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Some Basic Notation.

(a) Numbers.

• $\mathbb{N}$ natural numbers (also $\mathbb{N}_0$):
odd/even numbers
prime numbers; prime factor; decomposition in prime factors
zero, one, two, three, ...
0, 1, 2, 3, $\ldots$, 9: digits, numerals, ciphers, figures
zero = null = naught = "oh" = love
q divides p; divisor;
$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$ "n-factorial"
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ "n choose k"
n^2, n^3, n^k: "n-square",

• $\mathbb{Z}$ integer numbers:
positive and negative numbers; absolute value

• $\mathbb{Q}$ rational numbers:
fraction p/q with $p \in \mathbb{Z}$, $q \in \mathbb{N}$. "p over q"

• $\mathbb{R}$ real numbers:
decimal places, ...
square root of x for $x \geq 0$.
binary representation of numbers

• $\mathbb{C}$ complex numbers:
i imaginary unit
$z = x + \mathrm{i}y$, real and imaginary part,
conjugate of z: $\overline{z} = x - \mathrm{i}y$, "$z$-bar"
absolute value of z, $|z| = \sqrt{z\overline{z}}$, "modulus of z", "mod z"
polar representation $z = r\mathrm{e}^{\mathrm{i}\vartheta}$
$\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are *fields.*

$x + y$, x^2, x^3, x^4 "x to the 4-th," or "x to the power 4."

Prime Number Theorem. [W. Rudin, Functional Analysis]
For any positive number x, let $\pi(x)$ denote the number of primes p that satisfy $p \leq x$. Then

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

Loosely speaking, this means that $\pi(x)$ behaves asymptotically like $\frac{x}{\log x}$. In this form, the Prime Number Theorem was first proved (independently) by Hadamard and de la Vallée-Poussin in 1896.

It is considerably simpler to show that there are infinitely many primes. This was already known to the Greeks.

(b) Sets.

Let X be a set.
$x \in X$: "x element of X," or "x in X;"
$x \notin X$: "x not an element of X."
$\forall x \in X$: "for all $x \in X$ ...", "for any $x \in X$ ..."
$\exists x \in X$: "there exists $x \in X$ such that ..."'
$A \subset X$: "A subset of X" or "A is contained in X."
For $A, B \subset X$ we define

$$A \setminus B := \{x \in A; x \notin B\}.$$

$\{\ldots\}$: curly brackets, braces
The set of *ordered pairs* (x, y) with $x \in X$, $y \in Y$ is called the *Cartesian product* and denoted as

$$X \times Y := \{(x, y); x \in X, y \in Y\};$$

here we may define the pair (x, y) as $\{x, \{y\}\}$ or as $\{x, \{x, y\}\}$, etc.

• Now let X be a topological space (e.g., a metric space). We then have

$$M \subset X \text{ closed} \quad \Longleftrightarrow \quad X \setminus M \text{ open}.$$

For an arbitrary subset $M \subset X$ we define $\overline{M}$, the closure of M, as the smallest closed subset of X containing M, i.e.,

$$\overline{M} = \bigcap_{\substack{X \supset N \supset M \\ X \setminus N \text{ open}}} N.$$

• (c) Functions.

Let X, Y be sets. We write

$$f : X \to Y,$$

or

$$x \mapsto f(x)$$

to denote a function (or a mapping) from X to Y. (The function f associates to each $x \in X$ precisely one value $y = f(x) \in Y$.) For $A \subset X$ we write

$$f(A) := \{f(x); x \in A\} \subset Y,$$

and for $B \subset Y$,

$$f^{-1}(B) := \{x \in X; f(x) \in B\} \subset X;$$

$f(X)$ is called the *range* or the *image* of f,

$$\mathrm{Ran}(f) = \{f(x); x \in X\} = f(X) \subset Y.$$

f is called *surjective* or *onto* $\quad :\Longleftrightarrow \quad f(X) = Y$;
f is called *injective* or *one-to-one* $\quad :\Longleftrightarrow \quad (f(x) = f(y) \Leftrightarrow x = y)$;
f is called *bijective* $\quad :\Longleftrightarrow \quad f$ injective and surjective.
The *restriction* of $f : X \to Y$ to $A \subset X$ is denoted as $f \restriction_A$.
Let $A \subset X$. Then χ_A denotes the *characteristic function of* A,

$$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Let X, Y be topological spaces. A mapping $f : X \to Y$ is called *continuous*, if it enjoys the following property:

$$V \subset Y \text{ offen} \Rightarrow f^{-1}(V) \subset X \text{ offen}.$$

• **(d) Relations.**

Definition. Let X be a set. A *relation of* X is a subset $\mathcal{R}$ of $X \times X$.
For $(x, y) \in \mathcal{R}$ we say that x is in $\mathcal{R}$-relation to y and write $x\mathcal{R}y$.

Definition. A relation $\mathcal{R} \subset X \times X$ is called an *equivalence relation*, provided it has the following properties:
(i) $\mathcal{R}$ is *reflexive* (i.e., $\forall x \in X : x\mathcal{R}x$);
(ii) $\mathcal{R}$ is *symmetric* (i.e., $x\mathcal{R}y$ implies $y\mathcal{R}x$);
(iii) $\mathcal{R}$ is *transitive* (i.e., $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$).

For $x \in X$ and $\mathcal{R}$ an equivalence relation the set of all $y \in X$ satisfying $y\mathcal{R}x$ is called the *equivalence class of* x,

$$[x] := \{y \in X; y\mathcal{R}x\}.$$

$[\ldots]$ square brackets

Theorem. *Let X be a set and let $\mathcal{R}$ be an equivalence relation on X. We then have: Any $x \in X$ belongs to precisely one equivalence class; in other words: $\mathcal{R}$ leads to a decomposition of X into pairwise disjoint equivalence classes.*

Instead of $\mathcal{R}$ one frequently uses the notation $\sim$.
The set of equivalence classes is written $X/\mathcal{R}$.

0.4. Examples.

(1) Let $X := \mathbb{Z}$ and define

$$x\mathcal{R}y \;:\Longleftrightarrow\; x - y \in 3\mathbb{Z}.$$

Then X is decomposed into the three equivalence classes

$$\begin{aligned}
[0] &= \{\ldots, -6, -3, 0, 3, 6, \ldots\}, \\
[1] &= \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \\
[2] &= \{\ldots, -4, -1, 2, 5, 8, \ldots\}
\end{aligned}$$

(2) The real projective line.
Let $X := \mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$. We define a relation $\mathcal{R}$ on X by

$$x\mathcal{R}y \;:\Longleftrightarrow\; \exists \alpha \in \mathbb{R} : x = \alpha y,$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The corresponding equivalence classes can be visualized as straight lines through the origin $(0,0)$ vorstellen (omitting the point $(0,0)$).
(3) Cauchy sequences of rational numbers.
A sequence $(\alpha_n) \subset \mathbb{Q}$ is called a *Cauchy sequence* if the following holds:

$$\forall \varepsilon > 0,\; \exists N_\varepsilon \in \mathbb{N},\, \forall n, m \geq N_\varepsilon :\; |\alpha_n - \alpha_m| < \varepsilon.$$

Let us denote the set of all Cauchy sequences in $\mathbb{Q}$ by $\mathcal{C}(\mathbb{Q})$. We introduce a relation $\mathcal{R}$ on $\mathcal{C}(\mathbb{Q})$ in the following way:

$$a\mathcal{R}b \;:\Longleftrightarrow\; \lim_{n\to\infty} (\alpha_n - \beta_n) = 0,$$

where $a = (\alpha_n)$ and $b = (\beta_n)$.
Lemma. $\mathcal{R}$ is an equivalence relation. (easy to prove)
Remark. The set of equivalence classes in $\mathcal{C}(\mathbb{Q})$ is nothing else but the set of real numbers (including the distance metric of $\mathbb{R}$),

$$\mathbb{R} := \mathcal{C}(\mathbb{Q})/\mathcal{R}.$$

Elementary Geometric Notation.

point; line or straight line; line segment; circle; ellipse; hyperbola;
intersection of ...;
curve (but: line integral); arc, path; arcwise connected set; path integral (Brownian path)
sphere; ball;
ellipsoid; paraboloid; hyperboloid;

• Triangles: angles, sides, vertices;
right angle;
area; circumference; hypotenuse;
equilateral triangle;
isosceles triangle;
right triangle; (?)
perpendicular to;
parallel lines;
distance;
Pythagorean Theorem;

• Euclid's Axioms;
parallel axiom: two lines intersect unless they are parallel.

Some basic notions in numerical analysis.

absolute error
accelerated steepest descent
adaptive scheme
algorithm
approximation
band matrix
base
basis
B spline
Cauchy-Schwarz inequality
convex function
definite integral
dice problem
differential equation
discretization method
divided differences

dot product
double precision
elimination
error
exponential series
extrapolation
fixed point
fractional part
fundamental theorem of calculus
Gaussian quadrature formulas
geometric series
gradient vector
Hessian matrix
implicit method
indefinite integrals
integer part
interpolation
inverse
interative methods
knots
least squares
level sets
linearization
locating roots of an equation
loss of precision
machine number
mantissa
minimization of functions
Monte Carlo methods
multiplicity of zeros
natural cubic spline
Newton algorithm for polynomial interpolation
nonlinear problems
numerical differentiation/integration
odd function (even function)
ordering
order of convergence
orthogonal

partial differential equation
partition
periodic function
pivoting
point of attraction
positive definite matrix
quadrature
random number generator
recurrence relation
relative error
remainder
roots of an equation
saddlepoint
secant
shooting method
simplex method
simple zero
single precision
smoothing of data
to solve, solution
sparse system
splines
spurious zeros
stability
standard deviation
stationary point
steepest descent
stiff differential equation
support of a function
tables
Taylor series
three-term recurrence relation
trapezoid rule
transpose (of a matrix)
truncation error
unconstrained minimization
uniform spacing
upper triangular matrix

variable
variance
vertex
vibrating string
weight function

Notice, note that
recall that
we emphasize that
it is important/crucial to note that
we make the following observation
our proof uses the following basic idea
we extend the above result to a more general context
we are mostly concerned with the following problem
More precisely
roughly speaking
Let $\varepsilon > 0$
Suppose we are given a continuous function
Assume for a contradiction
We then have ...
Then the following holds:
Then the following properties are equivalent
shortly: in Kürze, bald (auf die Zeit bezogen)
in short, briefly, in brief: kurz, zusammengefaßt
combining eqns. $(*)$ and $(**)$ we obtain
now it follows from eqn. $(*)$ that
we conclude from eqn. $(*)$ and Lemma xy that
Since A enjoys the property xy, we may assume that
we may now simplify the integral on the RHS
Let us consider the following ...
we may deduce xxx from yyy
we may infer from xxx that yyy is true
this leads to a contradiction since
this concludes the proof
this gives the desired result
let H_0 denote the associated ...
the relevant equations are given by
we study the properties of
so that the result of xxx can be applied in the present situation

• In a proof, there is quite a number of possibilities to express that something is the logical consequence of the preceding line or equation:

thus; hence; therefore;

so that; from which we deduce; whence;

this implies that;

it now follows that;

we therefore see that;

from which we obtain ...

we conclude that

we finally obtain that

which is proved by using (1.10) and (1.11)

Since ... we see that ...

If ... we deduce that ...

• In a proof, we may have to define new objects:

If we set; we put; we let; let

• we have; we observe; the following holds; we also have;

in addition, we have ...

the statement follows from the fact that ...

$\wedge$ wedge

$\oplus$ direct sum

$\otimes$ tensor product

$\int$ integral

$\bar{g}$ "g-bar"

$\hat{g}$ "g-hat"

$\tilde{g}$ "g-tilda", "g-twiddle" (Brit.)

$\vec{a}$ vector a

$\#$ sharp

f^* "f-star"

$f * g$ "convol'ution product"

Δ Laplacian

∇ gradient

$\nabla\cdot$ divergence

$\nabla\times$ curl (Rotation)

$\circ$ circle

$\dagger$ dagger

$\exists$ there exists

$\forall$ for all

$\hbar$ h-bar, Plancksches Wirkungsquantum

$\aleph, \aleph_0$ Aleph, Aleph-naught

$\rightarrow$ arrow

$\iff$ if and only if

∞ inifinity

$\wedge$, $\vee$, $\neg$ logical and, logical or, negation

$\mapsto$ maps to

$\pm$ plus-minus

$\notin$ not in

∂ partial

$\perp$ perpendicular

f' f-prime, f'' f-double-prime

$\sqrt[5]{1+x^2}$ 5-th root of ...

$\sqrt{2}$ square root of 2

$\#$

$\sim$ similar

sin, sinh sine, hyperbolic sine

$*$ star, asterisque

$\triangle$ triangle, symmetric difference

$\uparrow, \downarrow$ up-arrow, down-arrow

• Let $(x_k)_{k\in\mathbb{N}}$ be a sequence and let A be a proposition.

We say "the statement $A(x_k)$ is *frequently* true iff for any $K \in \mathbb{N}$ there is some $k > K$ sucht that $A(x_k)$ is true.

We say "the statement $A(x_k)$ is *eventually* true iff there exists $K \in \mathbb{N}$ such that $A(x_k)$ is true for all $k \geq K$.

Achtung: "eventually" heißt auf Deutsch nicht "eventuell" oder "gelegentlich", sondern **schließlich.**

x_0 x-naught