

# A note on the product of two skew-Hamiltonian matrices and the product of two skew-symmetric matrices

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## Abstract

We show that the product  $C$  of two skew-Hamiltonian matrices obeys the Stenzel conditions. If at least one of the factors is nonsingular, then the Stenzel conditions amount to the requirement that each elementary divisor for a nonzero eigenvalue of  $C$  occur an even number of times. The same properties are valid for the product of two skew-pseudosymmetric matrices. We observe that the method proposed by Van Loan for calculating the eigenvalues of real Hamiltonian and skew-Hamiltonian matrices can be extended to complex skew-Hamiltonian matrices. Finally, we show that the calculation of the eigenvalues of a product of two skew-symmetric matrices can be reduced to the calculation of the eigenvalues of a similar skew-Hamiltonian matrix.

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## 1 On the product of two skew-Hamiltonian matrices

Let  $n = 2m$  be an even positive integer. We regard  $\mathbb{R}^n(\mathbb{C}^n)$  as a real (respectively, complex) linear space equipped with the scalar product

$$\langle x, y \rangle = (Jx, y), \quad (1)$$

where

$$(x, y) = x_1y_1 + \dots + x_ny_n$$

is the conventional Euclidean scalar product and

$$J = \begin{bmatrix} & 0 & I_m \\ -I_m & & 0 \end{bmatrix}. \quad (2)$$

In other words,  $\mathbb{R}^n(\mathbb{C}^n)$  is considered as a real (respectively, complex) symplectic space. Matrices in  $M_n(\mathbb{R})$  or  $M_n(\mathbb{C})$  are treated as linear operators acting on  $\mathbb{R}^n$  and  $\mathbb{C}^n$ , respectively.

Two important classes of matrices in symplectic spaces are Hamiltonian and skew-Hamiltonian matrices defined by the relations

$$\langle Ax, y \rangle = -\langle x, Ay \rangle \quad \forall x, y, \quad (3)$$

and

$$\langle Bx, y \rangle = \langle x, By \rangle \quad \forall x, y, \quad (4)$$

respectively. These relations show that skew-Hamiltonians are symmetric operators of the symplectic space, while Hamiltonians are skew-symmetric operators. It makes one think that 'skew' is erroneously used in the term for the matrix  $B$  in (4).

The spectrum  $\rho(A)$  of a Hamiltonian matrix  $A$  is not entirely arbitrary: if  $\lambda \in \rho(A)$ , then  $-\lambda \in \rho(A)$  as well. However, the spectrum of a skew-Hamiltonian matrix  $B$  has an even more remarkable property, which, for brevity, we call a double spectrum. This means that, in the Jordan form of  $B$ , each Jordan block occurs an even number of times. In particular, each eigenvalue of  $B$  has an even multiplicity. (For examples of the partial double spectrum property, meaning the evenness of the number of Jordan blocks for a particular eigenvalue, see, for instance, Theorems 6 and 7 in (HM99).)

The facts cited above are well known. For instance, the double spectrum property underlies the orthogonal method proposed by Van Loan for calculating the eigenvalues of a real Hamiltonian matrix  $A$  (see (Van84)). Van Loan observes that  $B = A^2$  is a skew-Hamiltonian matrix whose eigenvalues are the squares of the eigenvalues of  $A$ . Then, he cleverly modifies the Householder reduction of  $B$  to its Hessenberg form so that the resulting Hessenberg matrix  $H$  is block triangular

$$H = \begin{bmatrix} H_m & G_m \\ 0 & H_m^T \end{bmatrix}$$

This effectively reduces the order of the eigenvalue problem by half.

Although the double spectrum property by itself is well known, the actual reason behind this property is known much less. Note that definition (4) is equivalent to the matrix relation

$$JB = B^T J. \tag{5}$$

Since  $J^T = -J$ , relation (5) means that

$$K = JB$$

is a skew-symmetric matrix. Thus,  $B$  is a skew-Hamiltonian matrix if and only if  $B$  is a product of the form

$$B = JL, \tag{6}$$

where both factors  $J$  and  $L(= -K)$  are skew-symmetric matrices.

The fact that the product of two skew-symmetric matrices has remarkable spectral properties was repeatedly rediscovered during the last half-century (e.g., see (Dra52; And74; GL84; Djo91)). However, the story of this fact goes back much further. The authors were able to find its formulation in a paper of 1922 by H. Stenzel (Ste22). This formulation is more thorough than in many of the later papers.

**Theorem 1 (Stenzel)** *An  $n \times n$  matrix  $B$  can be represented as the product of two skew-symmetric matrices if and only if the following two properties hold:*

- a) *Each elementary divisor of a nonzero eigenvalue of  $B$  occurs an even number of times.*

b) Let  $\lambda^{m_1}, \lambda^{m_2}, \dots, \lambda^{m_s}$  be the elementary divisors for  $B$  for the zero eigenvalue ordered so that

$$m_1 \geq m_2 \geq \dots \geq m_s. \quad (7)$$

Then

$$m_{2i-1} - m_{2i} \leq 1, \quad i = 1, 2, \dots \quad (8)$$

(If  $s$  is odd, then we set  $m_{s+1} = 0$  in (8).)

Observe that, in Theorem 1,  $n$  can be an arbitrary positive integer and both factors in the skew-symmetric representation of  $B$  can be singular. For the skew-Hamiltonian case, which we discuss here,  $n$  is even and at least the matrix  $J$  in representation (6) is nonsingular. This simplifies the above conditions a) and b), reducing them to the double spectrum property.

One of the purposes of this short paper is to state an even less known fact contained in the theorem below. The authors think that this is a new fact, but one can never be sure.

**Theorem 2** *Let  $B_1$  and  $B_2$  be skew-Hamiltonian matrices of order  $n$ . Then, the matrix*

$$C = B_1 B_2 \quad (9)$$

*has both Stenzel properties, that is*

- a) *Each elementary divisor of a nonzero eigenvalue of  $C$  occurs an even number of times.*
- b) *Let  $\lambda^{m_1}, \lambda^{m_2}, \dots, \lambda^{m_s}$  be the elementary divisors for  $C$  for the zero eigenvalue arranged as in (7). Then, relations (8) hold.*

*If at least one of the matrices  $B_1$  and  $B_2$  is nonsingular, then  $C$  has the double spectrum property.*

**Proof:** According to (6),  $B_1$  and  $B_2$  can be written as

$$B_1 = J L_1 \quad \text{and} \quad B_2 = J L_2,$$

where  $L_1$  and  $L_2$  are skew-symmetric. It follows that

$$C = B_1 B_2 = (J L_1)(J L_2) = (J L_1 J) L_2$$

is the product of the skew-symmetric matrices  $J L_1 J$  and  $L_2$ . It remains to apply the Stenzel theorem.

If both  $B_1$  and  $B_2$  are nonsingular, then  $C$  is nonsingular as well and the double spectrum property reduces to property a) in the Stenzel list. Assume that, say,  $B_1$  is nonsingular, while  $B_2$  is singular. One can easily show that, for each positive integer  $k$ , the rank of  $C^k$  is an even number. This implies that each elementary divisor for a zero eigenvalue occurs an even number of times, which, combined with a), yields the double spectrum property.  $\square$

**Corollary** *Consider the generalized eigenvalue problem*

$$B_1x = \lambda B_2x, \tag{10}$$

*where both  $B_1$  and  $B_2$  are skew-Hamiltonian matrices. If at least one of the matrices  $B_1$  and  $B_2$  is nonsingular, then problem (10) has the double spectrum property.*

Indeed, if, say  $B_2$  is nonsingular, then (10) is equivalent to the ordinary eigenvalue problem

$$Cx = B_2^{-1}B_1x = \lambda x$$

with  $C$  being the product of the skew-Hamiltonian matrices  $B_2^{-1}$  and  $B_1$ .

The same result can be obtained from the normal form of a pencil of two skew-symmetric pencils as given in (Tho91) (just multiply both  $B_1$  and  $B_2$  by  $J$  to obtain skew-symmetric matrices). From the normal form one can conclude what happens in the case that both matrices are singular.

In closing, we note that the argument used in Theorem 2 essentially remains the same in the more general situation, where, instead of (1), the scalar product in  $\mathbb{R}^n$  or  $\mathbb{C}^n$  is defined by

$$\langle x, y \rangle_P = (Px, y) \tag{11}$$

with an orthogonal symmetric or skew-symmetric matrix  $P$  and  $B_1$  and  $B_2$  are skew-symmetric or, respectively, symmetric operators with respect to scalar product (11). For instance, suppose that

$$P = I_p \oplus (-I_q), \tag{12}$$

where  $p > 0, q > 0, p + q = n$ . In this case, skew-symmetric operators are the so-called skew-pseudosymmetric matrices. Thus, the product of skew-pseudosymmetric matrices  $B_1$  and  $B_2$  has the same spectral properties as matrix (9) in Theorem 2.

## 2 On the computation of the eigenvalues of a product of two skew-symmetric matrices

Let  $n = 2m$  be an even positive integer and partition a matrix  $Q \in M_n(\mathbb{C})$  into  $m \times m$  blocks

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad Q_{ij} \in M_m(\mathbb{C}).$$

Then symplecticity ( $Q^T J Q = J$ ) and unitarity ( $Q^* Q = I$ ) imply

$$JQ = QJ, \tag{13}$$

which is the same as

$$Q_{22} = \overline{Q}_{11}, \quad Q_{21} = -\overline{Q}_{12}.$$

Thus a unitary symplectic matrix must have the form

$$Q = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}, \quad A, B \in M_m(\mathbb{C}). \tag{14}$$

Two especially simple types of matrices in class (14) are

- (1)  $Q = H_m \oplus \overline{H}_m$ , where  $H_m$  is a complex Householder reflection.
- (2)  $A = \text{diag}(1, \dots, 1, c, 1, \dots, 1)$ ,  $B = \text{diag}(1, \dots, 1, s, 1, \dots, 1)$  where  $c, s \in \mathbb{C}$ ,  $|c|^2 + |s|^2 = 1$ , and  $c$  and  $s$  are at position  $k$  on the diagonal of  $A$  and  $B$ , for any  $k = 1, \dots, m$ . Such a  $Q$  is a particular type of the complex rotation.

Hence, with obvious modifications, the Van Loan method for computing the eigenvalues of a real skew-Hamiltonian matrix (Van84) can be extended to complex skew-Hamiltonian matrices (4).

Now, let  $K_1$  and  $K_2$  be skew-symmetric matrices of order  $n$ , and, say,  $K_1$  be nonsingular. From Theorem 1, we have that the matrix  $A = K_1 K_2$  has the double spectrum property.

**Theorem 3** *The calculation of the eigenvalues of  $A$  can be reduced to the calculation of the eigenvalues of a similar skew-Hamiltonian matrix.*

**Proof:** The nonsingular skew-symmetric matrix  $K_1$  is congruent to  $J$ ; that is,

$$PK_1PT = J \tag{15}$$

for an appropriate nonsingular matrix  $P$ . This matrix can be found as the product of

- the matrix that effects the Bunch reduction (Bun82) of  $K_1$  to block diagonal form

$$L_1 = \begin{bmatrix} 0 & a_1 \\ -a_1 & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & a_m \\ -a_m & 0 \end{bmatrix}, \quad a_1, \dots, a_m \neq 0; \tag{16}$$

- the diagonal matrix that makes (16) the sum of  $m$  identical blocks

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

- and the permutation matrix that transforms this sum to  $J$ .

Suppose that  $P$  has been found. Then

$$B = PAP^{-1} = PK_1P^TP^{-T}K_2P^{-1} = J(P^{-T}K_2P^{-1}).$$

Since  $P^{-T}K_2P^{-1}$  is skew-symmetric,  $B$  is a skew-Hamiltonian matrix.  $\square$

The eigenvalues of  $B$  can be computed with the modified Van Loan method mentioned above.

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