AN INVERSE EIGENVALUE PROBLEM AND AN ASSOCIATED APPROXIMATION PROBLEM FOR GENERALIZED K-CENTROHERMITIAN MATRICES

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Abstract: A partially described inverse eigenvalue problem and an associated optimal approximation problem for generalized K-centrohermitian matrices are considered. It is shown under which conditions the inverse eigenproblem has a solution. An expression of its general solution is given. In case a solution of the inverse eigenproblem exists, the optimal approximation problem can be solved. The formula of its unique solution is given.
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1. Introduction. Inverse eigenvalue problems concern the reconstruction of a matrix from prescribed spectral data. To be more specific, given a set of m (not necessarily linearly independent) vectors $x_j \in \mathbb{F}^n, j = 1, \ldots, m$ $(n > m)$ and a set of scalars $\lambda_j \in \mathbb{F}, j = 1, \ldots, m$, find a matrix $A \in \mathbb{F}^{n\times n}$ such that

$$Ax_j = \lambda_j x_j \tag{1.1}$$

for $j = 1, \ldots, m$. Here $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ denotes the field of real or complex numbers. See [6, 4] for a general review on inverse eigenproblems. For a structured inverse eigenproblem, A is subject to additional constraints, typically given in the form that $A \in \Omega$ is required, where Ω denotes a certain subset of $n\times n$ matrices. Several different kinds of sets Ω have already been dealt with in the literature: Jacobi matrices [8], symmetric matrices [10], anti-symmetric matrices [23], anti-persymmetric matrices [22, 24], unitary matrices [1, 2], centro-symmetric matrices [25], (generalized) Toeplitz matrices [9, 20], symmetric anti-bidiagonal matrices [14]. This is by far not a complete list, see [7] for a recent review, a number of applications and an extensive list of references.

Here we will consider the inverse eigenproblem for generalized K-centrohermitian matrices [13, 12, 17, 19, 21] (see also [3]): A matrix $A \in \mathbb{C}^{n\times n}$ is said to be

- generalized K-centrosymmetric if $A = KAK$,
- generalized K-centrohermitian if $A = K\overline{A}K$

where $K \in \mathbb{I}^{n\times n}$ can be any permutation matrix (i.e., $K^2 = I$ and $K = K^T$). Hermitian block Toeplitz matrices are a special class of generalized K-centrohermitian

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matrices with

$$K = \begin{bmatrix} & & & I_p \\ & & \cdot & \\ & I_p & & \\ \cdot & & & \\ I_p & & & \end{bmatrix}.$$

Moreover, the Kronecker product of two Hermitian Toeplitz matrices is a special generalized K-centrohermitian matrix with

$$K = \begin{bmatrix} J_p & & & & \\ & \ddots & & & \\ & & J_p & & \\ & & & \ddots & \\ & & & & J_p \end{bmatrix},$$

where J_p is the $p \times p$ exchange matrix (that is, J_p has only p nonzero entries $j_{\ell,n-\ell+1} = 1, \ell = 1, \dots, p$).

A problem closely related to the inverse eigenproblem (1.1) is the following optimal approximation problem: Given a matrix $\widetilde{A} \in \mathbb{C}^{n \times n}$, find a matrix S with some prescribed spectral data that gives the best approximation to $\widetilde{A}$ in the Frobenius norm, that is,

$$||\widetilde{A} - S||_F = \inf_{A \in \mathcal{S}} ||\widetilde{A} - A||_F, \tag{1.2}$$

where $\mathcal{S}$ denotes the set of all possible solutions of (1.1). Such a problem may arise, e.g., when a preconditioner with a specific structure is sought in order to solve linear systems of equations efficiently, see,.e.g., [5]. If a structured inverse eigenproblem (1.1) is considered, that is, A is required to be in some set Σ, then we obtain a structured optimal approximation problem, where in addition to (1.2) $A \in \Omega$ is required.

In this paper we consider the inverse eigenvalue problem (1.1) and the optimal approximation problem (1.2) for generalized K-centrohermitian matrices. That is, we require $A \in \Omega$, where Ω denotes the set of generalized K-centrohermitian matrices. In Section 2 some facts about generalized K-centrohermitian matrices, which will be used later on, are stated. Section 3 deals with the inverse eigenproblem, Section 4 with the optimal approximation problem. The derivations easily lead to algorithms for solving the two problems discussed. In the last section, we compare those algorithms to the ones used for the unstructured problems showing that the structured approach used here ensures significant savings in computational costs.

2. Generalized K-centrohermitian matrices. In [17] it is shown that every $n \times n$ generalized K-centrohermitian matrix can be reduced to a $n \times n$ real matrix by a simple unitary similarity transformation. As we will make explicit use of this reduction, we briefly recall the construction of the unitary transformation matrix.

K is a permutation matrix ($K^2 = I$ and $K = K^T$). Hence, without loss of generality, we can assume that

$$K = P_{j_1,\kappa(j_1)} P_{j_2,\kappa(j_2)} \cdots P_{j_l,\kappa(j_l)}, \quad l \leq n,$$

where P_{ij} is the transposition which interchanges the rows i and j and $j_i \neq \kappa(j_i)$ for $i = 1, \cdots, l$ (that is we do not allow for $P_{u,\kappa(u)} = I$, when $u = \kappa(u)$).

Define $Q^{(j_i,\kappa(j_i))}$ as the matrix that differs from the identity in the 4 entries

$$\begin{bmatrix} Q_{j_i,j_i} & Q_{j_i,\kappa(j_i)} \\ Q_{\kappa(j_i),j_i} & Q_{\kappa(j_i),\kappa(j_i)} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}. \tag{2.1}$$

$Q^{(j_i,\kappa(j_i))}$ in (2.1) is an orthogonal matrix and for i, $s = 1, \cdots, l$,

$$Q^{(j_i,\kappa(j_i))}Q^{(j_s,\kappa(j_s))} = Q^{(j_s,\kappa(j_s))}Q^{(j_i,\kappa(j_i))}.$$

The product of all these rank-two modifications of the identity

$$\tilde{Q} = Q^{(j_1,\kappa(j_1))}Q^{(j_2,\kappa(j_2))} \dots Q^{(j_l,\kappa(j_l))} \tag{2.2}$$

yields an orthogonal matrix $\tilde{Q}$. Let $\tilde{P}$ be a permutation matrix such that in $Q = \tilde{Q}\tilde{P}$ the columns of $\tilde{Q}$ are interchanged such that the columns $\kappa(j_1)$, $\kappa(j_2)$, $\dots$, $\kappa(j_l)$ of $\tilde{Q}$ become the columns $n-l+1$, $n-l+2$, $\cdots$, n of a new matrix Q. Partition Q as

$$Q = \tilde{Q}\tilde{P} = [Q_1, Q_2], \tag{2.3}$$

where Q_1 denotes the matrix consisting of the first $n-l$ columns of Q and Q_2 denotes the matrix consisting of the last l columns of Q.

Finally, define

$$U = [Q_1, iQ_2]. \tag{2.4}$$

U will reduce every $n \times n$ generalized K-centrohermitian matrix to an $n \times n$ real matrix by a simple unitary similarity transformation.

LEMMA 2.1. *[17, Theorem 3] Let K be a permutation matrix of order n and U be defined as in (2.4). Then A is a generalized K-centrohermitian matrix if and only if*

$$B = U^H A U = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{2.5}$$

holds, where $B_{11} \in \mathbb{R}^{(n-l)\times(n-l)}, B_{12} \in \mathbb{R}^{(n-l)\times l}, B_{21} \in \mathbb{R}^{l\times(n-l)}$ and $B_{22} \in \mathbb{R}^{l\times l}$ with $l = rank(I - K)$.

Moreover, we have the following result.

LEMMA 2.2. *Let K be a permutation matrix of order n and U be defined as in (2.4). A matrix $M \in \mathbb{C}^{n\times m}$ satisfies the relation $KM = \overline{M}$ if and only if $U^H M \in \mathbb{R}^{n\times m}$ holds.*

3. The inverse eigenproblem. Here we will deal with the following structured inverse eigenvalue problem: Given $X = [x_1, x_2, \cdots, x_m] \in \mathbb{C}^{n\times m}$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$, find an $n \times n$ generalized K-centrohermitian matrix A such that

$$AX = X\Lambda. \tag{3.1}$$

Most of the derivations in this section follow the ideas of [25] where the structured inverse eigenvalue problem for centrosymmetric matrices is considered.

Let $X = X_1 + iX_2$ with

$$X_1 = \frac{X + K\bar{X}}{2}, \quad X_2 = \frac{X - K\bar{X}}{2i}, \tag{3.2}$$

and

$$\Lambda = \Lambda_1 + i\Lambda_2, \tag{3.3}$$

where Λ_1 and Λ_2 denote the real and the imaginary part of Λ, respectively. Then the equation (3.1) can be rewritten as follows:

$$A(X_1 + iX_2) = (X_1 + iX_2)(\Lambda_1 + i\Lambda_2).$$

Left-multiplying by U^H (2.4) gives that

$$U^H AUU^H(X_1 + iX_2) = U^H(X_1 + iX_2)(\Lambda_1 + i\Lambda_2).$$

Denoting

$$\begin{aligned} B &= U^H AU, \\ Y_1 &= U^H X_1, \\ Y_2 &= U^H X_2, \end{aligned} \tag{3.4}$$

yields

$$B(Y_1 + iY_2) = (Y_1 + iY_2)(\Lambda_1 + i\Lambda_2).$$

This can be rewritten as

$$BY = Y\widehat{\Lambda}, \tag{3.5}$$

where

$$Y = [Y_1,\ Y_2] \qquad \text{and} \qquad \widehat{\Lambda} = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ -\Lambda_2 & \Lambda_1 \end{bmatrix}. \tag{3.6}$$

Then, by Lemma 2.1 and Lemma 2.2, the matrices B, Y_1, Y_2 are all real.

Thus we can always reduce the complex structured inverse eigenproblem (3.1) into one in the real field (3.5). As usual, when turning a complex-valued problem into a real-valued one, the size of the problem is doubled (here $2m$ instead of m).

(3.5) is just a standard inverse eigenproblem without any structural restrictions. This has already been considered in [15]. For the convenience of the reader, we state the general expression for the solution of (3.5) from [15] slightly modified to suit the notation used here. This expression involves a Moore-Penrose inverse, denoted by Y^+ for a matrix Y.

LEMMA 3.1. *Let $Y \in \mathbb{R}^{n\times 2m}$ and $\widehat{\Lambda} \in \mathbb{R}^{2m\times 2m}$ be given matrices. Assume that rank$(Y) = r \le 2m$. Denote the singular value decomposition of Y by*

$$Y = W \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^T = W_1 \Sigma V_1^T, \tag{3.7}$$

where

$$[W = [W_1,\ W_2] \in \mathbb{R}^{n\times n}, \qquad V = [V_1,\ V_2] \in \mathbb{R}^{2m\times 2m}$$

are orthogonal matrices with $W_1 \in \mathbb{R}^{n\times r}, V_1 \in \mathbb{R}^{2m\times r}$, and

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r), \qquad \textit{with} \ \ \sigma_i > 0, \ \ 1 \le i \le r.$$

Then $BY = Y\widehat{\Lambda}$ *has a real solution* $B \in \mathbb{R}^{n\times n}$ *if and only if*

$$Y\widehat{\Lambda}Y^{+}Y = Y\widehat{\Lambda}.$$

Its general solution can be expressed as

$$B = Y\widehat{\Lambda}Y^{+} + NW_2^T, \qquad \text{for all } N \in \mathbb{R}^{n\times(n-r)}. \tag{3.8}$$

Using Lemma 3.1, we can give a general expression for the solution of (3.1).

THEOREM 3.2. *Given* $X \in \mathbb{C}^{n\times m}$ *and* $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^{m\times m}$. *Let* $U, X_1, X_2, \Lambda_1, \Lambda_2, Y_1, Y_2, Y, \widehat{\Lambda}, W_1, W_2, V_1, V_2$ *be as defined above. Then* $AX = X\Lambda$ *has a generalized* K*-centrohermitian solution if and only if*

$$Y\widehat{\Lambda}Y^{+}Y = Y\widehat{\Lambda}, \tag{3.9}$$

and its general solution can be expressed as

$$A = U(Y\widehat{\Lambda}Y^{+} + NW_2^T)U^H, \qquad \text{for all } N \in \mathbb{R}^{n\times(n-r)}. \tag{3.10}$$

Proof. For all $N \in \mathbb{R}^{n\times(n-r)}$, $Y\widehat{\Lambda}Y^{+} + NW_2^T$ is a real matrix, which implies by Lemma 2.1 that $A = U(Y\widehat{\Lambda}Y^{+} + NW_2^T)U^H$ is a generalized K-centrohermitian matrix.

Assume that $Y\widehat{\Lambda}Y^{+}Y = Y\widehat{\Lambda}$. For any matrix $N \in \mathbb{R}^{n\times(n-r)}$, we now show that $A = U(Y\widehat{\Lambda}Y^{+} + NW_2^T)U^H$ is a solution of (3.1). Let

$$C = \begin{bmatrix} I_m \\ iI_m \end{bmatrix}. \tag{3.11}$$

Note that $U^HX = YC$, $\widehat{\Lambda}C = C\Lambda$ and $W_2^TY = 0$. Therefore we have

$$\begin{aligned} AX &= U(Y\widehat{\Lambda}Y^{+})U^HX + UNW_2^TU^HX = U(Y\widehat{\Lambda}Y^{+}Y)C + UNW_2^TYC \\ &= UY\widehat{\Lambda}C = UYC\Lambda = UU^HX\Lambda \\ &= X\Lambda, \end{aligned}$$

where we used the assumption $Y\widehat{\Lambda}Y^{+}Y = Y\widehat{\Lambda}$. Hence A as in (3.10) is a solution of (3.1).

Assume that $AX = X\Lambda$ has a generalized K-centrohermitian solution A. By Lemma 2.1 and equations (3.2)-(3.6), we obtain that $BY = Y\widehat{\Lambda}$. Using Lemma 3.1 then gives that $Y\widehat{\Lambda}Y^{+}Y = Y\widehat{\Lambda}$, and its general solution can be expressed as

$$B = Y\widehat{\Lambda}Y^{+} + NW_2^T, \qquad \text{for all } N \in \mathbb{R}^{n\times(n-r)}.$$

A recovery process (left-multiplying by U and right-multiplying by U^H on both sides of the above equality) shows that

$$A = U(Y\widehat{\Lambda}Y^{+} + NW_2^T)U^H, \qquad \text{for all } N \in \mathbb{R}^{n\times(n-r)}.$$

Thus the proof is complete. □

Please note, that the set of all possible solutions $\mathcal{S}$ to the problem (3.1) may be empty.

4. The optimal approximation problem. Here we will deal with the following structured optimal approximation problem: Given a matrix $\widetilde{A} \in \mathbb{C}^{n\times n}$, find a matrix $S \in \mathcal{S}$ that gives the best approximation to $\widetilde{A}$ in the Frobenius norm, that is,

$$||\widetilde{A} - S||_F = \inf_{A\in\mathcal{S}} ||\widetilde{A} - A||_F, \tag{4.1}$$

where $\mathcal{S}$ denotes the set of all possible solutions of (3.1). If $\mathcal{S}$ is nonempty, we have the following result.

THEOREM 4.1. *Given $\widetilde{A} \in \mathbb{C}^{n\times n}$. Under the assumptions of Theorem 1 and if $\mathcal{S}$ is nonempty, the problem (4.1) has a unique solution S, which can be expressed as*

$$S = U[B_0 + (B_1 - B_0)W_2W_2^T]U^H \tag{4.2}$$

where $B_0 = Y\widehat{\Lambda}Y^+$, $B_1 = U^H\widetilde{A}_1U$ with

$$\widetilde{A}_1 = \frac{\widetilde{A} + K\overline{\widetilde{A}}K}{2},$$

and U is defined as in (2.4) and W_2 as in (3.7).

Proof. From the hypothesis and by Theorem 3.2, we know that if $\mathcal{S}$ is nonempty, then any of its elements can be expressed as

$$A = U(B_0 + NW_2^T)U^H, \qquad \text{for all} \quad N \in \mathbb{R}^{n\times(n-r)},$$

where $B_0 = Y\widehat{\Lambda}Y^+$ and U as in (2.4).

Next, we observe that $\widetilde{A}$ can be expressed as the sum of two unique K-centrohermitian matrices $(\widetilde{A}_1, \widetilde{A}_2)$ such that

$$\widetilde{A} = \widetilde{A}_1 + i\widetilde{A}_2. \tag{4.3}$$

It follows immediately that

$$\widetilde{A}_1 = \frac{\widetilde{A} + K\overline{\widetilde{A}}K}{2}, \qquad \widetilde{A}_2 = \frac{\widetilde{A} - K\overline{\widetilde{A}}K}{2i}$$

are K-centrohermitian and satisfy (4.3). The uniqueness can be proven by showing that there does not exists another pair of K-centrohermitian matrices $(\widetilde{M}_1, \widetilde{M}_2)$ such that $\widetilde{A} = \widetilde{M}_1 + i\widetilde{M}_2$. Assuming that such a pair of matrices exists, we have

$$\widetilde{A}_1 - \widetilde{M}_1 = i(\widetilde{A}_2 - \widetilde{M}_2).$$

Taking the complex-conjugate of this equation, and pre- and postmultiplying by K yields

$$\widetilde{A}_1 - \widetilde{M}_1 = -i(\widetilde{A}_2 - \widetilde{M}_2).$$

Hence, $\widetilde{A}_1 = \widetilde{M}_1$ and $\widetilde{A}_2 = \widetilde{M}_2$.

Therefore we can express $\widetilde{A}$ uniquely as in (4.3) $||\widetilde{A} - A||_F = ||(\widetilde{A}_1 - A) + i\widetilde{A}_2||_F$. By Lemma 2.1 and the unitary invariance of the Frobenius norm, we have that

$$\begin{aligned} ||\widetilde{A} - A|| &= ||(\widetilde{A}_1 - A) + i\widetilde{A}_2|| \\ &= ||U^H(\widetilde{A}_1 - U(B_0 + NW_2^T)U^H)U + iU^H\widetilde{A}_2U|| \\ &= ||[(B_1 - B_0) - NW_2^T] - iB_2|| \\ &= ||(B_1 - B_0) - NW_2^T|| + ||B_2||, \end{aligned}$$

where $B_1 = U^H \widetilde{A}_1 U$ and $B_2 = U^H \widetilde{A}_2 U$. By Lemma 2.1, B_1 and B_2 are $n \times n$ real matrices. Thus, the problem $\min_{A \in \mathcal{S}} ||\widetilde{A} - A||$ is equivalent to

$$\min_{N \in \mathbb{R}^{n \times (n-r)}} ||(B_1 - B_0) - NW_2^T||.$$

As W is orthogonal, we have $W_1^T W_2 = 0$, $W_2^T W_2 = I_{n-r}$, and

$$\begin{aligned} ||(B_1 - B_0) - NW_2^T||^2 &= ||(B_1 - B_0)W - NW_2^T W||^2 \\ &= ||(B_1 - B_0)W_1||^2 + ||(B_1 - B_0)W_2 - N||^2. \end{aligned}$$

Therefore, if we choose

$$N = (B_1 - B_0)W_2,$$

we will minimize $||(B_1 - B_0) - NW_2^T||$.

Since the matrix $\widetilde{A}_1$ obtained from $\widetilde{A}$ is unique, so is the matrix B_1, and thus the matrix N is unique. That is to say that the solution of (4.1) is unique. This completes our proof. □

5. Concluding remarks. The common approach to compute the general unstructured solution $A \in \mathbb{C}^{n \times n}$ of the inverse eigenproblem (1.1) $AX = \Lambda X$ for a given matrix $X \in \mathbb{C}^{n \times m}$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$ involves the singular value decomposition (SVD) of X. Assume that X has rank $\hat{r}$ and the SVD of the matrix X is given by

$$X = \hat{U} \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \hat{V}^H = \hat{U}_1 \hat{\Sigma} \hat{V}_1^{\,H}, \tag{5.1}$$

where $\hat{U} = [\hat{U}_1,\ \hat{U}_2]$, $\hat{V} = [\hat{V}_1,\ \hat{V}_2]$ are, respectively, $n \times n$ and $m \times m$ unitary matrices with $\hat{U}_1 \in \mathbb{C}^{n \times \hat{r}}$, $\hat{V}_1 \in \mathbb{C}^{m \times \hat{r}}$, $\hat{\Sigma} = \text{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_{\hat{r}})$, $\hat{\sigma}_i > 0$, $1 \le i \le \hat{r}$. Then, using Lemma 3.1 we have that (1.1) is solvable if and only if

$$X \Lambda X^+ X = X \Lambda,$$

and its general solution can be written as

$$A = X \Lambda \hat{V}_1 \hat{\Sigma}^{-1} \hat{U}_1^H + C \hat{U}_2^H, \quad \text{for all } C \in \mathbb{C}^{n \times (n-\hat{r})}. \tag{5.2}$$

Hence an algorithm for computing the solution of (1.1) consists of two steps: the computation of the SVD of the matrix X according to (5.1) and the set up of A according to (5.2).

When the solution of the structured inverse eigenproblem (3.1) for generalized K-centrohermitian matrices is sought, we have to compute A as in (3.10). For this, we have to set up X_1 and X_2 (3.2), Λ_1 and Λ_2 (3.3), Y_1 and Y_2 (3.4), Y (3.6) and $\hat{\Lambda}$ (3.6). Next, the SVD of the matrix Y according to (3.7) has to be computed. Finally, A can be formed as $A = UBU^H$ where $B = Y \hat{\Lambda} Y^+ + NW_2^T$.

A careful flop count reveals that for $n > 2m$, the structured algorithm is about 8 times cheaper than the standard one.

A comparison of the standard algorithm for solving (1.2) for general matrices and our structured approach for generalized K-centrohermitian matrices reveals similar computational savings as in the previously discussed case.

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