Proper Orthogonal Decomposition for steady aerodynamic applications

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An approach combining proper orthogonal decomposition (POD) with linear regression, which is called gappy POD, is used to obtain complete flow solutions in steady aerodynamic applications from knowledge of a (suitable) POD basis and the solution at very few points. In aerodynamics the partial or gappy data can be gathered by wind tunnel experiments. The effectiveness of the Gappy POD will be demonstrated on an *industrial* testcase of the wing-body configuration MEGAFLUG.

1 Proper Orthogonal Decomposition

Suppose a set of m snapshots $\{u_1, \dots, u_m\}$ is given. In our application, these snapshot vectors $u_i \in \mathbb{R}^{d \cdot n},\ i = 1, \dots, m$ contain d flow unknowns (e.g. pressure and density) at each of the n grid points of the computational mesh and are solutions to the Navier-Stokes equations. Each $u_i(\alpha, Ma)$ represents the *steady* flow for a different combination of the two parameters angle of attack α and Mach number Ma.
The goal of POD is to find a set of orthonormal basis vectors $\{\phi_i\}_{i=1}^k$ with $k \leq m$ such that the minimization problem

$$\arg\min_{\{\phi_i\}_{i=1}^k} \sum_{j=0}^{m} \left\| u_j - \sum_{i=0}^{k} (u_j^T \cdot \phi_i)\, \phi_i \right\|_2^2 \tag{1}$$

is solved. In other words, a set of basis vectors $\{\phi_i\}_{i=1}^k$ for a k-dimensional subspace is sought such that the difference between the snapshots and their orthogonal projection onto the subspace is minimized.
The solution to this problem is given by the left singular vectors of the snapshot matrix $U = [u_1 \dots u_m] \in \mathbb{R}^{d \cdot n \times m}$ (cf. [1]). The singular value decomposition (SVD) is given by

$$U = \Phi \Sigma \Psi^T,$$

where $\Phi = [\phi_1 \dots \phi_r] \in \mathbb{R}^{d \cdot n \times r}$ and $\Psi = [\psi_1 \dots \psi_r] \in \mathbb{R}^{m \times r}$ are matrices whose columns are orthonormal and $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ with $\sigma_1 \geq \dots \geq \sigma_r$ and the rank of U being $r \leq \min(d \cdot n, m)$. Note that ϕ_i and ψ_i are called the left and right singular vectors respectively and σ_i are called singular values. We choose the first k vectors $\{\phi_i\}_{i=1}^k$ of Φ as our POD basis such that the ratio $\sum_{i=1}^k \sigma_i^2 / \sum_{i=1}^r \sigma_i^2$ is greater than a certain percentage p.

2 Gappy Proper Orthogonal Decomposition

The Gappy POD (GPOD) was first developed by Everson and Sirovich [2] in the context of reconstructing human face images. In [3] the gappy POD methodology was extended to fluid dynamic applications. The basic idea of GPOD is that the POD basis together with gappy data (which is data given at very few of the grid points) are used to reconstruct the flow vector for the entire grid.
To understand the concept of gappyness, consider a flow vector $u \in \mathbb{R}^n$ for one flow unknown, e.g. density or pressure. Each entry u_j of this flow vector stands for the value of the flow unknown at the jth grid point. Suppose the gappy data is given at the grid points $j_1, \dots, j_{\tilde{n}}$, where $\tilde{n}$ is the number of points where data is given. By defining the selection or mask matrix

$$P = \begin{pmatrix} e_{j_1} & \cdots & e_{j_{\tilde{n}}} \end{pmatrix} \in \mathbb{R}^{n \times \tilde{n}},$$

where e_j stands for the jth unit vector, only the known entries of a flow vector can be selected via $P^T \cdot u = \begin{pmatrix} u_{j_1} & \cdots & u_{j_{\tilde{n}}} \end{pmatrix}^T$.
Define $\tilde{\Phi}_k := P^T \Phi_k \in \mathbb{R}^{\tilde{n} \times k}$ and $\tilde{u} := P^T \cdot u \in \mathbb{R}^{\tilde{n}}$. The goal of GPOD is to find a solution within the POD subspace, which can be expressed as $\Phi_k \cdot a$ with the POD basis Φ_k and the coefficient vector $a(\alpha, Ma) \in \mathbb{R}^k$, such that it matches the data points as closely as possible. Hence, GPOD computes the vector a that satisfies

$$\min_{a \in \mathbb{R}^k} \|\tilde{\Phi}_k \cdot a - \tilde{u}\|. \tag{2}$$

A solution to this so-called *least squares* or *linear regression* problem is given by a linear system of the form

$$L \cdot a = f, \tag{3}$$

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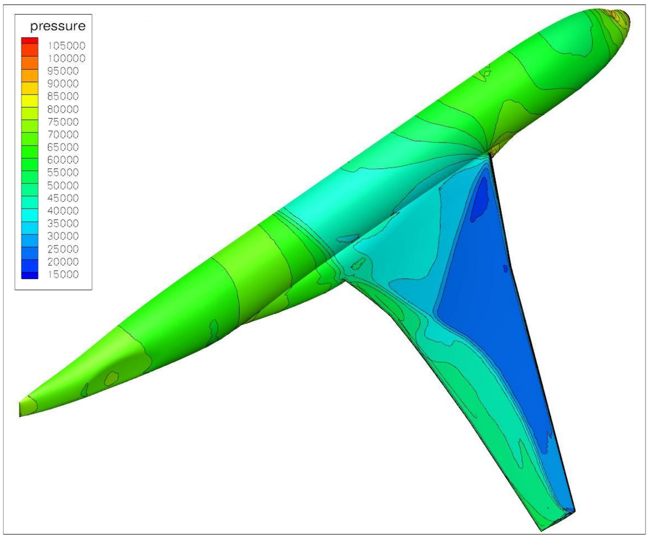

Fig. 1 The TAU reference solution at $\alpha = 7°$.

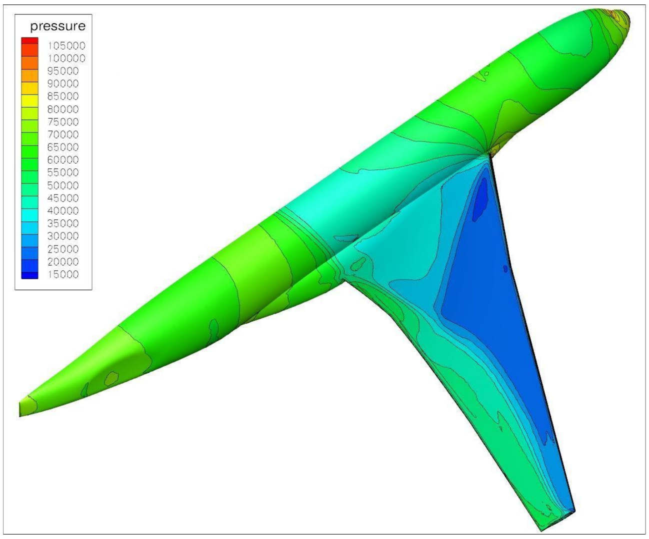

Fig. 2 The Gappy POD solution at $\alpha = 7°$.

where $L = \tilde{\Phi}_k^T \tilde{\Phi}_k \in \mathbb{R}^{k \times k}$ and $f = \tilde{\Phi}_k^T \tilde{u} \in \mathbb{R}^k$. The solution a of (3) then defines a flow vector $\Phi_k \cdot a$ for which a value is assigned for each grid point.

3 Results

As a major contribution of this article Gappy POD is tested on an *industrial* testcase. The transonic flow over a wing-body configuration called MEGAFLUG[1] is considered. The MEGAFLUG testcase consists of about six million grid points for the entire flow domain and $n = 163,204$ surface nodes. Snapshots of steady flows at a Mach number of $Ma = 0.9$ are computed for six different angles of attack $\alpha \in \{2°, 4°, 6°, 8°, 10°, 12°\}$ with the flow solver TAU [4, 5] using Navier-Stokes equations and the Spalart-Allmaras turbulence model. Gappy POD is then used to obtain the surface pressure distribution[2] at the angle of attack of $\alpha = 7°$ and the same Mach number. As gappy data the pressure values at seven randomly selected surface grid points is chosen.
Figure 1 shows surface pressure distribution computed with TAU. It is used as a reference solution. Figure 2 displays the result obtained with Gappy POD. When comparing the two solutions, only small differences can be observed. Note that it took about five hours to compute the solution with TAU in parallel on 32 domains on a state-of-the-art PC, while the Gappy POD produced results in less than a minute. However, it has to be stressed that the TAU solver has to compute the flow on the entire domain to get the surface pressure distribution, whereas Gappy POD worked only with the surface nodes.

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[1] by courtesy of Airbus
[2] Note that only the surface points of the snapshots need to be considered for the POD basis construction. However, to compute the snapshots the Navier-Stokes equations for the entire flow domain have to be solved.