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Missing Point Estimation for steady aerodynamic applications

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Model reduction in aerodynamic applications usually makes use of Proper Orthogonal Decomposition (POD). In this work a POD-based method, called Missing Point Estimation (MPE), will be applied to steady state flows with variation of the flow parameter angle of attack. The basic idea of MPE is to select a subset of the computational grid points (control volumes) and limit the governing equations to these. Subsequently, the remaining equations are projected onto the POD subspace. This approach has the advantage that the nonlinear right hand side of the governing equations has to be evaluated only for some selected points (control volumes) which makes the MPE faster than classical Galerkin projection.

1 Proper Orthogonal Decomposition

Suppose a set of m discrete steady flow solutions $\{\vec{w}_1, \ldots, \vec{w}_m\}$, called *snapshots*, are given. The snapshots are solutions to the Euler equations on a computational grid with n points for the four conservative variables density ρ, Cartesian momentum densities ρv_x and ρv_y, as well as the total energy density ρE, where v_x and v_y are the velocity components and E the total energy [1]. To each grid point a value of all four conservative variables is assigned and they are stored in the snapshot vectors in the form $\vec{w}^T = (\vec{\rho}^T \ \overrightarrow{\rho v_x}^T \ \overrightarrow{\rho v_y}^T \ \overrightarrow{\rho E}^T) \in \mathbb{R}^{1\times 4n}$. Note that in this work the snapshots are steady state solutions for different choices of the flow parameter angle of attack α, that is $\vec{w}_i = \vec{w}(\alpha_i)$. They are computed with TAU [2], which is an unstructered finite volume CFD solver employing a cell-vertex scheme with dual control volumes [1, Section 5.2.2]. The dual control volumes are constructed such that each grid point is surrounded by a control volume.

After computing the average of the snapshots $\bar{\vec{w}} = \frac{1}{m}\sum_{i=1}^{m} \vec{w}_i$, we can construct the so-called snapshot matrix as $Y = \left[\, (\vec{w}_1 - \bar{\vec{w}}) \ \cdots \ (\vec{w}_m - \bar{\vec{w}}) \,\right]$. While *Proper Orthogonal Decomposition* (POD) is often applied in the Euclidean space, in this work POD is used with an inner product of the discrete L_2 space [3, example 1.7], which is given by $(\vec{w}_i, \vec{w}_j)_\Omega = \vec{w}_i^T \Omega \vec{w}_j$ with $\vec{w}_i, \vec{w}_j \in \mathbb{R}^{4n}$, where $\Omega = \operatorname{diag}(V, V, V, V) \in \mathbb{R}^{4n\times 4n}$ with $V = \operatorname{diag}(V_1, \ldots, V_n) \in \mathbb{R}^{n\times n}$, where each V_k is the volume of the control volume associated with the kth grid point. The idea of POD is to find a set of basis vectors, which optimally describe the snapshots. Mathematically formulated this translates to the maximization problem [3]

$$\max_{u_1,\ldots,u_d \in \mathbb{R}^{4n}} \sum_{i=0}^{d} \sum_{j=0}^{m} |(w_j, u_i)_\Omega|^2 \quad \text{s.t.} \quad (u_k, u_l) = \delta_{kl} \quad \text{for} \quad 1 \le k, l \le d. \tag{1}$$

The solution to this problem is related to the *Singular Value Decomposition* (SVD) of the weighted snapshot matrix $\bar{Y} = \Omega^{1/2} Y \in \mathbb{R}^{4n\times m}$. The SVD is given by $\bar{Y} = \bar{U}\Sigma\bar{V}$, where $\bar{U} = [\bar{u}_1 \ldots \bar{u}_{4n}] \in \mathbb{R}^{4n\times 4n}$ and $\bar{V} = [\bar{v}_1 \ldots \bar{v}_m] \in \mathbb{R}^{m\times m}$ are orthogonal matrices containing the left and right singular vectors, respectively, and where $\Sigma = \operatorname{diag}\left(\sigma_1, \ldots, \sigma_{\min\{4n,m\}}\right)$ is a diagonal matrix with the singular values as entries, ordered such that $\sigma_1 \ge \ldots \ge \sigma_{\min\{4n,m\}}$. The maximization problem (1) is solved by the vectors $u_i = \Omega^{1/2}\bar{u}_i$ with $i = 1, \ldots, d$; see Theorem 1.8 in [3]. These vectors form the so-called *POD basis*. Note that the dimension of the affine POD subspace spanned by the POD basis is chosen such that $d = \arg\min_k \left\{ \sum_{i=1}^{k} \sigma_i^2 / \sum_{i=1}^{\min\{4n,m\}} \sigma_i^2 \ge \frac{99.9999}{100} \right\}$. It will be assumed that the solutions $\vec{w}^*$, which are to be computed, reside in this subspace. In other words, they can be expressed approximately in the form $\vec{w}^* \approx U\vec{a} + \bar{\vec{w}}$, where $\vec{a}$ is a vector of suitable coefficients for the POD basis $U = [u_1 \ldots u_d]$.

2 Missing Point Estimation

In this work the Euler equations are considered, which can be expressed in the form $\frac{d}{dt}\vec{w}(t) = -\Omega^{-1}\vec{R}(\vec{w}(t))$, where $\vec{w}$ and Ω are as defined in Section 1 and $\vec{R}$ is the *residual* vector, which is the spatial discretization of the fluxes over the boundaries of the control volumes; see [1] for details.

The idea of Missing Point Estimation [4] is to construct a reduced order model of the governing equations, which does not have to evaluate the residual vector at every point (control volume), but rather only at a subset of all points $\mathbb{X} = \{j_1, \ldots, j_{\tilde{n}}\} \subset \{1, \ldots, n\}$, where $\tilde{n}$ is the number of the selected points. For this purpose we define the *selection matrix* $\tilde{P} = [i_{j_1} \ \cdots \ i_{j_{\tilde{n}}}] \in \mathbb{R}^{n\times \tilde{n}}$, with the jth unit vector $i_j \in \mathbb{R}^n$ for one and $P = \operatorname{diag}(\tilde{P}, \tilde{P}, \tilde{P}, \tilde{P}) \in \mathbb{R}^{4n\times 4\tilde{n}}$ for all variables. A projection onto the

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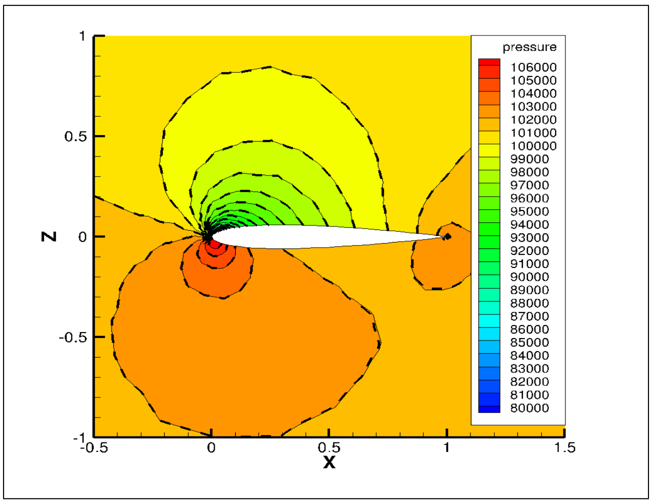

Fig. 1 The TAU reference solution (solid line) and the MPE solution (dashed line) at $\alpha = 7^\circ$.

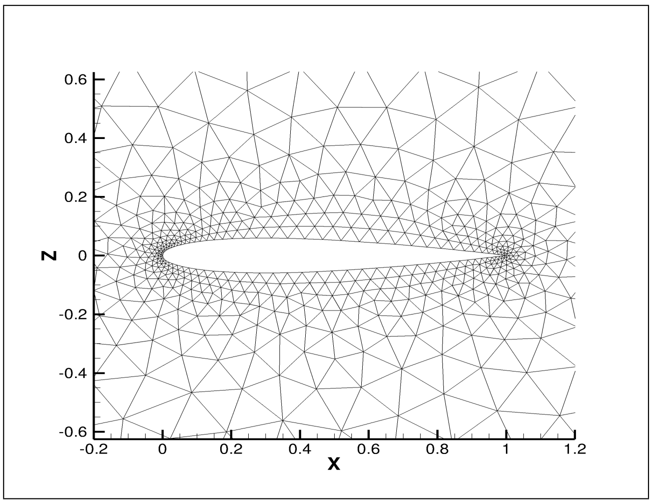

Fig. 2 The unstructured grid used for the computations for the NACA 0012 airfoil.

chosen points $\mathbb{X} = \{j_1, \ldots, j_{\tilde{n}}\}$ is then given by $\Pi_P = PP^T$. We apply this projection to the governing equations and insert the POD representation $\vec{w} = U\vec{a}(t) + \vec{\bar{w}}$. The later operation will introduce an error ϵ. We thus have

$$PP^T U \frac{d}{dt}\vec{a}(t) = -PP^T \Omega^{-1} \vec{R}(U\vec{a}(t) + \vec{\bar{w}}) + \epsilon.$$

Note that the average of the flow $\vec{\bar{w}}$ is time independent, hence $\frac{d}{dt}\vec{\bar{w}} = 0$. Finally, we impose orthogonality conditions onto the system by forcing the projected equation to be orthogonal to the POD subspace in the discrete L_2 space, that is $\left(u_i, PP^T \left[\frac{d}{dt}U\vec{a}(t) + \Omega^{-1}\vec{R}(U\vec{a}(t) + \vec{\bar{w}})\right]\right)_\Omega = 0$ with $i = 1, \ldots d$. Recasting this equation into matrix representation and pre-multiplying the equations with $(U^T PP^T \Omega U)^{-1}$ finally yields

$$\frac{d}{dt}\vec{a}(t) = -(U^T PP^T \Omega U)^{-1} U^T PP^T \vec{R}(U\vec{a}(t) + \vec{\bar{w}}). \tag{2}$$

This is the reduced system of the Missing Point Estimation (MPE).

3 Results

The Missing Point Estimation is tested on the NACA 0012 airfoil at a subsonic flow speed at Mach number $M_\infty = 0.3$. Snapshots are computed for the angles of attack $\alpha \in \{0^\circ, 2^\circ, 4^\circ, 6^\circ, 8^\circ, 10^\circ, 12^\circ\}$ with an unstructured grid as shown in Figure 2 consisting of $n = 2280$ points. After conducting the POD only two basis vectors ($d = 2$) from the seven snapshots ($m = 7$) are retained. Since we are looking for steady state solutions, the right hand side in (2) has to become zero. For this reason a root finding method (Powell's Dog Leg method) is employed for solving (2), instead of integrating the system.

Figure 1 shows the results of the original model (solid line) and the reduced system (dashed line) at the angle of attack $\alpha = 7^\circ$. Note that the solution at this angle is predicted, since it is not included in the snapshot set. Obviously, the two solutions are very close to each other. The points for MPE are chosen such that the condition number of the matrix $U^T PP^T \Omega U$ is minimized [4], which can be intepreted as forcing the projection of the basis to be as orthogonal as possible. For this test case only one single point (control volume), i.e. $\tilde{n} = 1$, lying on the farfield of the grid is selected, for which the residual (the fluxes over boundaries of the control volume) is evaluated. Furthermore, only nine residual evaluations are conducted, since a Newton-type of method is employed. This is typically considerably fewer than needed for a CFD computation for this mesh.

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References

[1] J. Blazek, Computational Fluid Dynamics: Principles and Applications, first edition (Elsevier, 2001).
[2] T. Gerhold, O. Friedrich, J. Evans, and M. Galle, AIAA paper **167** (1997).
[3] S. Volkwein, Model reduction using proper orthogonal decomposition, Lecture Notes, Institute of Mathematics and Scientific Computing, University of Graz. URL: http://www.uni-graz.at/imawww/volkwein/POD.pdf.
[4] P. Astrid, S. Weiland, K. Willcox, and T. Backx, IEEE Transactions on Automatic Control **53**(10), 2237–2251 (2008).

