A modified adaptive-order rational Arnoldi method for model order reduction

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A Greedy-type expansion point selection for moment-matching methods in model order reduction mainly depends on the computation of a sequence of reduced order models. Typically, the adaptive-order rational Arnoldi (AORA) method resembles an efficient way for the computation of a Galerkin projection corresponding to a set of expansion points. We will provide an extension of the AORA method, in order to reuse the orthonormal basis from previous calls of the AORA method.

1 Introduction

We will discuss the application of moment-matching methods for model order reduction of linear dynamical systems

\[ \tilde{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \]

where \( E, A \in \mathbb{R}^{N \times N}, B \in \mathbb{R}^{N \times m} \) and \( C \in \mathbb{R}^{p \times N} \). Moreover, we denote the state variable via \( x(t) \in \mathbb{R}^N \), while \( u(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^p \) refer to the input and output variable of the descriptor system. The transfer function of the dynamical system is given as \( \mathcal{H}(s) = C(sE - A)^{-1}B \). In general, the rational idea behind model order reduction results from the application of a Galerkin projection \( \Pi = V_n V_n^T \) to (1), in order to provide a reduced order quadruplet \( (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \) of significant smaller dimension \( n \ll N \). For moment-matching methods, the columns of the orthonormal matrix \( V_n \in \mathbb{C}^{N \times n} \) span the input Krylov subspace \( \mathcal{K}_n(- (s_l E - A)^{-1}E, (s_l E - A)^{-1}B) \) with \( s_l \in \mathbb{C} \), in order to preserve the input-output behaviour of the dynamical system in the reduced order model [1]. Since the accuracy of the reduced order model remains limited for a single expansion point, we usually investigate an adequate set of expansion points \( \{s_1, \ldots, s_p\} \subset \mathbb{C} \). The missing a-priori error estimation for moment-matching methods poses the problem of the reliable choice of a set of expansion points. Nevertheless, the accuracy of the reduced order model might be obtained from a heuristic error estimation introduced by Grimme et al. [5].

2 Adaptive-order rational Arnoldi method

In general, the Greedy-type expansion point selection for moment-matching methods follows from the computation of a sequence of reduced order models \( \mathcal{H}_1(s), \ldots, \mathcal{H}_k(s) \) including different sets of expansion points \( S_l = S_{l-1} \cup \{s_l\}, s_l \in \mathbb{C} \), for all \( l \geq 1 \), see [3]. If we assume that the dimension of the reduced order model has been chosen a-priori, each reduced order model \( \mathcal{H}_i(s) \) follows from a rational Arnoldi-type method [7]. Thereby, the span of the orthonormal matrix \( V_n \in \mathbb{C}^{N \times n} \) fulfills

\[ \text{span}(V_n) = \bigoplus_{l=1}^i \mathcal{K}_{j_l}(- (s_l E - A)^{-1}E, (s_l E - A)^{-1}B), \]

where \( n = j_1 + \cdots + j_i \), see [4].

An efficient way for the adaptive computation of each dimension \( j_l \geq 0 \) \( l = 1, \ldots, i \) remains from the adaptive-order rational Arnoldi (AORA) method [6]. Let \( Y^{(j)}(s_l) = CX^{(j)}(s_l) \) denote the output moment of the linear dynamical system (1), where \( X^{(j)}(s_l) = - (s_l E - A)^{-1}E \) refers to the state moment. If \( \tilde{Y}^{(j)}(s) = \tilde{C}X^{(j)}(s) \) determines the output moment of the reduced order model, the AORA method increases the Krylov subspace of the expansion point \( s_{\text{max}} = \arg \max_{s \in S_{l-1}} |Y^{(j)}(s) - \tilde{Y}^{(j)}(s)| \) with an additional vector in each iteration step. We remark that the computation of the expansion point \( s_{\text{max}} \in \mathbb{C} \) from the transfer function error follows as a by-product of the rational Arnoldi method. Moreover, the modified Gram-Schmidt procedure already ensures the computation of an orthonormal basis \( V_n \in \mathbb{C}^{N \times n} \).

3 Extension of rational Arnoldi-type methods

Of course, the drawback of the subsequent calls of the AORA method results from the fact, that the computation of the orthonormal basis \( V_n^{(k)} \in \mathbb{C}^{N \times n} \) corresponding to the \( k \)-th call does not consider any previous orthonormal bases. Since we

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simply add a single expansion point $s_1 \in \mathbb{C}$ during each call of the AORA method, the idea of the modified adaptive-order rational Arnoldi (mAORA) method is given as follows: At first, we call the AORA method subsequently with the expansion points $\{s_1\}, \{s_1, s_2\}, \ldots, \{s_1, \ldots, s_p\}$. Here, each call returns an orthonormal matrix $V_n^{(l)} \in \mathbb{C}^{N \times n}$ spanning the subspace (2) of dimension $n = j_{1,i} + \cdots + j_{l,i}$ ($i = 1, \ldots, l$). Moreover, as long as $j_{i,i+1} \leq j_{i,i}$ ($i = 1, \ldots, l$), we are able to reuse the orthonormal vectors from the previous call of the AORA method. Hence, we only have to explicitly solve a shifted linear system, whenever the new expansion point $s_{l+1} \in \mathbb{C}$ has been selected or the dimension $j_{i,i+1} > j_{i,i}$ exceeds the previous number of orthonormal vectors of the expansion point $s_i \in \mathbb{C}$ ($i = 1, \ldots, l$).

We point out that the AORA and mAORA method coincide for at most two expansion points. More expansion points lead to a sufficient approximation of the AORA method avoiding the complete recomputation of the orthonormal basis $V_n^{(l)} \in \mathbb{C}^{N \times n}$ due to the reuse of previous orthonormal vectors from $V_n^{(l-1)} \in \mathbb{C}^{N \times n}$. In general, both methods lead to a comparable sequence $n = j_{1,i} + \cdots + j_{l,i}$ ($i = 1, \ldots, p$) in each subsequent call.

4 Numerical experiment

Finally, we will provide a numerical example for a Coplanar Waveguide\(^1\) resulting from the time-harmonic, first-order Maxwell’s equations

$$i\omega(\mathbf{E}) = -\sigma \mathbf{E} + \nabla \times \mathbf{H}, \quad i\omega(\mathbf{H}) = \nabla \times \mathbf{E},$$

where $\mathbf{E}$ and $\mathbf{H}$ refer to the electric and magnetic field strength. The Coplanar Waveguide resembles a single-input, single-output dynamical system with the frequency range $[f_{\min}, f_{\max}] = [0.6, 3.0]$ GHz and $N = 32924$ degrees of freedom, cf. Figure 1(a). On the boundary of the computational domain, we have employed the PEC boundary condition $\mathbf{E} \times \mathbf{n} = 0$. However, the relative error $\epsilon_{rel}(\omega) = |H(\omega) - \tilde{H}(\omega)|/|H(\omega)|$, $\omega = 2\pi f$ and $f \in [f_{\min}, f_{\max}]$, in Figure 1(b) allows a comparison between the AORA and mAORA method for eight expansion points and the dimension $n = 25$. The expansion points have been determined adaptively on the basis of the rational Krylov residual, see [2].

In general, the numerical experiments indicate that the computational effort for the subsequent calls of the mAORA method reduces about a factor of three. This is due to saving a substantial amount of systems solves with large-scale and highly indefinite systems of the form $(s\mathbf{E} - \mathbf{A})x = f$, $s \in \mathbb{C}$.

![Fig. 1: Adaptive-order rational Arnoldi method in computational electromagnetism.](image)

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**References**


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