\mathcal{C}_{+1}(Q)$ and $\mathcal{C}_{-1}(Q)$ defined by

$$\begin{aligned} \mathcal{C}_{+1}(Q)(\mu) &:= (1-\mu)^k Q\left(\frac{1+\mu}{1-\mu}\right) \\ \text{and} \quad \mathcal{C}_{-1}(Q)(\mu) &:= (\mu+1)^k Q\left(\frac{\mu-1}{\mu+1}\right). \end{aligned} \tag{4.1}$$

This choice of definition was motivated in [17] by the observation that the Möbius transformations $\frac{\mu-1}{\mu+1}$ and $\frac{1+\mu}{1-\mu}$ map reciprocal pairs $(\mu, \frac{1}{\mu})$ to plus/minus pairs $(\lambda, -\lambda)$, as well as conjugate reciprocal pairs $(\mu, 1/\overline{\mu})$ to conjugate plus/minus pairs $(\lambda, -\overline{\lambda})$. When viewed as maps on the space of $n \times n$ matrix polynomials of degree k, the Cayley transformations in (4.1) can be shown by direct calculation to be inverses of each other up to a scaling factor [17], that is,

$$\mathcal{C}_{+1}(\mathcal{C}_{-1}(Q)) = \mathcal{C}_{-1}(\mathcal{C}_{+1}(Q)) = 2^k \cdot Q, \quad \text{where} \quad 1 \le k = \deg Q.$$

The next lemma gives some straightforward observations that are helpful in relating structure of a matrix polynomial to that of its Cayley transform.

LEMMA 4.1. *Let Q be a matrix polynomial of degree k. Then*

$$\operatorname{rev}\mathcal{C}_{+1}(Q)(\mu) = (\mu-1)^k Q\left(\frac{\mu+1}{\mu-1}\right) = (-1)^k \mathcal{C}_{-1}(Q)(-\mu) \tag{4.2}$$

and

$$\operatorname{rev}\mathcal{C}_{-1}(Q)(\mu) = (\mu+1)^k Q\left(\frac{1-\mu}{1+\mu}\right) = \mathcal{C}_{+1}(Q)(-\mu). \tag{4.3}$$

Proof. Relations (4.2) and (4.3) follow directly from the definitions and some simple algebraic manipulations. For (4.2) with $\mu \neq 1$ we have

$$\operatorname{rev}\mathcal{C}_{+1}(Q)(\mu) = \mu^k \mathcal{C}_{+1}(Q)\left(\frac{1}{\mu}\right) = \mu^k (1-\tfrac{1}{\mu})^k Q\left(\frac{1+\frac{1}{\mu}}{1-\frac{1}{\mu}}\right) = (\mu-1)^k Q\left(\frac{\mu+1}{\mu-1}\right).$$

That this is equal to $(-1)^k \mathcal{C}_{-1}(Q)(-\mu)$ then follows immediately from Definition 4.1; equality extends to $\mu = 1$ by continuity, since each expression represents a polynomial.

In a similar manner we have for (4.3) with $\mu \neq -1$ that

$$\operatorname{rev}\mathcal{C}_{-1}(Q)(\mu) = \mu^k \mathcal{C}_{-1}(Q)\left(\frac{1}{\mu}\right) = \mu^k (\tfrac{1}{\mu}+1)^k Q\left(\frac{\frac{1}{\mu}-1}{\frac{1}{\mu}+1}\right) = (\mu+1)^k Q\left(\frac{1-\mu}{1+\mu}\right),$$

which in turn is seen to be equal to $\mathcal{C}_{+1}(Q)(-\mu)$ directly from Definition 4.1. Equality again extends to $\mu = -1$ by continuity. ∎

We are now in a position to prove the following theorem relating structure in $Q(\lambda)$ to that of its Cayley transforms.

THEOREM 4.2 (Structure of Cayley transforms). *Let $Q(\lambda)$ be a matrix polynomial of degree k and let P be a real involution.*

1. *If $Q(\lambda)$ is (anti-)PCP, then the Cayley transforms of Q are PCP-even or PCP-odd. More precisely, if $Q(\lambda) = \pm P \cdot \mathrm{rev}\,\overline{Q}(\lambda) \cdot P$ then*

$$\mathcal{C}_{+1}(Q)(\mu) = \pm P \cdot \mathcal{C}_{+1}(\overline{Q})(-\mu) \cdot P\,,$$
$$\mathcal{C}_{-1}(Q)(\mu) = \pm(-1)^k P \cdot \mathcal{C}_{-1}(\overline{Q})(-\mu) \cdot P\,.$$

2. *If $Q(\lambda)$ has PCP-even/odd structure, then the Cayley transforms of Q are (anti-)PCP. Specifically, if $Q(\lambda) = \pm P \cdot \overline{Q}(-\lambda) \cdot P$ then*

$$\mathcal{C}_{+1}(Q)(\mu) = \pm(-1)^k P \cdot \mathrm{rev}\,(\mathcal{C}_{+1}(\overline{Q})(\mu)) \cdot P\,,$$
$$\mathcal{C}_{-1}(Q)(\mu) = \pm P \cdot \mathrm{rev}\,(\mathcal{C}_{-1}(\overline{Q})(\mu)) \cdot P\,.$$

Proof. To prove part (1), suppose that

$$Q(\lambda) \,=\, \pm P \cdot \mathrm{rev}\,\overline{Q}(\lambda) \cdot P \,=\, \pm P \cdot \lambda^k \overline{Q}\big(\tfrac{1}{\lambda}\big) \cdot P\,. \tag{4.4}$$

Setting $\lambda = \frac{1+\mu}{1-\mu}$ in (4.4) and multiplying by $(1-\mu)^k$ yields

$$\begin{aligned}(1-\mu)^k Q\left(\frac{1+\mu}{1-\mu}\right) &= \pm(1-\mu)^k P \cdot \left(\frac{1+\mu}{1-\mu}\right)^k \overline{Q}\left(\frac{1-\mu}{1+\mu}\right) \cdot P\\ &= \pm P \cdot (1+\mu)^k \overline{Q}\left(\frac{1-\mu}{1+\mu}\right) \cdot P\,,\end{aligned}$$

which by (4.3) says that $\mathcal{C}_{+1}(Q)(\mu) = \pm P \cdot \mathcal{C}_{+1}(\overline{Q})(-\mu) \cdot P$. On the other hand, setting $\lambda = \frac{\mu-1}{\mu+1}$ in (4.4) and multiplying with $(\mu+1)^k$ yields

$$\begin{aligned}(\mu+1)^k Q\left(\frac{\mu-1}{\mu+1}\right) &= \pm(\mu+1)^k P \cdot \left(\frac{\mu-1}{\mu+1}\right)^k \overline{Q}\left(\frac{\mu+1}{\mu-1}\right) \cdot P\\ &= \pm P \cdot (\mu-1)^k \overline{Q}\left(\frac{\mu+1}{\mu-1}\right) \cdot P\,,\end{aligned}$$

which by (4.2) says that $\mathcal{C}_{-1}(Q)(\mu) = \pm(-1)^k P \cdot \mathcal{C}_{-1}(\overline{Q})(-\mu) \cdot P$. This completes the proof of part (1) of this theorem.

For part (2), we now begin by supposing that

$$Q(\lambda) \,=\, \pm P \cdot \overline{Q}(-\lambda) \cdot P\,. \tag{4.5}$$

Setting $\lambda = \frac{1+\mu}{1-\mu}$ in (4.5), multiplying with $(1-\mu)^k$ and using (4.2) yields

$$\begin{aligned}\mathcal{C}_{+1}(Q)(\mu) \,=\, (1-\mu)^k Q\left(\frac{1+\mu}{1-\mu}\right) &= \pm(1-\mu)^k P \cdot \overline{Q}\left(-\frac{1+\mu}{1-\mu}\right) \cdot P\\ &= \pm(-1)^k P \cdot \mathrm{rev}\,\mathcal{C}_{+1}(\overline{Q})(\mu) \cdot P\,.\end{aligned}$$

On the other hand, setting $\lambda = \frac{\mu-1}{\mu+1}$ in (4.5), multiplying with $(\mu+1)^k$ and using (4.3) yields

$$\begin{aligned}\mathcal{C}_{-1}(Q)(\mu) \,=\, (\mu+1)^k Q\left(\frac{\mu-1}{\mu+1}\right) &= \pm(\mu+1)^k P \cdot \overline{Q}\left(-\frac{\mu-1}{\mu+1}\right) \cdot P\\ &= \pm P \cdot \mathrm{rev}\,\mathcal{C}_{-1}(\overline{Q})(\mu) \cdot P\,.\end{aligned}$$

This completes the proof of the theorem. ∎

Analogous relationships between palindromic and even/odd matrix polynomials have been observed in [17]. Table 4.1 summarizes all these results.

Table 4.1
Cayley transformations

$Q(\lambda)$	$\mathcal{C}_{-1}(Q)(\mu)$		$\mathcal{C}_{+1}(Q)(\mu)$	
	k even	k odd	k even	k odd
palindromic $\star$-palindromic	even $\star$-even	odd $\star$-odd	even $\star$-even	
anti-palindromic $\star$-anti-palindromic	odd $\star$-odd	even $\star$-even	odd $\star$-odd	
PCP anti-PCP	PCP-even PCP-odd	PCP-odd PCP-even	PCP-even PCP-odd	
even $\star$-even	palindromic $\star$-palindromic		palindromic $\star$-palindromic	anti-palindromic $\star$-anti-palindromic
odd $\star$-odd	anti-palindromic $\star$-anti-palindromic		anti-palindromic $\star$-anti-palindromic	palindromic $\star$-palindromic
PCP-even PCP-odd	PCP anti-PCP		PCP anti-PCP	anti-PCP PCP

5. Structured linearizations. Following the strategy in [17], we will consider the vector spaces $\mathbb{L}_1(Q)$ and $\mathbb{L}_2(Q)$ introduced in [16, 18],

$$\mathbb{L}_1(Q) := \left\{ L(\lambda) = \lambda X + Y : L(\lambda)\cdot(\Lambda \otimes I_n) = v \otimes Q(\lambda),\, v \in \mathbb{C}^k \right\}, \tag{5.1}$$

$$\mathbb{L}_2(Q) := \left\{ L(\lambda) = \lambda X + Y : (\Lambda^T \otimes I_n)\cdot L(\lambda) = w^T \otimes Q(\lambda),\, w \in \mathbb{C}^k \right\}, \tag{5.2}$$

$$\text{where} \qquad \Lambda = [\, \lambda^{k-1} \;\lambda^{k-2} \;\cdots\; \lambda \;\; 1\,]^T,$$

as sources of structured linearizations for our structured polynomials. The vector v in (5.1) is called the right ansatz vector of $L(\lambda) \in \mathbb{L}_1(Q)$, while w in (5.2) is called the left ansatz vector of $L(\lambda) \in \mathbb{L}_2(Q)$. We recall some of the key results known about these spaces for the convenience of the reader.

The pencil spaces $\mathbb{L}_i(Q)$ are generalizations of the first and second companion forms (1.4); direct calculations show that $C_i(\lambda) \in \mathbb{L}_i(Q)$, with ansatz vector e_1 in both cases. These spaces can be represented using the column-shifted sum and row-shifted sum defined as follows. Viewing X and Y as block $k \times k$ matrices, partitioned into $n \times n$ blocks X_{ij}, Y_{ij}, the column shifted sum $X \boxplus\!\!\!\rightarrow Y$ and the row shifted sum $X \underset{\downarrow}{\boxplus} Y$ are defined to be

$$X \boxplus\!\!\!\rightarrow Y := \begin{bmatrix} X_{11} & \cdots & X_{1k} & 0 \\ \vdots & & \vdots & \vdots \\ X_{k1} & \cdots & X_{kk} & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_{11} & \cdots & Y_{1k} \\ \vdots & \vdots & & \vdots \\ 0 & Y_{k1} & \cdots & Y_{kk} \end{bmatrix},$$

$$X \underset{\downarrow}{\boxplus} Y := \begin{bmatrix} X_{11} & \cdots & X_{1k} \\ \vdots & & \vdots \\ X_{k1} & \cdots & X_{kk} \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ Y_{11} & \cdots & Y_{1k} \\ \vdots & & \vdots \\ Y_{k1} & \cdots & Y_{kk} \end{bmatrix},$$

where the zero blocks are also $n \times n$. An alternate characterization [18],

$$\mathbb{L}_1(Q) = \left\{ \lambda X + Y : X \boxplus\!\!\!\rightarrow Y = v \otimes [\, B_k \ B_{k-1} \ \cdots \ B_0 \,],\ v \in \mathbb{C}^k \right\}, \tag{5.3}$$

$$\mathbb{L}_2(Q) = \left\{ \lambda X + Y : X \underset{\downarrow}{\boxplus} Y = w^T \otimes \begin{bmatrix} B_k \\ \vdots \\ B_0 \end{bmatrix},\ w \in \mathbb{C}^k \right\}, \tag{5.4}$$

now shows that like the companion forms, pencils $L(\lambda) \in \mathbb{L}_i(Q)$ are easily constructible from the data in $Q(\lambda)$.

The spaces $\mathbb{L}_i(Q)$ are fertile sources of linearizations: having nearly half the dimension of the full pencil space (they are both of dimension $k(k-1)n^2 + k$ [18, Cor 3.6]), almost all pencils in these spaces are strong linearizations when Q is regular [18, Thm 4.7]. Furthermore, eigenvectors of $Q(\lambda)$ are easily recoverable from those of $L(\lambda)$. For an eigenvalue λ of Q, the correspondence $x \leftrightarrow \Lambda \otimes x$ is an isomorphism between right eigenvectors x of $Q(\lambda)$ and those of any linearization $L(\lambda) \in \mathbb{L}_1(Q)$. Similar observations hold for linearizations in $\mathbb{L}_2(Q)$ and left eigenvectors [18, Thms 3.8 and 3.14].

It is natural to consider pencils in

$$\mathbb{DL}(Q) \ := \ \mathbb{L}_1(Q) \cap \mathbb{L}_2(Q)\,,$$

since for such pencils both right and left eigenvectors of Q are easily recovered. It is shown in [18, Thm 5.3] that the right and left ansatz vectors v and w must coincide for pencils $L(\lambda) \in \mathbb{DL}(Q)$, and that every $v \in \mathbb{C}^k$ uniquely determines X and Y such that $\lambda X + Y$ is in $\mathbb{DL}(Q)$. Thus $\mathbb{DL}(Q)$ is a k-dimensional space of pencils, almost all of which are strong linearizations for Q [18, Thm 6.8].

Furthermore, all pencils in $\mathbb{DL}(Q)$ are block-symmetric [7]; in particular, the set of all block-symmetric pencils in $\mathbb{L}_1(Q)$ is precisely $\mathbb{DL}(Q)$. Here a block $k \times k$ matrix A with $n \times n$ blocks A_{ij} is said to be block-symmetric if $A^{\mathcal{B}} = A$, where $A^{\mathcal{B}}$ denotes the block transpose of A, that is, $A^{\mathcal{B}}$ is the block $k \times k$ matrix with $n \times n$ blocks defined by $(A^{\mathcal{B}})_{ij} := A_{ji}$. See [7] for more on symmetric linearizations of matrix polynomials and their connection to $\mathbb{DL}(Q)$.

The existence of other types of structured linearization in $\mathbb{L}_1(Q)$, in particular for $\star$-(anti)-palindromic and $\star$-even/odd polynomials Q, has been established in [16] and [17] by showing how they may be constructed from $\mathbb{DL}(Q)$-pencils. A second method for building these structured pencils using the shifted sum was presented in [16]. In the following subsections we develop analogous methods to construct PCP-structured linearizations in $\mathbb{L}_1(Q)$, $\mathbb{L}_2(Q)$ and $\mathbb{DL}(Q)$ for all the types of PCP-structured polynomials considered in this paper.

It is important to point out that linearizations other than the ones in $\mathbb{L}_1(Q)$ and $\mathbb{L}_2(Q)$ discussed here are also possible. Indeed, several other methods for constructing block-symmetric linearizations of matrix polynomials have appeared previously in the literature, see [7, Section 4] for more details.

5.1. Structured linearizations of (anti-)PCP polynomials. We now turn to the problem of finding structured linearizations for general (anti-)PCP polynomials, that is, for $Q(\lambda) = \sum_{i=1}^{k} \lambda^i B_i$ satisfying $B_i = \pm P \overline{B}_{k-i} P$ for some $n \times n$ real involution P. Our search for these structured linearizations will take place in the spaces $\mathbb{L}_1(Q)$, $\mathbb{L}_2(Q)$, and $\mathbb{DL}(Q)$.

In this context a linearization $L(\lambda) = \lambda X + Y$ for Q will be considered structure-preserving if it satisfies

$$\widehat{P} \cdot \operatorname{rev} \overline{L}(\lambda) \cdot \widehat{P} \; = \; \pm L(\lambda)\,, \quad \text{equivalently} \quad Y = \pm \widehat{P} \cdot \overline{X} \cdot \widehat{P}\,, \tag{5.5}$$

for some $kn \times kn$ real involution $\widehat{P}$. It is not immediately obvious, though, what we should use for $\widehat{P}$. One might reasonably expect that an appropriate $\widehat{P}$ would incorporate the original involution P in some way. An apparently natural choice,

$$I_k \otimes P \; = \; \begin{bmatrix} P & & \\ & \ddots & \\ & & P \end{bmatrix},$$

works only when the coefficient matrices B_i of Q are very specifically tied to one another; e.g., for $k = 2$, Q would be constrained by $B_1 = P\overline{B}_2 P + B_2 = B_0 + B_2$, and a structured $L(\lambda) \in \mathbb{L}_1(Q)$ would have to have right ansatz vector $v = [\,1 \;\; 1\,]^T$. Things work out better if we use instead the involution

$$\widehat{P} \; := \; R \otimes P \; = \; \begin{bmatrix} & & P \\ & \cdot^{\cdot^{\cdot}} & \\ P & & \end{bmatrix} \qquad \text{where} \qquad R = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}_{k \times k}. \tag{5.6}$$

Note that $\widehat{P} = R \otimes P$ is symmetric whenever P is, a property that will be important in Section 6.

Fixing the involution $\widehat{P} = R \otimes P$ for the rest of this section, we begin by observing that if a pencil $\lambda X^{(1)} + Y^{(1)}$ is (anti-)PCP with respect to $\widehat{P}$, then from (5.5) $Y^{(1)} \; = \; \pm \widehat{P}\, \overline{X}^{(1)} \widehat{P}$ is uniquely determined by $X^{(1)}$, so it suffices to specify all the admissible $X^{(1)}$. Partitioning $X^{(1)}$ and $Y^{(1)}$ into $n \times n$ blocks $X^{(1)}_{ij}$ and $Y^{(1)}_{ij}$ with $i, j = 1, \ldots, k$, we obtain from (5.5) and (5.6) that these blocks satisfy

$$Y^{(1)}_{ij} \; = \; \pm P \overline{X}^{(1)}_{k-i+1,k-j+1} P\,. \tag{5.7}$$

For $\lambda X^{(1)} + Y^{(1)}$ to be a pencil in $\mathbb{L}_1(Q)$, we know from (5.3) that

$$X^{(1)} \boxplus\!\!\rightarrow Y^{(1)} \; = \; v \otimes [\, B_k \; B_{k-1} \; \cdots \; B_0 \,] \; =: \; Z \quad \text{for some} \quad v \in \mathbb{C}^k. \tag{5.8}$$

It follows immediately from the definition of the column shifted sum $\boxplus\!\!\rightarrow$ that if Z is partitioned conformably into $n \times n$ blocks $Z_{i\ell}$ with $\ell = 1, \ldots, k+1$, then

$$Z_{i\ell} \; = \; v_i B_{k-\ell+1} \; = \; \begin{cases} X^{(1)}_{i1} & \ell = 1\,, \\ X^{(1)}_{i\ell} + Y^{(1)}_{i,\ell-1} & 1 < \ell < k+1\,, \\ Y^{(1)}_{ik} & \ell = k+1\,. \end{cases} \tag{5.9}$$

Invoking (5.7) with $j = k$, (5.9) with $\ell = 1$ and $\ell = k+1$, together with the PCP-structure of Q yields

$$\begin{aligned} v_i B_0 \; = \; Y^{(1)}_{ik} \; &= \; \pm P \overline{X}^{(1)}_{k-i+1,1} P \\ &= \; \pm P \big(\overline{v_{k-i+1} B_k} \big) P \\ &= \; \overline{v}_{k-i+1} (\pm P \overline{B}_k P) \; = \; \overline{v}_{k-i+1} B_0 \end{aligned} \tag{5.10}$$

for all i. Hence $v_i = \overline{v}_{k-i+1}$, equivalently $Rv = \overline{v}$, is a necessary condition for the right ansatz vector v of any PCP-pencil in $\mathbb{L}_1(Q)$.

The first block column of $X^{(1)}$ is completely determined by (5.9) with $\ell = 1$,

$$X_{i1}^{(1)} = v_i B_k \, , \tag{5.11}$$

while (5.9) for $2 \leq \ell = j \leq k$ together with (5.7) provides a pairwise relation

$$X_{ij}^{(1)} \;=\; v_i B_{k-j+1} - Y_{i,j-1}^{(1)} \;=\; v_i B_{k-j+1} \mp P\overline{X}_{k-i+1,k-j+2}^{(1)} P \tag{5.12}$$

among the remaining $k(k-1)$ blocks of $X^{(1)}$ in block columns 2 through k. Because the "centrosymmetric" pairing of indices in (5.12)

$$(i, j) \;\longleftrightarrow\; (k - i + 1, k - j + 2) \quad \text{with} \quad j \geq 2$$

has no fixed points, (5.12) is always a relation between distinct blocks of $X^{(1)}$. One block in each of these centrosymmetric pairs can be chosen arbitrarily; then (5.12) uniquely determines the rest of the blocks $X_{ij}^{(1)}$ with $j \geq 2$. Gathering (5.11) and (5.12) together with the conditions on the blocks of $Y^{(1)}$ that follow from (5.7) gives us the following blockwise specification

$$X_{ij}^{(1)} = \begin{cases} v_i B_k & j = 1 \\ v_i B_{k-j+1} \mp P\overline{X}_{k-i+1,k-j+2}^{(1)} P & j > 1 \, , \end{cases} \tag{5.13}$$

$$Y_{ij}^{(1)} = \begin{cases} v_i B_{k-j} - X_{i,j+1}^{(1)} & j < k \\ v_i B_0 & j = k \, . \end{cases} \tag{5.14}$$

of an (anti-)PCP-pencil $\lambda X^{(1)} + Y^{(1)}$. These pencils can now all be shown to be in $\mathbb{L}_1(Q)$ by a straightforward verification of property (5.8).

Thus we see that for any $v \in \mathbb{C}^k$ satisfying $Rv = \overline{v}$, there always exist pencils $L(\lambda) \in \mathbb{L}_1(Q)$ with right ansatz vector v and (anti-)PCP structure. These pencils are far from unique — the above analysis shows that for each admissible v there are $k(k-1)n^2/2$ (complex) degrees of freedom available for constructing (anti-)PCP-pencils in $\mathbb{L}_1(Q)$ with v as right ansatz vector. Indeed, the set of all PCP-pencils in $\mathbb{L}_1(Q)$ can be shown to be a real subspace of $\mathbb{L}_1(Q)$ of *real* dimension $k + k(k-1)n^2$. This is quite different from the palindromic structures considered in [18, Thm 3.5], where for each suitably restricted right ansatz vector there was shown to be a unique structured pencil in $\mathbb{L}_1(Q)$.

A similar analysis can be used to develop formulas for the set of all (anti-)PCP-structured pencils $\lambda X^{(2)} + Y^{(2)}$ in $\mathbb{L}_2(Q)$, using the row shifted sum characterization (5.4) as a starting point in place of (5.8). We find that the left ansatz vector w of any (anti-)PCP-pencil in $\mathbb{L}_2(Q)$ is restricted, just as it was for (anti-)PCP-pencils in $\mathbb{L}_1(Q)$, to ones satisfying $Rw = \overline{w}$. Partitioning $X^{(2)}$ and $Y^{(2)}$ into $n \times n$ blocks $X_{ij}^{(2)}$ and $Y_{ij}^{(2)}$ as before now forces the first block row of $X^{(2)}$ to be

$$X_{1j}^{(2)} = w_j B_k \, ,$$

while the remaining blocks of $X^{(2)}$ in block rows 2 through k must pairwise satisfy the relations

$$X_{ij}^{(2)} = w_j B_{k-i+1} \mp P\overline{X}_{k-i+2,k-j+1}^{(2)} P \quad \text{for} \quad 2 \leq i \leq k \, , \tag{5.15}$$

analogous to (5.13) for (anti-)PCP-pencils in $\mathbb{L}_1(Q)$. Here the pairing of indices for blocks of $X^{(2)}$ is

$$(i,j) \longleftrightarrow (k-i+2, k-j+1) \quad \text{for} \quad i \geq 2\,.$$

Once again we have a pairing with no fixed points, allowing one block in each block pair to be chosen arbitrarily, while the other is then uniquely specified by (5.15). Thus we obtain the following blockwise specification for a general (anti-)PCP pencil in $\mathbb{L}_2(Q)$,

$$X^{(2)}_{ij} = \begin{cases} w_j B_k & i = 1 \\ w_j B_{k-i+1} \mp P\overline{X}^{(2)}_{k-i+2,k-j+1} P & i > 1\,, \end{cases} \tag{5.16}$$

$$Y^{(2)}_{ij} = \begin{cases} w_j B_{k-i} - X^{(2)}_{i+1,j} & i < k \\ w_j B_0 & i = k\,, \end{cases} \tag{5.17}$$

analogous to (5.13) and (5.14) for (anti-)PCP pencils in $\mathbb{L}_1(Q)$.

An alternative way to generate (anti-)PCP pencils in $\mathbb{L}_2(Q)$ is to use the block transpose linear isomorphism [7, Thm 2.2]

$$\begin{aligned} \mathbb{L}_1(Q) &\longrightarrow \mathbb{L}_2(Q) \\ L(\lambda) &\longmapsto L(\lambda)^{\mathcal{B}}\,, \end{aligned}$$

between $\mathbb{L}_1(Q)$ and $\mathbb{L}_2(Q)$. For any (anti-)PCP pencil $\lambda X \pm \widehat{P}\,\overline{X}\widehat{P}$ with the particular involution $\widehat{P} = R \otimes P$ we can show that

$$\left(\lambda X \pm \widehat{P}\,\overline{X}\widehat{P}\right)^{\mathcal{B}} = \lambda X^{\mathcal{B}} \pm \left(\widehat{P}\,\overline{X}\widehat{P}\right)^{\mathcal{B}} = \lambda X^{\mathcal{B}} \pm \widehat{P}\,\overline{X}^{\,\mathcal{B}}\widehat{P}\,.$$

Thus block transpose preserves (anti-)PCP structure, and hence restricts to an isomorphism between the (real) subspaces of all (anti-)PCP pencils in $\mathbb{L}_1(Q)$ and all (anti-)PCP pencils in $\mathbb{L}_2(Q)$.

We now know how to generate lots of (anti-)PCP pencils in $\mathbb{L}_1(Q)$ and in $\mathbb{L}_2(Q)$ for each admissible right or left ansatz vector. But what about $\mathbb{DL}(Q) = \mathbb{L}_1(Q) \cap \mathbb{L}_2(Q)$? Are there any (anti-)PCP pencils in this very desirable subspace of pencils? The following theorem answers this question in the affirmative, and also gives a uniqueness result analogous to the ones for the palindromic structures considered in [18].

Theorem 5.1 (Existence/Uniqueness of PCP-Structured Pencils in $\mathbb{DL}(Q)$). *Suppose $Q(\lambda)$ is an (anti-)PCP-polynomial with respect to the involution P. Let $v \in \mathbb{C}^k$ be any vector such that $Rv = \overline{v}$, and let $L(\lambda)$ be the unique pencil in $\mathbb{DL}(Q)$ with ansatz vector v. Then $L(\lambda)$ is an (anti-)PCP-pencil with respect to the involution $\widehat{P} = R \otimes P$.*

Proof. Our strategy is to show that the pencil $\widehat{L}(\lambda) := \pm\widehat{P}\,\mathrm{rev}\,\overline{L}(\lambda)\widehat{P}$ (using $+$ when Q is PCP and $-$ when Q is anti-PCP) is also in $\mathbb{DL}(Q)$, with the same ansatz vector v as $L(\lambda)$. Then from the unique determination of $\mathbb{DL}(Q)$-pencils by their ansatz vectors, see [7, Thm 3.4] or [18, Thm 5.3], we can conclude that $\widehat{L}(\lambda) \equiv L(\lambda)$, and hence that $L(\lambda)$ is (anti-)PCP with respect to $\widehat{P}$.

We begin by showing that $L(\lambda) \in \mathbb{L}_1(Q)$ with right ansatz vector v implies that $\widehat{L}(\lambda) \in \mathbb{L}_1(Q)$ with right ansatz vector v. From the defining identity (in the variable λ) for a pencil in $\mathbb{L}_1(Q)$ we have

$$L(\lambda) \cdot (\Lambda \otimes I) = v \otimes Q(\lambda) = v \otimes \left[\pm P\,\mathrm{rev}\,\overline{Q}(\lambda) P\right].$$

Taking rev of both sides of this identity, and using the fact that $\operatorname{rev}\Lambda = R\Lambda$, we get

$$\operatorname{rev} L(\lambda) \cdot (R\Lambda \otimes I) \;=\; \pm v \otimes \left[P\overline{Q}(\lambda)P\right].$$

Multiplying on the right by the involution $1 \otimes P = P$ and simplifying yields

$$\begin{aligned} \pm\operatorname{rev} L(\lambda) \cdot (R\Lambda \otimes P) \;&=\; \Big(v \otimes \left[P\overline{Q}(\lambda)P\right]\Big)(1 \otimes P) \\ \Longrightarrow \quad \pm\operatorname{rev} L(\lambda) \cdot (R \otimes P)(\Lambda \otimes I) \;&=\; v \otimes P\overline{Q}(\lambda). \end{aligned}$$

Now multiply on the left by $R \otimes P$, and use the hypothesis $Rv = \overline{v}$ to obtain

$$\begin{aligned} \pm(R \otimes P) \cdot \operatorname{rev} L(\lambda) \cdot (R \otimes P)(\Lambda \otimes I) \;&=\; Rv \otimes \overline{Q}(\lambda) \\ \Longrightarrow \qquad \left[\pm\widehat{P} \operatorname{rev} L(\lambda)\, \widehat{P}\right] \cdot (\Lambda \otimes I) \;&=\; \overline{v} \otimes \overline{Q}(\lambda). \end{aligned}$$

Finally conjugate both sides, and replace $\overline{\lambda}$ by λ in the resulting identity:

$$\left[\pm\widehat{P} \operatorname{rev} \overline{L}(\overline{\lambda})\, \widehat{P}\right] (\overline{\Lambda} \otimes I) = v \otimes Q(\overline{\lambda}) \quad \Longrightarrow \quad \left[\pm\widehat{P} \operatorname{rev} \overline{L}(\lambda)\, \widehat{P}\right] (\Lambda \otimes I) = v \otimes Q(\lambda).$$

Thus $\widehat{L}(\lambda) \cdot (\Lambda \otimes I) = v \otimes Q(\lambda)$, and so $\widehat{L}(\lambda) \in \mathbb{L}_1(Q)$ with right ansatz vector v.

A similar computation starts from the defining identity

$$(\Lambda^T \otimes I) \cdot L(\lambda) \;=\; v^T \otimes Q(\lambda)$$

for a pencil $L(\lambda)$ to be in $\mathbb{L}_2(Q)$, and shows that whenever $L(\lambda) \in \mathbb{L}_2(Q)$ has left ansatz vector v, then $\widehat{L}(\lambda)$ is also in $\mathbb{L}_2(Q)$ with left ansatz vector v. Thus $\widehat{L}(\lambda) \in \mathbb{DL}(Q)$ with ansatz vector v, hence $\widehat{L}(\lambda) \equiv L(\lambda)$, and so $L(\lambda)$ is a PCP-pencil with respect to the involution $\widehat{P} = R \otimes P$. ◻

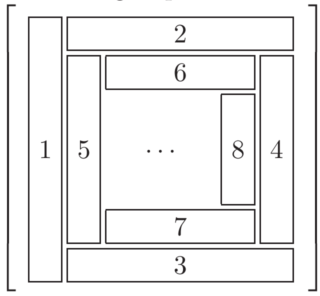
Now that we know there exists a unique structured pencil in $\mathbb{DL}(Q)$ for each admissible ansatz vector, how can we go about constructing it in a simple and effective manner? Perhaps the simplest answer is just to use either of the explicit formulas for $\mathbb{DL}(Q)$ pencils given in [7] and [18, Thm 5.3]. An alternative is to adapt the procedures used in [16] for constructing $\star$-palindromic and $\star$-even/odd pencils in $\mathbb{DL}(Q)$, as follows.

Given a vector $v \in \mathbb{C}^k$ such that $Rv = \overline{v}$, our goal is to construct the pencil $\lambda X + Y$ in $\mathbb{DL}(Q)$ with ansatz vector v that is (anti-)PCP with respect to the involution $\widehat{P} = R \otimes P$. Recall that it suffices to determine X, since the (anti-)PCP structure forces Y to be $\pm\widehat{P}\,\overline{X}\widehat{P}$. We now construct X one group of blocks at a time, alternating between using the fact that X comes from a pencil in $\mathbb{DL}(Q)$ and hence is block-symmetric, and the fact that it comes from a pencil that is (anti-)PCP in $\mathbb{L}_1(Q)$ and so satisfies the conditions in (5.13).

1. the first block column of X is determined by (5.13) to be $X_{i1} = v_i B_k$.
2. the first block row of X is now forced to be $X_{1j} = v_j B_k$ by block-symmetry.
3. (5.13) now determines the last block row of X from the first block row.
4. the last block column of X is now determined by block-symmetry.
5. (5.13) determines the second block column of X from the last block column.
6. the second block row of X follows by block-symmetry.
7. (5.13) determines the next-to-last block row of X from the second block row.
8. the next-to-last block column of X is now determined by block-symmetry.
9. and so on ...

The order of construction for the various groups of blocks in X follows the pattern

$X =$



(5.18)

similar to that in [16, Section 7.3.2] for $\star$-even and $\star$-odd linearizations.

The matrix X resulting from this construction is necessarily block-symmetric, since all the blocks in the even-numbered panels $2, 4, 6, 8, \ldots$ are determined by imposing the condition of block-symmetry. Since $\left(\widehat{P}\,\overline{X}\widehat{P}\right)^{\mathcal{B}} = \widehat{P}\,\overline{X}^{\mathcal{B}}\widehat{P} = \widehat{P}\,\overline{X}\widehat{P}$, we see that the pencil $\lambda X \pm \widehat{P}\,\overline{X}\widehat{P}$ as a whole is block-symmetric, and hence is in $\mathbb{DL}(Q)$.

Example 5.2 (Quadratic case). *To illustrate this procedure we find all the structured pencils in $\mathbb{DL}(Q)$ for the quadratic PCP-polynomial $Q(\lambda) = \lambda^2 B_2 + \lambda B_1 + B_0$, where $B_1 = P\overline{B}_1 P$ and $B_0 = P\overline{B}_2 P$. An admissible ansatz vector $v \in \mathbb{C}^2$ must satisfy $Rv = \overline{v}$, i.e., must be of the form $v = [\,\alpha, \overline{\alpha}\,]^T$. The matrix X in the structured $\mathbb{DL}(Q)$-pencil $\lambda X + Y$ with ansatz vector v is then constructed in three steps:*

$$\textit{first} \begin{bmatrix} \alpha B_2 & * \\ \overline{\alpha} B_2 & * \end{bmatrix}, \ \textit{then} \begin{bmatrix} \alpha B_2 & \overline{\alpha} B_2 \\ \overline{\alpha} B_2 & * \end{bmatrix}, \ \textit{and finally} \begin{bmatrix} \alpha B_2 & \overline{\alpha} B_2 \\ \overline{\alpha} B_2 & \overline{\alpha} B_1 - \alpha P \overline{B}_2 P \end{bmatrix},$$

resulting in the structured pencil $\lambda X + \widehat{P}\overline{X}\widehat{P}$ given by

$$\lambda \begin{bmatrix} \alpha B_2 & \overline{\alpha} B_2 \\ \overline{\alpha} B_2 & \overline{\alpha} B_1 - \alpha P \overline{B}_2 P \end{bmatrix} + \begin{bmatrix} \alpha B_1 - \overline{\alpha} B_2 & \alpha P \overline{B}_2 P \\ \alpha P \overline{B}_2 P & \overline{\alpha} P \overline{B}_2 P \end{bmatrix}.$$

So far in this section we have shown how to construct many structured pencils in $\mathbb{L}_1(Q)$, $\mathbb{L}_2(Q)$, and $\mathbb{DL}(Q)$. But which, if any, of these pencils are actually linearizations for the structured polynomial Q that we began with? It is known that when $Q(\lambda)$ is regular, then any regular pencil in $\mathbb{L}_1(Q)$ or $\mathbb{L}_2(Q)$ is a (strong) linearization for Q [18, Thm 4.3]. Although there is a systematic approach [18] for determining the regularity of a pencil $L(\lambda)$ in $\mathbb{L}_1(Q)$ or $\mathbb{L}_2(Q)$, there is in general no connection between its regularity and the right (or left) ansatz vector of $L(\lambda)$. By contrast, for pencils in $\mathbb{DL}(Q)$ the Eigenvalue Exclusion Theorem [18, Thm 6.7] characterizes regularity directly in terms of the ansatz vector : $L(\lambda) \in \mathbb{DL}(Q)$ with ansatz vector $v = [v_i] \in \mathbb{C}^k$ is regular, and hence a (strong) linearization for $Q(\lambda)$, if and only if no root of the scalar v-polynomial $\mathsf{p}(x; v) := v_1 x^{k-1} + v_2 x^{k-2} + \ldots + v_{k-1} x + v_k$ is an eigenvalue of $Q(\lambda)$. Among the ansatz vectors v satisfying $Rv = \overline{v}$, there will always be many choices such that the roots of the v-polynomial

$$\mathsf{p}(x; v) \ = \ v_1 x^{k-1} + v_2 x^{k-2} + \ldots + \overline{v}_2 x + \overline{v}_1$$

are disjoint from the eigenvalues of $Q(\lambda)$, thus providing many structured pencils in $\mathbb{DL}(Q)$ that are indeed linearizations for $Q(\lambda)$.

One might also wish to choose the ansatz vector v so that the desired eigenvalues are optimally conditioned. Although the problem of determining the best conditioned linearization in $\mathbb{DL}(Q)$ for an unstructured polynomial Q has been investigated in [8], up to now it is not clear how to do this for structured linearizations of structured polynomials Q.

Remark 1: Consider again the general quadratic PCP-polynomial Q as discussed in Example 5.2. In this case admissible ansatz vectors have the form $v = [\,\alpha, \overline{\alpha}\,]^T$ with corresponding v-polynomial $\mathsf{p}(x; v) = \alpha x + \overline{\alpha}$. So to obtain a linearization we need only choose $\alpha \in \mathbb{C}$ so that the number $-\overline{\alpha}/\alpha$ on the unit circle is not an eigenvalue of $Q(\lambda)$. Clearly this can always be done. ◇

Remark 2: In this section our structured linearizations have been of the same type as the structured polynomial — we linearize a PCP-polynomial with a PCP-pencil, and an anti-PCP-polynomial with an anti-PCP-pencil. It should be noted, however, that "crossover" linearizations are also possible. Small modifications of the constructions given in this section show that any PCP-polynomial can be linearized by an anti-PCP-pencil, and any anti-PCP-polynomial by a PCP-pencil. The admissibility condition for the ansatz vectors of these crossover linearizations is now $Rv = -\overline{v}$ rather than $Rv = \overline{v}$. From the point of view of numerical computation such crossover linearizations are just as useful, since spectral symmetries are still preserved. ◇

Remark 3: It is not yet clear whether the choice of $\widehat{P} = R \otimes P$ as the involution for our structured linearizations is the only one possible, or if there might be other choices for $\widehat{P}$ that work just as well. ◇

5.2. Structured linearizations of PCP-even/odd polynomials. Next we consider the linearization of PCP-even/odd polynomials by PCP-even/odd pencils in $\mathbb{L}_1(Q)$, $\mathbb{L}_2(Q)$, and $\mathbb{DL}(Q)$. Recall that $Q(\lambda) = \sum_{i=1}^{k} \lambda^i B_i$ is PCP-even/odd if $Q(\lambda) = \pm P\overline{Q}(-\lambda)P$, equivalently if $B_i = \pm(-1)^i P\overline{B}_i P$, for some real involution P. Thus a pencil $L(\lambda) = \lambda X + Y$ is PCP-even/odd if there is some involution $\widehat{P}$ such that

$$X = \mp \widehat{P}\,\overline{X}\,\widehat{P} \quad \text{and} \quad Y = \pm \widehat{P}\,\overline{Y}\,\widehat{P}\,. \tag{5.19}$$

Now just as in Section 5.1, the first issue is to decide which $\widehat{P}$ to use; certainly we want $\widehat{P}$ such that structured pencils which *linearize* $Q(\lambda)$ can always be found. The first two possibilities that spring to mind, $I_k \otimes P$ and $R_k \otimes P$, turn out to work only for structured Q having additional restrictions on its coefficient matrices. We will see, however, that choosing

$$\widehat{P} := \Sigma_k \otimes P = \begin{bmatrix} \ddots & & & \\ & -P & & \\ & & P & \\ & & & -P & \\ & & & & P \end{bmatrix} \quad \text{where} \quad \Sigma_k := \begin{bmatrix} (-1)^{k-1} & & & \\ & (-1)^{k-2} & & \\ & & \ddots & \\ & & & (-1)^0 \end{bmatrix}_{k \times k} \tag{5.20}$$

works for any PCP-even/odd $Q(\lambda)$. Fixing $\widehat{P} = \Sigma_k \otimes P$ for the rest of this section, and partitioning X and Y like $\widehat{P}$ into $n \times n$ blocks X_{ij} and Y_{ij}, we obtain from (5.19) that

$$X_{ij} = \mp(-1)^{i+j} P\overline{X}_{ij}P \quad \text{and} \quad Y_{ij} = \pm(-1)^{i+j} P\overline{Y}_{ij}P\,. \tag{5.21}$$

Now we know from (5.3) that a pencil $\lambda X^{(1)} + Y^{(1)}$ is in $\mathbb{L}_1(Q)$ exactly when

$$X^{(1)} \boxplus\!\!\rightarrow Y^{(1)} = v \otimes [\,B_k \; B_{k-1} \; \cdots \; B_0\,] \quad \text{for some} \quad v \in \mathbb{C}^k. \tag{5.22}$$

Thus the blocks of such a pencil have to satisfy the conditions

$$X_{ij}^{(1)} = \begin{cases} v_i B_k & j = 1 \\ v_i B_{k-j+1} - Y_{i,j-1}^{(1)} & j > 1\,, \end{cases} \tag{5.23}$$

$$Y_{ij}^{(1)} = \begin{cases} Y_{ij}^{(1)} & j < k \\ v_i B_0 & j = k\,, \end{cases} \tag{5.24}$$

for an arbitrary choice of the blocks $Y_{ij}^{(1)}$ for $1 \le j \le k-1$ and $v \in \mathbb{C}^k$. For $\lambda X^{(1)} + Y^{(1)}$ to be a structured pencil in $\mathbb{L}_1(Q)$, it remains to determine how these arbitrary choices can be made so that all the relations in (5.21) hold.

To satisfy (5.21) for $Y_{ij}^{(1)}$ with $j = k$, i.e. for $Y_{ik}^{(1)} = v_i B_0$, we must have

$$v_i B_0 \;=\; \pm(-1)^{i+k} P\big(\overline{v_i B_0}\big)P \;=\; (-1)^{i+k}\overline{v}_i B_0 \qquad \text{for all } i\,. \tag{5.25}$$

Hence the right ansatz vector v must satisfy $v_i = (-1)^{i+k}\overline{v}_i$, or equivalently $\Sigma_k v = \overline{v}$. Choosing the rest of the $Y_{ij}^{(1)}$ for $1 \le j \le k-1$ in *any* way such that (5.21) holds clearly yields $Y^{(1)}$ such that $Y^{(1)} = \pm\widehat{P}\,\overline{Y}^{(1)}\widehat{P}$. The matrix $X^{(1)}$ is now completely determined by (5.23), and a straightforward, albeit tedious, verification shows that all the relations in (5.21) hold for this $X^{(1)}$. Thus we have obtained a complete description of all the PCP-even/odd pencils in $\mathbb{L}_1(Q)$.

Remark 4: It is interesting to note an unexpected consequence of this characterization: when Q is PCP-even, a small variation of the first companion form linearization $C_1(\lambda)$ is structure-preserving! Letting Z denote the $k \times k$ cyclic permutation

$$Z \;=\; \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & & & 0 \end{bmatrix},$$

we see that the block-row-permuted companion form $(Z \otimes I)C_1(\lambda)$ is a PCP-even pencil in $\mathbb{L}_1(Q)$ with right ansatz vector $v = e_k$. $\diamond$

A similar analysis, starting from the row shifted sum characterization (5.4) in place of (5.22), yields the following description of all the PCP-even/odd pencils $\lambda X^{(2)} + Y^{(2)}$ in $\mathbb{L}_2(Q)$ with left ansatz vector w. The blocks of such a structured pencil satisfy

$$X_{ij}^{(2)} = \begin{cases} w_j B_k & i = 1 \\ w_j B_{k-i+1} - Y_{i-1,j}^{(2)} & i > 1\,, \end{cases} \tag{5.26}$$

$$Y_{ij}^{(2)} = \begin{cases} Y_{ij}^{(2)} & i < k \\ w_j B_0 & i = k\,, \end{cases} \tag{5.27}$$

where once again the left ansatz vector w is restricted to ones such that $\Sigma_k w = \overline{w}$, and the blocks $Y_{ij}^{(2)}$ for $1 \le i \le k-1$ are chosen in *any* way satisfying (5.21). The matrix $X^{(2)}$ is then determined by (5.26), and the resulting pencil $\lambda X^{(2)} + Y^{(2)} \in \mathbb{L}_2(Q)$ is guaranteed to be PCP-even/odd.

When we look in $\mathbb{DL}(Q)$ for pencils that are PCP-even/odd, we find a situation very much like the one described in Theorem 5.1 for PCP-polynomials. The following theorem shows that PCP-even/odd pencils in $\mathbb{DL}(Q)$ are uniquely defined by any admissible ansatz vector v, i.e. by any v that satisfies $\Sigma_k v = \overline{v}$.

THEOREM 5.3 (Existence/Uniqueness of PCP-Even/Odd Pencils in $\mathbb{DL}(Q)$). *Suppose $Q(\lambda)$ is a PCP-even/odd polynomial with respect to the involution P. Let $v \in \mathbb{C}^k$ be any vector such that $\Sigma_k v = \overline{v}$, and let $L(\lambda)$ be the unique pencil in $\mathbb{DL}(Q)$ with ansatz vector v. Then $L(\lambda)$ is PCP-even/odd with respect to the involution $\widehat{P} = \Sigma_k \otimes P$.*

Proof. The strategy for this proof parallels that used in the proof of Theorem 5.1. We define the auxiliary pencil $\widehat{L}(\lambda) := \pm \widehat{P}\,\overline{L}(-\lambda)\widehat{P}$ (using $+$ when Q is PCP-even and $-$ when Q is PCP-odd), and then show that $\widehat{L}(\lambda)$ is in $\mathbb{DL}(Q)$ with the same ansatz vector as $L(\lambda)$. The unique determination of $\mathbb{DL}(Q)$-pencils by their ansatz vectors, see [7, Thm 3.4] or [18, Thm 5.3], then implies that $\widehat{L}(\lambda) \equiv L(\lambda)$, and hence that $L(\lambda)$ is PCP-even/odd with respect to $\widehat{P}$.

We begin by showing that $L(\lambda) \in \mathbb{L}_1(Q)$ with right ansatz vector v implies that $\widehat{L}(\lambda) \in \mathbb{L}_1(Q)$ with right ansatz vector v. From the defining identity (in the variable λ) for a pencil in $\mathbb{L}_1(Q)$ we have

$$L(\lambda) \cdot (\Lambda \otimes I) \;=\; v \otimes Q(\lambda) \;=\; v \otimes \left[\pm P\,\overline{Q}(-\lambda)P\right].$$

Replacing λ by $-\lambda$ in this identity, and using the fact that $\Lambda(-\lambda) = \Sigma_k \Lambda$, we get

$$L(-\lambda) \cdot (\Sigma_k \Lambda \otimes I) \;=\; \pm v \otimes \left[P\,\overline{Q}(\lambda)P\right].$$

Multiplying on the right by the involution $1 \otimes P = P$ and simplifying yields

$$\begin{aligned} \pm L(-\lambda) \cdot (\Sigma_k \Lambda \otimes P) \;&=\; \Big(v \otimes \left[P\,\overline{Q}(\lambda)P\right]\Big)(1 \otimes P) \\ \pm L(-\lambda) \cdot (\Sigma_k \otimes P)(\Lambda \otimes I) \;&=\; v \otimes P\,\overline{Q}(\lambda). \end{aligned}$$

Now multiply on the left by $\Sigma_k \otimes P$, and use the hypothesis $\Sigma_k v = \overline{v}$ to obtain

$$\begin{aligned} \pm(\Sigma_k \otimes P) \cdot L(-\lambda) \cdot (\Sigma_k \otimes P)(\Lambda \otimes I) \;&=\; \Sigma_k v \otimes \overline{Q}(\lambda), \\ \left[\pm \widehat{P}\, L(-\lambda)\, \widehat{P}\right] \cdot (\Lambda \otimes I) \;&=\; \overline{v} \otimes \overline{Q}(\lambda). \end{aligned}$$

Finally conjugate both sides, and replace $\overline{\lambda}$ by λ in the resulting identity:

$$\left[\pm \widehat{P}\,\overline{L}(-\overline{\lambda})\,\widehat{P}\right](\overline{\Lambda} \otimes I) = v \otimes Q(\overline{\lambda}) \quad\Longrightarrow\quad \left[\pm \widehat{P}\,\overline{L}(-\lambda)\,\widehat{P}\right](\Lambda \otimes I) = v \otimes Q(\lambda).$$

Thus $\widehat{L}(\lambda) \cdot (\Lambda \otimes I) = v \otimes Q(\lambda)$, and so $\widehat{L}(\lambda) \in \mathbb{L}_1(Q)$ with right ansatz vector v.

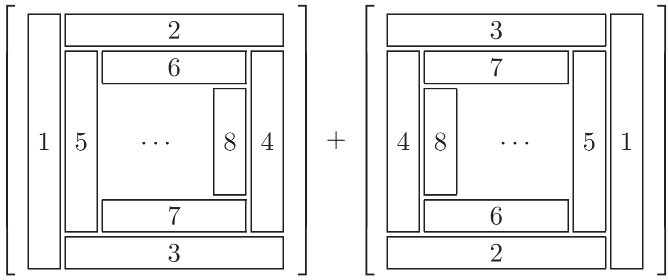
A similar computation starts from the defining identity

$$(\Lambda^T \otimes I) \cdot L(\lambda) \;=\; v^T \otimes Q(\lambda)$$

for a pencil $L(\lambda)$ to be in $\mathbb{L}_2(Q)$, and shows that whenever $L(\lambda) \in \mathbb{L}_2(Q)$ has left ansatz vector v, then $\widehat{L}(\lambda)$ is also in $\mathbb{L}_2(Q)$ with left ansatz vector v. Thus $\widehat{L}(\lambda) \in \mathbb{DL}(Q)$ with ansatz vector v, hence $\widehat{L}(\lambda) \equiv L(\lambda)$, and so $L(\lambda)$ is PCP-even/odd with respect to the involution $\widehat{P} = \Sigma_k \otimes P$. □

To construct these structured pencils $L(\lambda) \in \mathbb{DL}(Q)$ we once again have two main options — use the explicit formulas for general $\mathbb{DL}(Q)$-pencils given in [7] and [18], or alternatively build them up blockwise using a shifted sum construction analogous to the procedures used in [16, Section 7.3.2] for building $\star$-even and $\star$-odd linearizations. In this construction we alternate between using the fact that $L(\lambda) = \lambda X + Y$ is to be in $\mathbb{L}_1(Q)$ and so must satisfy the shifted sum condition (5.22), and invoking block-symmetry to ensure that $L(\lambda)$ is in $\mathbb{DL}(Q)$. (Recall that the set of all block-symmetric pencils in $\mathbb{L}_1(Q)$ is precisely $\mathbb{DL}(Q)$.) The determination of the blocks in $\lambda X + Y$ proceeds in the order indicated in the following diagram.

$\lambda X + Y \;=\; \lambda$



(5.28)

We start with a choice of ansatz vector v such that $\Sigma_k v = \overline{v}$. Then

1. $L(\lambda)$ being in $\mathbb{L}_1(Q)$ immediately determines the blocks in the panels labelled by (1); from (5.23) and (5.24) we know that $X_{i1} = v_i B_k$ and $Y_{ik} = v_i B_0$.
2. block-symmetry now forces the blocks in the panels labelled (2).
3. the shifted sum condition (5.22) next determines the blocks in the panels labelled (3).
4. block-symmetry now forces the blocks in the panels labelled (4).
5. (5.22) next determines the blocks in the panels labelled (5).
6. block-symmetry now forces the blocks in the panels labelled (6).
7. and so on ...

In summary, each panel labelled with an odd number is constructed using information from the shifted sum condition (5.22), while the panels labelled with an even number are constructed so as to maintain block-symmetry. Since the even-numbered panels comprise all the blocks above the diagonal in X and all the blocks below the diagonal in Y, we are guaranteed that the construction as a whole will produce a block-symmetric pencil, and hence a pencil in $\mathbb{DL}(Q)$.

The question of determining which of these structured pencils in $\mathbb{DL}(Q)$ is actually a linearization for Q is handled in the same way as it was in Section 5.1 — by using the Eigenvalue Exclusion Theorem [18, Thm 6.7] for pencils in $\mathbb{DL}(Q)$. For any admissible ansatz vector v such that the roots of the v-polynomial $\mathsf{p}(x; v)$ are disjoint from the eigenvalues of Q, the structured pencil in $\mathbb{DL}(Q)$ corresponding to v will be a linearization for Q. Clearly there will be many such v for which this is the case.

Finally it should be noted that remarks similar to the ones at the end of Section 5.1, e.g. on the existence of "crossover" structured linearizations and the possibility of there being other good choices of involution $\widehat{P}$, also apply here in the context of PCP-even/odd structure.

6. Structured Schur form for PCP-pencils. Once a PCP-polynomial has been linearized in a structure-preserving manner, then the eigenvalues of the resulting PCP-pencil should be computed in such a way that the reciprocal pairing of the spectrum (see Theorem 3.1) is guaranteed.

The generalized Schur decomposition $(S, T) = (QAZ, QBZ)$ of a matrix pair (A, B), where S and T are upper triangular and Q and Z are unitary, is the basis for most numerical approaches to computing eigenvalues and generalized invariant subspaces for the general linear eigenproblem $(\lambda A + B)x = 0$. In this section we discuss the computation of a *structured* Schur-type decomposition for the linear PCP-eigenproblem

$$(\lambda X + P\overline{X}P)v = 0\,, \tag{6.1}$$

where $X \in \mathbb{C}^{m\times m}$ and $P \in \mathbb{R}^{m\times m}$ is an involution. We begin by assuming that P is also symmetric; this is true for the involution in the quadratic PCP-eigenproblem arising from the stability analysis of time-delay systems discussed in Section 2, as well as in the structured linearizations for such problems described in Section 5.1.

Since P is an involution its eigenvalues are in $\{\pm 1\}$, so when P is symmetric it admits a Schur decomposition of the form

$$P = WDW^T, \qquad D = \begin{bmatrix} I_p & \\ & -I_{m-p} \end{bmatrix},$$

where $W \in \mathbb{R}^{m\times m}$ is orthogonal. With

$$\widehat{X} = W^T X W \quad \text{and} \quad \widehat{v} = W^T v$$

we can then simplify (6.1) to the linear PCP-eigenproblem

$$(\lambda\widehat{X} + D\overline{\widehat{X}}D)\,\widehat{v} \;=\; 0 \tag{6.2}$$

with involution D.

Using a Cayley transform and scaling yields

$$\tfrac{1}{2}\Big(\mathcal{C}_{+1}\big(\lambda\widehat{X} + D\overline{\widehat{X}}D\big)\Big)(\mu)\,\widehat{v} \;=:\; (\mu N + M)\,\widehat{v} \;=\; 0 \tag{6.3}$$

where $\mu = \frac{\lambda-1}{\lambda+1}$. By Theorem 4.2, the pencil $\mu N + M$ is PCP-even with involution D, hence $N = -D\overline{N}D$ and $M = D\overline{M}D$. These relations can also be directly verified from the defining equations $N := \frac{1}{2}(\widehat{X} - D\overline{\widehat{X}}D)$, and $M := \frac{1}{2}(\widehat{X} + D\overline{\widehat{X}}D)$. Note also that Theorem 3.1 guarantees symmetry of the spectrum of $\mu N + M$ with respect to the imaginary axis. Partitioning N and M conformably with D we have

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -\overline{N}_{11} & \overline{N}_{12} \\ \overline{N}_{21} & -\overline{N}_{22} \end{bmatrix} = -D\overline{N}D\,,$$
$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} \overline{M}_{11} & -\overline{M}_{12} \\ -\overline{M}_{21} & \overline{M}_{22} \end{bmatrix} = D\overline{M}D\,.$$

Hence the blocks N_{12}, N_{21}, M_{11}, and M_{22} are real while N_{11}, N_{22}, M_{12}, and M_{21} are purely imaginary.

Multiplying on both sides by $\widetilde{D} := \mathrm{diag}(I_p\,,\, -iI_{m-p})$ yields the equivalent *real* pencil

$$\begin{aligned}
&\widetilde{D}(\mu N + M)\widetilde{D} \\
&\quad = \left(\mu \begin{bmatrix} i\mathrm{Im}(\widehat{X}_{11}) & -i\mathrm{Re}(\widehat{X}_{12}) \\ -i\mathrm{Re}(\widehat{X}_{21}) & -i\mathrm{Im}(\widehat{X}_{22}) \end{bmatrix} + \begin{bmatrix} \mathrm{Re}(\widehat{X}_{11}) & \mathrm{Im}(\widehat{X}_{12}) \\ \mathrm{Im}(\widehat{X}_{21}) & -\mathrm{Re}(\widehat{X}_{22}) \end{bmatrix}\right) \\
&\quad = \left((-i\mu) \begin{bmatrix} -\mathrm{Im}(\widehat{X}_{11}) & \mathrm{Re}(\widehat{X}_{12}) \\ \mathrm{Re}(\widehat{X}_{21}) & \mathrm{Im}(\widehat{X}_{22}) \end{bmatrix} + \begin{bmatrix} \mathrm{Re}(\widehat{X}_{11}) & \mathrm{Im}(\widehat{X}_{12}) \\ \mathrm{Im}(\widehat{X}_{21}) & -\mathrm{Re}(\widehat{X}_{22}) \end{bmatrix}\right) =: (\nu X_1 + X_2)
\end{aligned}$$

with X_1, X_2 real and $\nu = -i\mu$. Here $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ denote the real and imaginary parts of X, respectively.

Now let $(\widetilde{S},\widetilde{T}) = (\widetilde{Q}X_1\widetilde{Z},\widetilde{Q}X_2\widetilde{Z})$ be a *real* generalized Schur form for the real pair (X_1, X_2); i.e., $\widetilde{Q}$ and $\widetilde{Z}$ are real orthogonal, $\widetilde{T}$ is real and upper triangular, and $\widetilde{S}$ is real and quasi-upper triangular with 1×1 and 2×2 blocks. Any 1×1 block in this real Schur form corresponds to a real eigenvalue of $\nu X_1 + X_2$, hence to a purely imaginary eigenvalue of $\mu N + M$, and thus to an eigenvalue of $\lambda X + P\overline{X}P$ on the unit circle. Similarly, any 2×2 block corresponds to a complex conjugate pair of eigenvalues for $\nu X_1 + X_2$, which in turn corresponds to an eigenvalue pair $(\mu, -\bar{\mu})$ for $\mu N + M$, and thence to a reciprocal pair of eigenvalues $(\lambda, 1/\bar{\lambda})$ for $\lambda X + P\overline{X}P$. Thus we see that the block structure in the real Schur form of the real pencil $\nu X_1 + X_2$ precisely mirrors the reciprocal pairing structure in the spectrum of the original PCP-pencil $\lambda X + P\overline{X}P$.

We recover a structured Schur form for $\lambda X + P\overline{X}P$ by re-assembling all the transformations together to obtain

$$(\underbrace{\widetilde{Q}\widetilde{D}W^T}_{Q})(\lambda X + P\overline{X}P)(\underbrace{W\widetilde{D}\widetilde{Z}}_{Z}) \;=\; \lambda\,\underbrace{(\widetilde{T} - i\widetilde{S})}_{S} + \underbrace{(\widetilde{T} + i\widetilde{S})}_{T}.$$

Since $\lambda S + T = \lambda S + \overline{S}$ this Schur form is again a PCP-pencil, but with respect to the involution $P = I$. This derivation can be summarized in the following algorithm.

Algorithm 1 (Structured Schur form for PCP-pencils).

Input: *$X \in \mathbb{C}^{m\times m}$ and $P \in \mathbb{R}^{m\times m}$ with $P^2 = I$ and $P^T = P$.*

Output: *Unitary $Q, Z \in \mathbb{C}^{m\times m}$ and block upper triangular $S \in \mathbb{C}^{m\times m}$ such that $QXZ = S$ and $QP\overline{X}PZ = \overline{S}$; the diagonal blocks of S are only of size 1×1 (corresponding to eigenvalues of magnitude 1) and 2×2 (corresponding to pairs of eigenvalues of the form $(\lambda, 1/\bar{\lambda})$).*

1: $P \to WDW^T$ *with* $D = \operatorname{diag}(I_p\,, -I_{m-p})$ [*find real symmetric Schur form*]

2: $\widehat{X} \leftarrow W^T XW$

3: $X_1 \leftarrow \begin{bmatrix} -\operatorname{Im}(\widehat{X}_{11}) & \operatorname{Re}(\widehat{X}_{12}) \\ \operatorname{Re}(\widehat{X}_{21}) & \operatorname{Im}(\widehat{X}_{22}) \end{bmatrix}$ *where* $\widehat{X}_{11} \in \mathbb{C}^{p\times p}$

4: $X_2 \leftarrow \begin{bmatrix} \operatorname{Re}(\widehat{X}_{11}) & \operatorname{Im}(\widehat{X}_{12}) \\ \operatorname{Im}(\widehat{X}_{21}) & -\operatorname{Re}(\widehat{X}_{22}) \end{bmatrix}$ *where* $\widehat{X}_{11} \in \mathbb{C}^{p\times p}$

5: $(X_1, X_2) \;\to\; (\widetilde{Q}^T\widetilde{S}\widetilde{Z}^T, \widetilde{Q}^T\widetilde{T}\widetilde{Z}^T)$ [*compute real generalized Schur form*]

6: $Q \leftarrow \widetilde{Q}\operatorname{diag}(I_p, -iI_{m-p})W^T\,,\quad Z \leftarrow W\operatorname{diag}(I_p, -iI_{m-p})\widetilde{Z}$

7: $S \leftarrow \widetilde{T} - i\widetilde{S}$

This algorithm has several advantages over the standard QZ algorithm applied directly to $\lambda X + P\overline{X}P$. First, it is faster, since the main computational work is the real QZ rather than the complex QZ algorithm. Second, structure preservation guarantees reciprocally paired eigenvalues; in particular, the presence of eigenvalues on the unit circle can be robustly detected. It is interesting to note that an algorithm with these properties (computation of a *structured* Schur form with the resulting guaranteed spectral symmetry, and greater efficiency than the standard QZ algorithm) is not yet available for the T- or $*$-palindromic eigenvalue problem.

In many applications it is also necessary to compute eigenvectors for a PCP-polynomial, e.g., in the stability analysis of time-delay systems described in Section 2. These can be found by starting with the eigenvalue problem $(\nu\widetilde{S} + \widetilde{T})x = 0$ in real generalized Schur form, and computing eigenvectors x using standard methods. It then follows that $\left(\frac{1+i\nu}{1-i\nu}S + T\right)x = \left(\frac{1+i\nu}{1-i\nu}S + \overline{S}\right)x = 0$, which in turn implies that $\left(\frac{1+i\nu}{1-i\nu}X + P\overline{X}P\right)(Zx) = 0$. In other words, $v = Zx$ is an eigenvector of (6.1) corresponding to the eigenvalue $\lambda = \frac{1+i\nu}{1-i\nu}$. If (6.1) was originally obtained as a structured linearization in $\mathbb{L}_1(Q)$ for a PCP-polynomial $Q(\lambda)$, then (as described in Section 5) v must be of the form $\Lambda \otimes u$ for some eigenvector u of $Q(\lambda)$ corresponding to the eigenvalue $\lambda = \frac{1+i\nu}{1-i\nu}$. Thus eigenvectors u of $Q(\lambda)$ are immediately recoverable from the eigenvectors v of (6.1).

Remark 5: Any real involution P that is not symmetric admits a Schur decomposition of the form

$$R = W^T P W = \begin{bmatrix} I_p & R_{12} \\ & -I_{m-p} \end{bmatrix}.$$

Defining

$$K := \begin{bmatrix} I_p & \frac{1}{2}R_{12} \\ & -I_{m-p} \end{bmatrix}$$

we have $K^{-1}RK = D = \mathrm{diag}(I_p\,, -I_{m-p})$, and so $P = \widetilde{W} D \widetilde{W}^{-1}$ with $\widetilde{W} = WK$. Thus if P is only mildly non-normal (i.e., $\|R_{12}\|$ is small), then there is a well-conditioned similarity transformation that brings P to the diagonal form D, and replacing W by $\widetilde{W}$ and W^T by $\widetilde{W}^{-1}$ in Algorithm 1 would still be a reasonable way to compute the eigenvalues and eigenvectors of a PCP-pencil. Note, however, that the output matrices Q and Z will no longer be unitary. $\diamond$

Remark 6: Note that with some minor modifications, Algorithm 1 can also be used on an anti-PCP-pencil to compute a structured Schur form that is anti-PCP, of the form $\lambda S - \overline{S}$. $\diamond$

7. Applications and Numerical Results. As we saw earlier in Section 2, eigenvalue problems with PCP-structure arise in the stability analysis of neutral linear time-delay systems. Such systems provide useful mathematical models in many physical application areas (see [12, 21] and the references therein); one example is circuits with delay elements, such as transmission lines and partial element equivalent circuits (PEEC's). A realistic problem motivated by the small PEEC model in Fig. 7.1 is given by

$$\mathcal{S} = \left\{ \begin{array}{rcll} D_1\dot{x}(t-h) + \dot{x}(t) & = & A_0x(t) + A_1x(t-h)\,, & t \geq 0 \\ x(t) & = & \varphi(t)\,, & t \in [-h, 0) \end{array} \right. \tag{7.1}$$

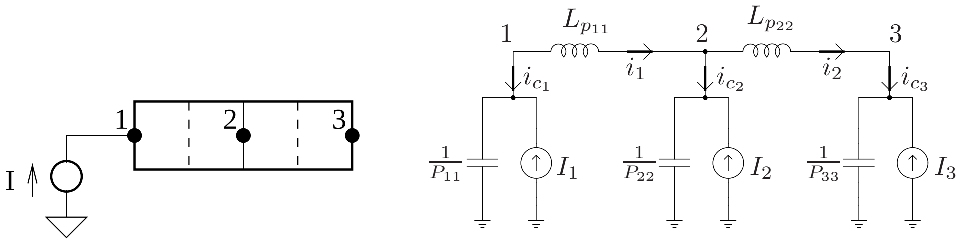
where

$$A_0 = 100 \cdot \begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \quad A_1 = 100 \cdot \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix},$$

$$D_1 = -\frac{1}{72} \cdot \begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}, \quad D_0 = I, \quad \text{and}$$

$$\varphi(t) = \left[\sin(t),\ \sin(2t),\ \sin(3t)\right]^T.$$

More details on this example can be found in [1].



(a) (b)

FIG. 7.1. *(a) Metal strip with two L_p cells (three capacitive cells dashed) and (b) small PEEC model for metal strip. Figures are redrawn from [1].*

The quadratic eigenproblem (2.14) for this example is $(z^2E + zF + G)u = 0$ with

$$\begin{aligned} E &= (D_0 \otimes A_1) + (A_0 \otimes D_1), \quad G = (D_1 \otimes A_0) + (A_1 \otimes D_0), \\ F &= (D_0 \otimes A_0) + (A_0 \otimes D_0) + (D_1 \otimes A_1) + (A_1 \otimes D_1). \end{aligned}$$

It is easy to verify that $E = P\overline{G}P$ and $F = P\overline{F}P$ hold for

$$P = \begin{bmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{bmatrix},$$

where M_{ij} denotes the 3×3 matrix with the entry 1 in position (i, j) and zeroes everywhere else.

The standard companion forms for this quadratic eigenproblem are

$$C_1(\lambda) = \lambda \begin{bmatrix} E & \\ & I \end{bmatrix} + \begin{bmatrix} F & G \\ -I & 0 \end{bmatrix} \quad \text{and} \quad C_2(\lambda) = \lambda \begin{bmatrix} E & \\ & I \end{bmatrix} + \begin{bmatrix} F & -I \\ G & 0 \end{bmatrix}.$$

A structured pencil in $\mathbb{L}_1(Q)$ (as discussed in Section 5.1) is given by

$$\lambda \begin{bmatrix} v_1E & -X_{12} \\ \overline{v}_1E & \overline{v}_1F + P\overline{X}_{12}P \end{bmatrix} + \begin{bmatrix} X_{12} + v_1F & v_1P\overline{E}P \\ -P\overline{X}_{12}P & \overline{v}_1P\overline{E}P \end{bmatrix}, \quad v_1 \in \mathbb{C}, \tag{7.2}$$

where X_{12} is arbitrary, while a structured pencil in $\mathbb{L}_2(Q)$ is given by

$$\lambda \begin{bmatrix} w_1E & \overline{w}_1E \\ X_{21} & \overline{w}_1F - P\overline{X}_{21}P \end{bmatrix} + \begin{bmatrix} w_1F - X_{21} & P\overline{X}_{21}P \\ w_1P\overline{E}P & \overline{w}_1P\overline{E}P \end{bmatrix}, \quad w_1 \in \mathbb{C}, \tag{7.3}$$

where X_{21} is arbitrary. With $w_1 = v_1$ and $X_{21} = \overline{v}_1E = -X_{12}$ the pencils (7.2) and (7.3) are the same, and give the unique structured pencil (up to choice of scalar v_1)

$$\lambda \begin{bmatrix} v_1E & \overline{v}_1E \\ \overline{v}_1E & \overline{v}_1F - v_1P\overline{E}P \end{bmatrix} + \begin{bmatrix} v_1F - \overline{v}_1E & v_1P\overline{E}P \\ v_1P\overline{E}P & \overline{v}_1P\overline{E}P \end{bmatrix} \tag{7.4}$$

that lies in the intersection $\mathbb{DL}(Q) = \mathbb{L}_1(Q) \cap \mathbb{L}_2(Q)$. By the Eigenvalue Exclusion Theorem [18, Thm 6.7] the pencil (7.4) is a structured linearization if and only if $-\overline{v}_1/v_1$ is not an eigenvalue of $Q(\lambda)$.

Choosing $v_1 = 1$ and applying Algorithm 1 we found that (7.4) has no eigenvalues on the unit circle, so the time-delay system $\mathcal{S}$ in (7.1) has no critical delays. The system $\mathcal{S}$ is stable for $h = 0$, since all eigenvalues of the pencil $L(\alpha) = \alpha(D_0 + D_1) - (A_0 + A_1)$ have negative real part. Continuity of the eigenvalues of $\mathcal{S}$ as a function of the delay h then implies that $\mathcal{S}$ is stable for every choice of the delay $h \geq 0$, a property known as *delay-independent stability.*

Our next example arises from the discretization of a partial delay-differential equation (PDDE), taken from Example 3.22 in [12, Sections 2.4.1, 3.3, 3.5.2]. It consists of the retarded time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + A_2 x(t - h_2) \tag{7.5}$$

where $A_0 \in \mathbb{R}^{n\times n}$ is tridiagonal and $A_1, A_2 \in \mathbb{R}^{n\times n}$ are diagonal with

$$\begin{aligned}
(A_0)_{kj} &= \begin{cases} -2(n+1)^2/\pi^2 + a_0 + b_0 \sin\big(j\pi/(n+1)\big) & \text{if} \quad k = j \\ \quad (n+1)^2/\pi^2 & \text{if} \ \ |k - j| = 1\,, \end{cases} \\
(A_1)_{jj} &= a_1 + b_1 \frac{j\pi}{n+1}\Big(1 - e^{-\pi\left(1 - j/(n+1)\right)}\Big)\,, \\
(A_2)_{jj} &= a_2 + b_2 \frac{j\pi^2}{n+1}\big(1 - j/(n+1)\big)\,.
\end{aligned}$$

Here a_ℓ, b_ℓ are real scalar parameters and $n \in \mathbb{N}$ is the number of uniformly spaced interior grid points in the discretization of the PDDE. We used the values

$$a_0 = 2,\ b_0 = 0.3,\ a_1 = -2,\ b_1 = 0.2,\ a_2 = -2,\ b_2 = -0.3$$

(as in [12]) and considered various values for n. With $\varphi_1 = -\pi/2$ (i.e. $e^{i\varphi_1} = i$) the quadratic PCP eigenvalue problem to solve is

$$(\lambda^2 E + \lambda F + P\overline{E}P)\, v \;=\; 0 \tag{7.6}$$

where

$$E \;=\; I \otimes A_2\,, \quad F \;=\; \big(I \otimes (A_0 - iA_1)\big) + \big((A_0 + iA_1) \otimes I\big),$$

and P is the $n^2 \times n^2$ permutation that interchanges the order of Kronecker products as in (2.17). Table 7.1 displays the results of our numerical experiments. Here n denotes the dimension of the time-delay system (7.5), $2n^2$ the dimension of the PCP-pencil (7.4), and $t_{\text{polyeig}}, t_{\text{QZ}}, t_{\text{PCP}}$ denote the execution times in seconds of the three tested methods:

1. solving the quadratic eigenvalue problem (7.6) using MATLAB's `polyeig` command, which applies the QZ algorithm to a (permuted) companion form,

TABLE 7.1

n	$2n^2$	t_{polyeig}	t_{QZ}	t_{PCP}	err_{polyeig}	err_{QZ}	$\#_{\text{polyeig}}$	$\#_{\text{QZ}}$	$\#_{\text{PCP}}$
5	50	0.02	0.02	0.01	5.5e-15	3.7e-15	4	4	4
10	200	0.50	0.55	0.28	6.5e-14	1.2e-13	4	4	4
15	450	5.5	6.3	3.0	4.4e-13	2.6e-13	4	3	4
20	800	33	36	20	1.3e-12	4.8e-13	3	0	4
25	1250	131	137	72	3.1e-12	6.6e-13	3	0	4
30	1800	413	435	227	1.1e-11	7.5e-13	0	0	4

2. solving the generalized eigenproblem for the PCP-pencil (7.4) using MATLAB's `QZ` algorithm, and
3. solving the eigenproblem for the PCP-pencil (7.4) using Algorithm 1.

All computations were done in MATLAB 7.5 (R2007b) under OpenSUSE Linux 10.2 (kernel 2.6.18, 64 bit) on a Core 2 Duo Processor E6850 3.0GHz with 4GB memory. The quantities err_{polyeig} and err_{QZ}, defined by

$$err \;=\; \max_{\lambda_j} \min_{\lambda_k} \frac{|\,\lambda_j - (1/\overline{\lambda}_k)\,|}{|\,\lambda_j\,|}$$

where λ_j, λ_k are (not necessarily distinct) eigenvalues of (7.6), measure the distance of the computed eigenvalues from being paired for the two unstructured methods. Note that this measure is zero for Algorithm 1 by construction. The numbers $\#_{\text{polyeig}}$, $\#_{\text{QZ}}$, and $\#_{\text{PCP}}$ denote the number of eigenvalues on the unit circle found by each method; for the unstructured methods this is the number of eigenvalues λ with $\big|\,|\lambda|-1\big| < 10^{-14}$, while for Algorithm 1 this is the number of 1×1 blocks in the structured Schur form.

As can be seen from the table, our structured method is about twice as fast as both unstructured methods. Note that the QZ algorithm applied to the PCP linearization (column t_{QZ}) is slightly slower than the QZ algorithm applied to a companion form linearization (column t_{polyeig}). On the other hand, the eigenvalues computed by `polyeig` are not as well paired as those computed by the QZ algorithm applied to the PCP linearization. In the time-delay setting the only eigenvalues of interest are those on the unit circle and in this respect the three methods perform very differently. All methods correctly find the number of eigenvalues of unit magnitude for $n = 5, 10$. For larger n the unstructured methods do not find all, and sometimes not any, of the desired eigenvalues. In particular, for $n = 30$ only the structured method finds all 4 eigenvalues on the unit circle whereas the unstructured methods find none.

As a third example, we tested PCP-pencils of the form $\lambda X + P\overline{X}P$ where X is randomly generated by the Matlab command `randn(n)+i*randn(n)` and P is the matrix R as defined in (5.6). We found that for values of $50 < n < 2000$ on this type of problem our Algorithm 1 performs 2.5 to 3 times faster than the QZ algorithm.

8. Concluding Summary. Motivated by a quadratic eigenproblem arising in the stability analysis of time-delay systems [12], we have identified a new type of matrix polynomial structure, termed PCP-structure, that is analogous to the palindromic structures investigated in [17]. The properties of these PCP-polynomials were investigated, along with those of three closely related structures — anti-PCP, PCP-even and PCP-odd polynomials. Spectral symmetries were revealed, and relationships between these structures were established via the Cayley transformation.

Building on the work in [18], we have shown how to construct structure-preserving linearizations for all these structured polynomials in the pencil spaces $\mathbb{L}_1(Q)$, $\mathbb{L}_2(Q)$, and $\mathbb{DL}(Q)$. In addition to preservation of eigenvalue symmetry, such linearizations also permit easy eigenvector recovery, which can be an important consideration in applications. Structured Schur forms for PCP and anti-PCP pencils were derived, along with a new algorithm for their computation that compares favorably with the QZ algorithm. Using a structure-preserving linearization followed by the computation of a structured Schur form thus allows us to solve the new structured eigenproblem efficiently, reliably, and with guaranteed preservation of spectral symmetries.

Acknowledgment. The authors thank Elias Jarlebring for many enlightening discussions about time-delay systems.

REFERENCES

[1] A. Bellen, N. Guglielmi, and A.E. Ruehli, *Methods for linear systems of circuit delay-differential equations of neutral type*, IEEE Trans. Circuits and Systems - I: Fundamental Theory and Applications, 46 (1999), pp. 212–216.

[2] R. Byers, D. S. Mackey, V. Mehrmann, and H. Xu, *Symplectic, BVD, and palindromic approaches to discrete-time control problems*, MIMS EPrint 2008.35, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, Mar. 2008. To appear in the "Jubilee Collection of Papers dedicated to the 60th Anniversary of Mihail Konstantinov" (P. Petkov and N. Christov, eds.), Bulgarian Academy of Sciences, Sofia, 2008.

[3] I. Gohberg, M.A. Kaashoek, and P. Lancaster, *General theory of regular matrix polynomials and band Toeplitz operators*, Integr. Eq. Operator Theory, 11 (1988), pp. 776–882.

[4] I. Gohberg, P. Lancaster, and I. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.

[5] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-differential Equations*, vol. 99 of Applied Mathematical Sciences, Springer-Verlag, New York, 1993.

[6] N.J. Higham, R-C. Li, and F. Tisseur, *Backward error of polynomial eigenproblems solved by linearization*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 1218–1241.

[7] N.J. Higham, D.S. Mackey, N. Mackey, and F. Tisseur, *Symmetric linearizations for matrix polynomials*, SIAM J. Matrix Anal. Appl., 29 (2006), pp. 143–159.

[8] N.J. Higham, D.S. Mackey, and F. Tisseur, *The conditioning of linearizations of matrix polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1005–1028.

[9] ———, *Definite matrix polynomials and their linearization by definite pencils*, MIMS EPrint 2007.97, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2008. To appear in SIAM J. Matrix Anal. Appl.

[10] N.J. Higham, D.S. Mackey, F. Tisseur, and S.D. Garvey, *Scaling, sensitivity and stability in the numerical solution of quadratic eigenvalue problems*, Int. J. of Numerical Methods in Engineering, 73 (2008), pp. 344–360.

[11] R.A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, reprinted paperback version ed., 1995.

[12] E. Jarlebring, *The Spectrum of Delay-Differential Equations: Numerical Methods, Stability and Perturbation*, PhD thesis, TU Braunschweig, Institut Computational Mathematics, Carl-Friedrich-Gauß-Fakultät, 38023 Braunschweig, Germany, 2008.

[13] P. Lancaster, *Lambda-Matrices and Vibrating Systems*, Pergamon Press, Oxford, 1966. reprinted by Dover, New York, 2002.

[14] P. Lancaster and P. Psarrakos, *A note on weak and strong linearizations of regular matrix polynomials*, MIMS EPrint 2006.72, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2005.

[15] P. Lancaster and L. Rodman, *The Algebraic Riccati Equation*, Oxford University Press, Oxford, 1995.

[16] D.S. Mackey, *Structured Linearizations for Matrix Polynomials*, PhD thesis, The University of Manchester, U.K., 2006. Available as MIMS-EPrint 2006.68, Manchester Institute for Mathematical Sciences, Manchester, England, 2006.

[17] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Structured polynomial eigenvalue problems: Good vibrations from good linearizations*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1029–1051.

[18] ———, *Vector spaces of linearizations for matrix polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 971–1004.

[19] V. Mehrmann, *A step toward a unified treatment of continuous and discrete-time control problems*, Linear Algebra Appl., 241-243 (1996), pp. 749–779.

[20] W. Michiels and S.-I. Niculescu, *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach*, vol. 12 of Advances in Design and Control, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2007.

[21] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, Springer-Verlag, London, 2001.

[22] C. Schröder, *Palindromic and Even Eigenvalue Problems - Analysis and Numerical Methods*, PhD thesis, Technical University Berlin, Germany, 2008.

[23] F. Tisseur and K. Merbergen, *A survey of the quadratic eigenvalue problem*, SIAM Review, 43 (2001), pp. 234–286.