QUASI-DIAGONALIZABLE AND CONGRUENCE-NORMAL MATRICES

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Abstract. A matrix $A \in \mathbb{C}^{n\times n}$ is unitarily quasi-diagonalizable if A can be brought by a unitary similarity transformation to a block diagonal form with 1×1 and 2×2 diagonal blocks. In particular, the square roots of normal matrices are unitarily quasi-diagonalizable.

A matrix $A \in \mathbb{C}^{n\times n}$ is congruence-normal if $B = A\overline{A}$ is a normal matrix. We show that every congruence-normal matrix A can be brought by a unitary congruence transformation to a block diagonal form with 1×1 and 2×2 diagonal blocks. Our proof emphasizes and exploits the likeness between the equations $X^2 = B$ and $X\overline{X} = B$ for a normal matrix B.

1. Introduction. A matrix $A \in \mathbb{C}^{n\times n}$ is normal if

$$AA^* = A^*A.$$

A is said to be conjugate-normal if

$$AA^* = \overline{A^*A}.$$

These two matrix classes with the definitions so closely resembling each other have indeed much in common as their lists of properties, compiled in [GJSW, EI] and [FI1], respectively, demonstrate. Both play an equally important role: the former class in the theory of unitary similarity and the latter in the theory of unitary congruence.

One aspect of their likeness important for this paper is the following. Every normal matrix A can be brought by a unitary similarity transformation to diagonal form, the diagonal entries being the eigenvalues of A. Every conjugate-normal matrix A can be brought by a unitary congruence transformation to a block diagonal form with 1×1 and 2×2 blocks (see [VHV, FI2]). From this form, the so-called coneigenvalues of A can easily be read out.

Our purpose here is to give a comparative analysis of another two matrix classes. One of them are unitarily quasi-diagonalizable matrices, that is, matrices that can be brought by a unitary similarity transformation to a block diagonal form with 1×1 and 2×2 diagonal blocks. For brevity, we omit 'unitarily' in this term and denote by $\mathcal{Q}_n$ the set of quasi-diagonalizable matrices of order n. All the normal matrices are quasi-diagonalizable, and the rest of $\mathcal{Q}_n$ are nonnormal matrices whose structure is most close to that of normal matrices. From other perspectives, though, matrices in $\mathcal{Q}_n$ can be arbitrarily far from normality. For instance, oblique (as well as orthogonal) projectors and involutions are members of $\mathcal{Q}_n$. More generally, $\mathcal{Q}_n$ contains all the matrices with quadratic minimal polynomials.

The second class are congruence-normal matrices introduced in [HLV].

DEFINITION 1.1. *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be congruence-normal if $B = A\overline{A}$ is a normal matrix.*

Thus, every congruence-normal matrix is a solution to the matrix equation

$$X\overline{X} = B \tag{1.1}$$

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for some normal matrix B. The set of $n \times n$ congruence-normal matrices will be denoted by $\mathcal{C}_n$. It can easily be shown that $\mathcal{C}_n$ contains all the $n \times n$ conjugate-normal matrices. On the other hand, the matrices

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

belong to $\mathcal{C}_2$ but are not conjugate-normal.

In Section 3, we show that a matrix $A \in \mathcal{C}_n$ can be brought by a unitary congruence transformation to a block diagonal form with 1×1 and 2×2 diagonal blocks. The difference from the conjugate-normal case is that 2×2 blocks may not be conjugate-normal.

It should be noted that a canonical form of congruence-normal matrices with respect to unitary congruences was already found in [HLV]. Thus, our contribution is not the canonical form itself but, rather, the approach to its derivation, which differs from that in [HLV].

We consider equation (1.1) as a congruence analogue of the equation

$$X^2 = B, \tag{1.2}$$

where B is again a normal matrix.

Equation (1.2) describes the square roots of normal matrices or, in a different nomenclature, quadratically normal matrices (see [K]). These matrices were examined in [RR], and the following result was obtained:

Theorem 1.2. *If $A \in \mathbb{C}^{n \times n}$ is the square root of a normal matrix, then A is unitarily similar to*

$$N = \begin{bmatrix} N_1 & N_2 \\ 0 & -N_1 \end{bmatrix} \oplus N_3, \tag{1.3}$$

where N_1, N_2, and N_3 are normal matrices and N_1 and N_2 commute.

Consider the unitary matrix

$$Q = Q_1 \oplus Q_1 \oplus Q_3, \tag{1.4}$$

where Q_3 brings N_3 to its diagonal form and Q_1 brings both N_1 and N_2 to their diagonal forms. (A simultaneous reduction is possible because N_1 and N_2 commute.) Then, the unitary congruence transformation

$$N \rightarrow M = Q^T N Q$$

produces the matrix

$$M = \begin{bmatrix} M_1 & M_2 \\ 0 & -M_1 \end{bmatrix} \oplus M_3, \tag{1.5}$$

such that M_1, M_2, and M_3 are diagonal matrices. By a simple symmetric rearrangement of the rows and columns in (1.5), the first direct summand can be made a direct sum of 2×2 blocks of the form

$$\begin{bmatrix} m_1 & m_2 \\ 0 & -m_1 \end{bmatrix}$$

and a diagonal matrix. We arrive at the following conclusion:

Theorem 1.3. *Every quadratically normal matrix is unitarily quasi-diagonalizable.*

In Section 2, we give our own proof of Theorem 1.3. It is this proof on which our derivation in Section 3 is patterned. Some comments on the block diagonal form obtained are given in the concluding Section 4.

2. Quadratically normal matrices.

We precede the proof of Theorem 1.3 by two auxiliary propositions.

Recall that $A \in \mathbb{C}^{n\times n}$ is a nilpotent matrix if $A^m = 0$ for some positive integer m. The minimal of such m is called the index of nilpotence.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be a nilpotent matrix of index two. Then, A is unitarily quasi-diagonalizable.*

Proof. Let $q_1, \ldots, q_k$ be an orthonormal basis of the null space $\mathcal{N}$ of A, and let Q be a unitary $n \times n$ matrix with $q_1, \ldots, q_k$ as its first k columns. Then,

$$\triangle = Q^* A Q = \begin{bmatrix} 0 & \triangle_{12} \\ 0 & 0 \end{bmatrix}. \tag{2.1}$$

The lower right block is zero because $A^2 = 0$ and, hence,

$$Aq_i \in \mathcal{N} = \operatorname{span}\{q_1, \ldots, q_k\}, \quad i = k+1, \ldots, n.$$

Let

$$\triangle_{12} = U\Sigma V^*, \qquad \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_\ell), \qquad \ell = \min\{k, n-k\}, \tag{2.2}$$

be the singular value decomposition of $\triangle_{12}$. Define

$$P = U \oplus V. \tag{2.3}$$

Then,

$$\Gamma = P^* \triangle P = \begin{bmatrix} 0 & \Sigma \\ 0 & 0 \end{bmatrix}. \tag{2.4}$$

Finally, a symmetric rearrangement of the rows and columns in Γ yields a block diagonal matrix whose 2×2 blocks have the form

$$\begin{bmatrix} 0 & \sigma_j \\ 0 & 0 \end{bmatrix} \tag{2.5}$$

and correspond to nonzero singular values of $\triangle_{12}$. Thus, A can be transformed to the desired block diagonal form, which means that $A \in \mathcal{Q}_n$. □

Corollary 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of the form*

$$A = \alpha I_n + N,$$

where N is a nilpotent matrix of index two. Then, $A \in \mathcal{Q}_n$.

Lemma 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix with exactly two distinct eigenvalues. Then, A is unitarily quasi-diagonalizable.*

Proof. Denote by λ_1 and λ_2 the eigenvalues of A and by k the multiplicity of λ_1. Let Q be a unitary matrix that transforms A by similarity to its Schur form such that the first k diagonal positions are occupied by λ_1. Then,

$$\triangle = Q^* A Q = \begin{bmatrix} \lambda_1 I_k & \triangle_{12} \\ 0 & \lambda_2 I_{n-k} \end{bmatrix}.$$

As in Lemma 2.1, consider the singular value decomposition (2.2) of the block $\triangle_{12}$ and define the unitary matrix (2.3). Then,

$$\Gamma = P^* \triangle P = \begin{bmatrix} \lambda_1 I_k & \Sigma \\ 0 & \lambda_2 I_{n-k} \end{bmatrix}. \tag{2.6}$$

A symmetric rearrangement of rows and columns makes Γ a block diagonal matrix whose 2×2 blocks have the form

$$\begin{bmatrix} \lambda_1 & \sigma_j \\ 0 & \lambda_2 \end{bmatrix} \tag{2.7}$$

and correspond to nonzero singular values of $\triangle_{12}$. The block diagonal form obtained proves that $A \in \mathcal{Q}_n$. □

Lemmas 2.1 and 2.3 and Corollary 2.2 lead to the following conclusion:

Corollary 2.4. *Every matrix $A \in \mathbb{C}^{n \times n}$ having a quadratic minimal polynomial is unitarily quasi-diagonalizable.*

Consider a slightly more general equation than (1.2), namely,

$$f(X) = B, \tag{2.8}$$

where, as before, B is a normal matrix and f is a quadratic polynomial. Without loss of generality, f can be assumed to be monic:

$$f(z) = z^2 + az + b.$$

Theorem 2.5. *Every solution to matrix equation (2.8) is a unitarily quasi-diagonalizable matrix.*

Proof. Since B is a polynomial in the unknown X, both matrices should commute

$$BX = XB. \tag{2.9}$$

Let $\lambda_1, \ldots, \lambda_m$ be all the distinct eigenvalues of B and $k_1, \ldots, k_m$ $(k_1 + \ldots + k_m = n)$ be their multiplicities. Find a unitary matrix Q such that

$$C = Q^* B Q = \lambda_1 I_{k_1} \oplus \lambda_2 I_{k_2} \oplus \cdots \oplus \lambda_m I_{k_m}. \tag{2.10}$$

Define

$$Y = Q^* X Q.$$

Then,

$$CY = YC \tag{2.11}$$

and

$$f(Y) = C. \tag{2.12}$$

Relations (2.10) and (2.11) imply that

$$Y = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m. \tag{2.13}$$

The diagonal block Y_i $(i = 1, \ldots, m)$ has the size $k_i \times k_i$ and satisfies the equation

$$f(Y_i) = \lambda_i I_{k_i}.$$

Now, we omit the subscripts and consider two possible situations.

Case 1: $a^2 = 4(b - \lambda)$. In this case, Y solves the equation

$$g(Y) = 0, \tag{2.14}$$

where the polynomial

$$g(z) = f(z) - \lambda \tag{2.15}$$

has a zero discriminant and, hence, a unique root α:

$$g(z) = (z - \alpha)^2.$$

It follows that Y is a matrix of the type specified in Corollary 2.2. Consequently, Y is quasi-diagonalizable.

Case 2: $a^2 \neq 4(b - \lambda)$. Now, polynomials (2.15) has two distinct roots. By (2.14), $g(z)$ is the minimal polynomial of Y. According to Corollary 2.4, Y is quasi-diagonalizable.

We conclude that all the blocks Y_i $(i = 1, \ldots, m)$ in (2.13) are quasi-diagonalizable. It follows that $Y \in \mathcal{Q}_n$ and, hence, $X \in \mathcal{Q}_n$. $\square$

It remains to observe that Theorem 1.3 corresponds to a particular choice of the polynomial f, namely,

$$f(z) = z^2.$$

3. Congruence-normal matrices. In this section, we prove the following assertion:

THEOREM 3.1. *Every congruence-normal matrix can be brought by a unitary congruence transformation to a block diagonal form with* 1×1 *and* 2×2 *diagonal blocks.*

As already said in the introduction, a canonical form of congruence-normal matrices with respect to unitary congruences was first found in [HLV]. Our proof differs from the derivation in [HLV] in that it tries to emphasize and exploit the likeness between equations (1.1) and (1.2). However, we do borrow several clever ideas used in [HLV].

We begin with several preparatory propositions.

LEMMA 3.2. *Let* $A, B \in \mathbb{C}^{n \times n}$*, and let* B *be normal. Then, the relation*

$$A\overline{B} = BA \tag{3.1}$$

implies that

$$A\overline{B^*} = B^*A,$$

that is,

$$AB^T = B^*A. \tag{3.2}$$

Proof. It is well known that, for a normal matrix B, the Hermitian adjoint B^* can be represented as a polynomial in B:

$$B^* = f(B). \tag{3.3}$$

It is less known that a polynomial f with real coefficients can be chosen for relation (3.3) (see [L]). Let our f be such a polynomial. Then,

$$B^*A = f(B)A = Af(\overline{B}) = A\overline{f(B)} = A\overline{B^*} = AB^T.$$

□

Remark 3.3. Our Lemma 3.2 is Lemma 1 in [HLV], although the proofs are different.

Corollary 3.4. *If $A \in \mathcal{C}_n$, then*

$$AA^*A^T = A^TA^*A. \tag{3.4}$$

Proof. Recall that $A \in \mathcal{C}_n$ means that $A\overline{A}$ is a normal matrix. Now, equality (3.1) is trivially fulfilled for $B = A\overline{A}$. Since $B^T = A^*A^T$, relation (3.2) yields (3.4). □

Remark 3.5. As noted in [HLV], equality (3.4) actually is characteristic of congruence-normal matrices. Indeed, assume that (3.4) holds and set $B = A\overline{A}$. Then,

$$\begin{aligned} BB^* &= A\overline{A}A^TA^* = A\overline{(AA^*A^T)} = A\overline{(A^TA^*A)} = AA^*A^T\overline{A} \\ &= (AA^*A^T)\overline{A} = (A^TA^*A)\overline{A} = (A^TA^*)(A\overline{A}) = B^*B. \end{aligned}$$

Thus, B is normal, and $A \in \mathcal{C}_n$.

For a given A, define the matrices

$$C = A^T\overline{A} \tag{3.5}$$

and

$$D = AA^*. \tag{3.6}$$

Lemma 3.6. *If $A \in \mathcal{C}_n$, then*

$$CD = DC = BB^*. \tag{3.7}$$

Proof. The chain of equalities in Remark 3.5 contains the relation

$$BB^* = AA^*A^T\overline{A}. \tag{3.8}$$

The right-hand side is DC. Taking the adjoint of (3.8) we obtain

$$BB^* = A^T\overline{A}AA^*,$$

which is CD. □

Remark 3.7. For a conjugate-normal matrix A (see its definition in Section 1), C and D are the same matrix. According to Lemma 3.6, for an arbitrary $A \in \mathcal{C}_n$, the matrices C and D can be different but must commute.

We will also use the relation

$$A\overline{C} = DA, \tag{3.9}$$

which is immediate from (3.5) and (3.6).

Before we embark on the proof of Theorem 3.1, let us recall the special features of the spectra of matrices of the form $B = A\overline{A}$ (see [HJ1, p. 252–253]):

1. The spectrum of B is symmetric with respect to the real axis. Moreover, the eigenvalues λ and $\overline{\lambda}$ are of the same multiplicity.
2. The negative eigenvalues of B (if any) are necessarily of even multiplicity.

Now, we will present our proof of Theorem 3.1.

Proof. Let $A \in \mathcal{C}_n$ and $B = A\overline{A}$. The identity

$$A\overline{A}A = A\overline{A}A$$

can be rewritten as

$$BA = A\overline{B}. \tag{3.10}$$

This signifies a kind of commutation between A and B, which is preserved by unitary congruence transformations of A. We choose such a transformation

$$A \to Y = Q^T AQ \tag{3.11}$$

so as to make

$$\Lambda = Q^T A\overline{AQ} = (Q^T AQ)(Q^*\overline{AQ}) = Y\overline{Y} \tag{3.12}$$

a diagonal matrix with the following properties:

(a) if λ is an eigenvalue of B with the multiplicity $m > 1$, then the m copies of λ occupy consecutive positions on the main diagonal of Λ;

(b) if λ is a complex eigenvalue of B, then, on the main diagonal of Λ, the copies of $\overline{\lambda}$ (or λ) follow immediately after the copies of λ (or $\overline{\lambda}$).

Relation (3.10) transforms into

$$\Lambda Y = Y\overline{\Lambda}, \tag{3.13}$$

which implies that Y is a block diagonal matrix:

$$Y = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_k. \tag{3.14}$$

Each block Y_i ($1 \leq i \leq k$) corresponds either to a real eigenvalue of B or to a pair of complex conjugate eigenvalues. To prove Theorem 3.1, we must show that each Y_i can be brought by a unitary congruence transformation to a block diagonal form with 1×1 and 2×2 diagonal blocks.

We distinguish between four possible situations:

1. Y_i corresponds to $\lambda = 0$.
2. Y_i corresponds to a positive real eigenvalue of B.
3. Y_i corresponds to a negative real eigenvalue of B.
4. Y_i corresponds to a pair of complex eigenvalues $\lambda, \overline{\lambda}$.

We give separate analyses of these four cases. To simplify the notation, we drop the subscript i. The order of Y is denoted by t or (if even) by $2t$.

3.1. Case 1: $\lambda = 0$. In this case, the matrix Λ corresponding to Y is zero. In view of (3.12), we have

$$Y\overline{Y} = 0. \tag{3.15}$$

To bring Y to a block diagonal form, we use a construction similar to that in Lemma 2.1. Let $p_1, \ldots, p_k$ be an orthonormal basis of the null space $\mathcal{N}$ of Y (which means that $\overline{p_1}, \ldots, \overline{p_k}$ are a basis of the null space of $\overline{Y}$). Let P be a unitary $t \times t$ matrix with $p_1, \ldots, p_k$ as its first k columns. Then,

$$\triangle = P^T Y P = \left[\begin{array}{cc} 0 & \triangle_{12} \\ 0 & 0 \end{array} \right].$$

The lower right block is zero because of (3.15). Indeed, (3.15) implies that

$$Y p_i \in \overline{\mathcal{N}} = \operatorname{span}\{\overline{p_1}, \ldots, \overline{p_k}\}, \quad i = k+1, \ldots, t.$$

Let

$$\triangle_{12} = U \Sigma V^*, \quad \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_\ell), \ell = \min\{k, t-k\},$$

be the singular value decomposition of $\triangle_{12}$. Define

$$R = \overline{U} \oplus V.$$

Then,

$$\Gamma = R^T \triangle R = \left[\begin{array}{cc} 0 & \Sigma \\ 0 & 0 \end{array} \right].$$

Now, the desired block diagonal form is obtained by a symmetric rearrangement of the rows and columns in Γ. The 2×2 blocks have form (2.5) and correspond to nonzero singular values of $\triangle_{12}$.

3.2. Case 2: $\lambda > 0$. In this case, the basic equation is

$$Y\overline{Y} = \lambda I.$$

We can get rid of λ by introducing the new matrix

$$\widetilde{Y} = \frac{1}{\sqrt{\lambda}} Y.$$

To not complicate the notation, we drop the tilde and hereafter work with the equation

$$Y\overline{Y} = I. \tag{3.16}$$

Equation (3.16) tells us something about the singular values of Y. Indeed, $\overline{Y}$ has the same singular values as Y, while the singular values of Y^{-1} are the reciprocals of the singular values of Y. Since $\overline{Y} = Y^{-1}$, we conclude that, if $\gamma \neq 1$ is a singular value of Y, then γ^{-1} is a singular value too and both have the same multiplicity.

Matrices (3.5) and (3.6) transform into

$$S = Y^T \overline{Y} = \overline{Y^* Y}$$

and

$$T = Y Y^*,$$

while relation (3.7) transforms into

$$ST = I. \tag{3.17}$$

Now, we perform the unitary congruence transformation

$$Y \to Z = P^T Y P$$

choosing P so as to make

$$\Gamma = P^T S \overline{P} = P^T Y^T \overline{Y P} = (P^T Y^T P)(P^* \overline{Y P}) = Z^T \overline{Z}$$

a diagonal matrix. Moreover, we assume that, if γ is a multiple singular value of Y, then all copies of γ^2 occupy consecutive positions on the main diagonal of Γ and, if $\gamma > 1$, the copies of γ^{-2} follow immediately after the copies of γ^2.

In view of (3.17), relation (3.9) takes the form

$$Z\Gamma = \Gamma^{-1} Z.$$

In other words,

$$\gamma_j^2 z_{ij} = \frac{1}{\gamma_i^2} z_{ij} \qquad \forall i, j. \tag{3.18}$$

Equalities (3.18) lead us to the following conclusions:

1. Z is a block diagonal matrix:

$$Z = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_\ell. \tag{3.19}$$

2. If $\gamma = 1$ is a singular value of Y, then the direct sum (3.19) contains a block corresponding to this singular value. The order of this block is equal to the multiplicity of $\gamma = 1$.
3. Each of the other blocks in (3.19) corresponds to a pair γ_j, γ_j^{-1} of reciprocal singular values of Y. If s is the multiplicity of γ_j, then the associated block Z_j has the size $2s \times 2s$. Being partitioned into $s \times s$ subblocks, Z_j has the form

$$Z_j = \begin{bmatrix} 0 & G_j \\ H_j & 0 \end{bmatrix}. \tag{3.20}$$

To complete the analysis of case 2, it remains to show that each block Z_j can be brought to the desired form. We do this separately for $\gamma = 1$ and for a pair γ, γ^{-1}, where $\gamma > 1$. The index j is omitted (exactly as before the index i was dropped).

3.2.1. $\gamma = 1$**.** The corresponding Z has a unique singular value 1 and, hence, is a unitary matrix. On the other hand, relation (3.16) implies that

$$Z\overline{Z} = I. \tag{3.21}$$

Since $Z^{-1} = Z^*$ and $Z^{-1} = \overline{Z}$, we conclude that Z is symmetric. By Takagi's theorem (see [HJ1, Corollary 4.4.4]), the symmetric matrix Z can be brought by a unitary congruence transformation to diagonal form whose diagonal entries are the singular values of Z. In the case under discussion, this diagonal form is the identity matrix.

3.2.2. $\gamma > 1$**.** Our Z has form (3.20) (with the index j omitted) and again obeys condition (3.21). It follows that

$$G\overline{H} = I. \tag{3.22}$$

The relation

$$S = \overline{Y^* Y}$$

implies that

$$Z^* Z = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & \gamma^{-2} I \end{bmatrix},$$

that is,

$$H^* H = \gamma^2 I$$

and

$$G^* G = \frac{1}{\gamma^2} I. \tag{3.23}$$

Thus, H has a unique singular value γ, while G has a unique singular value γ^{-1}. A comparison of (3.22) and (3.23) shows that

$$G^{-1} = \overline{H} = \gamma^2 G^*,$$

which yields

$$H = \gamma^2 G^T. \tag{3.24}$$

Let

$$G = \gamma^{-1} U V^*$$

be the singular value decomposition of G. By (3.24) we have

$$H = \gamma \overline{V} U^T.$$

Define

$$R = \overline{U} \oplus V$$

and apply to Z the congruence transformation

$$Z \rightarrow W = R^T Z R.$$

This produces the matrix

$$W = \begin{bmatrix} 0 & \gamma^{-1} I \\ \gamma I & 0 \end{bmatrix}. \tag{3.25}$$

Finally, a symmetric rearrangement of the rows and columns in (3.25) makes W the direct sum of s 2×2 blocks of the form

$$\begin{bmatrix} 0 & \gamma^{-1} \\ \gamma & 0 \end{bmatrix}. \tag{3.26}$$

The analysis performed in this subsection can be summarized as the following proposition:

THEOREM 3.8. *Every solution Y to matrix equation (3.16) can be brought by a unitary congruence transformation to the direct sum of an identity matrix (perhaps, void) and 2×2 blocks of form (3.26). The identity matrix is present if $\gamma = 1$ is a singular value of Y. Its order is equal to the multiplicity of this singular value. Each block of form (3.26) corresponds to a pair γ, γ^{-1} of reciprocal singular values of Y, where $\gamma \neq 1$.*

REMARK 3.9. A matrix $E \in \mathbb{C}^{n \times n}$ is said to be coninvolutory if $E\overline{E} = I$ (see [HJ2, p. 477]. Our Theorem 3.8 describes a canonical form of coninvolutory matrices with respect to unitary congruences. Up to minor details, this is Theorem 1.5 in [HM].

3.3. Case 3: $\lambda < 0$. Recall that a negative eigenvalue of B is necessarily of even multiplicity. Up to this distinction, the analysis of the case $\lambda < 0$ is very similar to the one given in the preceding subsection. Therefore, we skip the details, focusing mainly on the differences from Section 3.2.

By introducing a new matrix, we can reduce the basic equation of the present case to the form

$$Y\overline{Y} = -I. \tag{3.27}$$

Similarly to (3.16), this equation shows that, if $\gamma \neq 1$ is a singular value of Y, then γ^{-1} is a singular value too and both have the same multiplicity. It follows that, if $\gamma = 1$ is a singular value, then it has an even multiplicity.

The matrices S and T are defined as in Section 3.2, and relation (3.17) is fulfilled. Nothing changes in the description of the unitary congruence transformation

$$Y \to Z = P^T Y P,$$

which again leads to equalities (3.18) implying the block diagonal form (3.19) for Z.

Suppose that $\gamma = 1$ is a singular value of Y. Then, one of the blocks on the right-hand side of (3.19) corresponds to this singular value. Dropping the index, assume that Z is this block. Thus, Z has a unique singular value 1 and, hence, is a unitary matrix. On the other hand, relation (3.27) implies that

$$Z\overline{Z} = -I. \tag{3.28}$$

Since $Z^{-1} = Z^*$ and $Z^{-1} = -\overline{Z}$, we conclude that Z is skew-symmetric. Every nonsingular skew-symmetric matrix can be brought by a unitary congruence transformation to the direct sum of 2×2 blocks of the form

$$\begin{bmatrix} 0 & \varsigma_j \\ -\varsigma_j & 0 \end{bmatrix}, \qquad \varsigma_j \neq 0$$

(see [HJ1, Section 4.4, Problem 26]). The scalars ς_j can be chosen positive and, in this case, are the singular values of the original matrix. For our matrix Z, the above reduction yields the direct sum of the s copies of the block

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.29}$$

Each of the other blocks Z_j in (3.19) corresponds to a pair γ_j, γ_j^{-1} of reciprocal singular values of Y and must have form (3.20). Again, we drop the index j and deduce from (3.28) the relation

$$G\overline{H} = -I, \tag{3.30}$$

which differs from (3.22). On the other hand, equality (3.23) remains true. Together, (3.23) and (3.30) imply that

$$H = -\gamma^2 G^T \tag{3.31}$$

(cf. (3.24)). Defining again the unitary matrix R in terms of the singular value decomposition of G and then performing the transformation

$$Z \to W = R^T Z R,$$

we obtain the matrix

$$W = \begin{bmatrix} 0 & \gamma^{-1} I \\ -\gamma I & 0 \end{bmatrix},$$

which, after a symmetric rearrangement of its rows and columns, becomes the direct sum of the s copies of the block

$$\begin{bmatrix} 0 & \gamma^{-1} \\ -\gamma & 0 \end{bmatrix}. \tag{3.32}$$

The discussion above can be summarized as the following proposition:

THEOREM 3.10. *Every solution Y to matrix equation (3.27) can be brought by a unitary congruence transformation to the direct sum of 2×2 blocks of type (3.32). The scalars γ_j are singular values of Y. If $\gamma \neq 1$ is a singular value of multiplicity s, then there are exactly s blocks in the direct sum associated with this γ. The remaining blocks (if any) correspond to $\gamma = 1$ and have form (3.29).*

REMARK 3.11. A matrix $E \in \mathbb{C}^{n \times n}$ is said to be skew-coninvolutory if $E\overline{E} = -I$. Our Theorem 3.10 describes a canonical form of skew-coninvolutory matrices with respect to unitary congruences. Up to minor details, this is Theorem 12 in [AMP].

3.4. Case 4: complex eigenpair $\lambda, \overline{\lambda}$. Recall that λ and $\overline{\lambda}$ are the eigenvalues of the original matrix $B = A\overline{A}$. Assume that both have the multiplicity t. Then, Y is a matrix of order $2t$ satisfying relation (3.13) with

$$\Lambda = \lambda I_t \oplus \overline{\lambda} I_t.$$

We partition Y into $t \times t$ blocks and deduce from (3.13) that Y has the form

$$Y = \begin{bmatrix} 0 & K \\ L & 0 \end{bmatrix}. \tag{3.33}$$

In view of (3.12), we have

$$K\overline{L} = \lambda I_t.$$

It follows that

$$L = \overline{\lambda K^{-1}}. \tag{3.34}$$

Let

$$K = U\Sigma V^*, \qquad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_t),$$

be the singular value decomposition of K. Then, (3.34) implies that

$$L = \overline{\lambda}\overline{V}\Sigma^{-1}U^T.$$

Define

$$R = \overline{U} \oplus V$$

and perform the congruence transformation

$$Y \to Z = R^T Y R,$$

which yields the matrix

$$Z = \left[\begin{array}{cc} 0 & \Sigma \\ \overline{\lambda}\Sigma^{-1} & 0 \end{array} \right].$$

By a symmetric rearrangement of its rows and columns, Z can be made the direct sum of the 2×2 blocks

$$\left[\begin{array}{cc} 0 & \sigma_j \\ \overline{\lambda}/\sigma_j & 0 \end{array} \right], \qquad j = 1, 2, \ldots, t. \tag{3.35}$$

This completes the analysis of case 4 and the entire proof of Theorem 3.1. □

4. Concluding Remarks. Let us look more closely at the constituents of the block diagonal matrix obtained in the proof of Theorem 3.1. The scalars σ_j in Sections 3.1 and 3.4 and the scalars γ_k in Sections 3.2-3.3 are singular values of the corresponding blocks Y_i and, ultimately, singular values of the original matrix A. The scalar λ in (3.35) is a complex eigenvalue of the matrix $B = A\overline{A}$. Thus, all the entries in the final block diagonal matrix are uniquely determined by A, and this matrix is actually a canonical form of A with respect to unitary congruences.

Up to insignificant distinctions, this is the same canonical form as in Theorem 3 of [HLV]. However, neither the term 'singular value' nor 'singular value decomposition' can be found in [HLV].

As was noted in the introduction, every conjugate-normal $n \times n$ matrix A belongs to $\mathcal{C}_n$; hence, Theorem 3.1 applies to such a matrix. However, not all 2×2 blocks listed in Section 3 may be present in the canonical form of a conjugate-normal matrix A. As is evident from their definitions, conjugate-normal matrices share with conventional normal matrices the following property: for each i ($1 \le i \le n$), the 2-norm of row i is equal to the 2-norm of column i. Being applied to the canonical form, this property excludes blocks of types (2.5) and (3.26), as well as blocks of type (3.32) with $\gamma \neq 1$. Moreover, for blocks of type (3.35), we must have

$$\sigma_j = |\lambda|/\sigma_j,$$

that is,

$$\sigma_j = |\lambda|^{\frac{1}{2}}.$$

When these restrictions are taken into account, the canonical form of conjugate-normal matrices described in [VHV] is obtained.

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