Locally Finite Products of Two Locally Nilpotent Groups

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Introduction

A group G is called *the product of its subgroups A and B* if G equals the set $AB = \{ab \mid a \in A, b \in B\}$. A subgroup S of $G = AB$ is *prefactorized* if S is the product of a subgroup of A and a subgroup of B, and in this case, S satisfies $S = (S \cap A)(S \cap B)$. A prefactorized subgroup of $G = AB$ is called *factorized* if, in addition, it contains $A \cap B$. In particular, if S is a subgroup of $G = AB$, then the intersection of all factorized subgroups containing S is a factorized subgroup of G. This subgroup, which is evidently the smallest factorized subgroup of G containing S, is called *the factorizer of S in $G = AB$*.

Products of two subgroups, and in particular products of two locally nilpotent subgroups have been studied by many authors. One of the fundamental results about such products is the theorem of Kegel [Keg61] and Wielandt [Wie58], which states that a product of finite nilpotent subgroups is soluble. Many further results about products of locally nilpotent subgroups, both finite and infinite, can be found in the monograph [AFG92].

Consider a periodic locally soluble group G which is the product of two locally nilpotent subgroups A and B. In order to investigate the structure of G, it is important to have a detailed knowledge about prefactorized and factorized subgroups of G, because this allows to reduce structural questions to certain subgroups of G. The present dissertation is concerned with finding conditions under which certain subgroups of G are prefactorized or factorized. In particular, we improve results obtained in [Hei90], [Fra91], [Hoe93], [AF94] and [AH94] for products of two finite nilpotent subgroups and extend them to various classes of locally finite groups.

Sylow theory

In Chapter 2, we investigate Sylow π-subgroups, i.e. maximal π-subgroups, of locally soluble groups G which are the product of two locally nilpotent subgroups A and B. It turns out that the problem of finding prefactorized Sylow π-subgroups is closely connected with the question whether the characteristic subgroups $O_\pi(G)$ and $O_{\pi',\pi}(G)$ of G are prefactorized or factorized; see Theorem 2.2.5 for details. If the group G is radical, i.e. if G possesses an ascending series whose factors are locally nilpotent, then the prefactorized Sylow p-subgroups of $G = AB$ often form a Sylow basis of G (for our definition of a Sylow basis, see Section 1.2 below). In particular, we obtain the following result; see Theorem 2.3.3 below for further details.

Theorem. *Let the periodic radical group G be the product of two locally nilpotent subgroups A and B. If, for every prime p, A_p and B_p denote the p-components of A and B, respectively, then the following statements are equivalent:*

(a) *$\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

(b) *For every prime p and every normal subgroup N of G, the set A_pB_pN/N is a maximal p-subgroup of G/N.*

(c) *The group G has an ascending series of prefactorized subgroups with locally nilpotent factors.*

(d) *For every normal subgroup N of G, the Hirsch-Plotkin radical $R(G/N)$ of G/N is factorized.*

In the sequel, prefactorized Sylow bases of the form $\{A_pB_p \mid p \in \mathbb{P}\}$ will play an important role. For instance, in Theorem 2.3.7, it is shown that such Sylow bases determine which subgroups of $G = AB$ may be prefactorized.

Theorem. *Let the periodic radical group G be the product of its locally nilpotent subgroups A and B, and suppose that the set $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G. If S is a prefactorized subgroup of G, then $\{A_pB_p \cap S \mid p \in \mathbb{P}\}$ is a Sylow basis of S, i.e. $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into S.*

In Section 2.4, we show that a product G of two locally nilpotent subgroups A and B possesses a prefactorized Sylow basis if G belongs to some class of periodic locally soluble groups for which a satisfactory Sylow theory has been developed. This is for instance the case when G is an FC- or a CC-group, a $\mathfrak{U}$-group in the sense of [GHT71], or if G satisfies the minimal condition for p-subgroups (min-p) for every prime p. Here a group G is an FC-group (a CC-group) if, for every $g \in G$, the factor group $G/C_G(x^G)$ is finite (a Černikov group). Furthermore, $\mathfrak{U}$ denotes the largest subgroup-closed class of periodic locally soluble groups such that for every $G \in \mathfrak{U}$ and every set π of primes, the Sylow π-subgroups of G are conjugate. In particular, the class $\mathfrak{U}$ contains all homomorphic images of periodic locally soluble linear groups and all periodic soluble locally nilpotent-by-finite groups.

Schunck classes of nilpotent-by-finite groups

The results of Chapter 3 do not deal with products of groups and may be of independent interest. Using the notion of a Schunck class introduced in [Tom95], we extend well-known results about Schunck classes of finite soluble groups to the class of all periodic soluble nilpotent-by-finite groups. For instance, in Proposition 3.1.1, we prove that if $\mathfrak{H}$ is a class of periodic soluble nilpotent-by-finite groups such that every periodic soluble nilpotent-by-finite group possesses $\mathfrak{H}$-projectors, then $\mathfrak{H}$ is a Schunck class. As a consequence, every local formation of periodic soluble nilpotent-by-finite group is a Schunck class; see Proposition 3.1.2. Our main result about Schunck classes of nilpotent-by-finite groups, contained in Theorem 3.2.6 and Corollary 3.2.7, can be summarized as follows.

Theorem. *Let $\mathfrak{H}$ be a Schunck class of nilpotent-by-finite groups and suppose that G is a periodic soluble nilpotent-by-finite group. Then G possesses $\mathfrak{H}$-projectors, and any two are conjugate. Moreover, if H is an $\mathfrak{H}$-projector of G and H is contained in a subgroup L of G, then H is also an $\mathfrak{H}$-projector of L.*

Thus the $\mathfrak{H}$-projectors of a periodic soluble nilpotent-by-finite group G are pronormal in G. In Proposition 3.3.1, it is shown that pronormal subgroups of periodic soluble nilpotent-by-finite groups can be characterized as in the finite case.

Factorizers of $\mathfrak{F}$-subgroups

Let the periodic locally soluble group G be the product of two locally nilpotent subgroups A and B. In Chapter 4, we study group-theoretical properties of a subgroup H of G which are inherited by the factorizer of H. Theorem 4.1.5 deals with the factorizers of $\mathfrak{H}$-subgroups of a nilpotent-by-finite product of two locally nilpotent subgroups, where $\mathfrak{H}$ is a Schunck class of nilpotent-by-finite groups.

Theorem. *Let $\mathfrak{H}$ be a Schunck class of nilpotent-by-finite groups whose characteristic is π, and suppose that the periodic soluble nilpotent-by-finite group G is the product of two locally nilpotent subgroups A and B. Further, let H be an $\mathfrak{H}$-subgroup of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces.*

(a) *If π contains $\pi(A) \cap \pi(B)$, then the factorizer of H is an $\mathfrak{H}$-group.*

(b) *If H is a π-group, then the factorizer of H in $A_\pi B_\pi$ is an $\mathfrak{H}$-group. Hence H is contained in a prefactorized $\mathfrak{H}$-subgroup of G.*

Here, the characteristic π of a group class $\mathfrak{X}$ is the set of primes p such that $\mathfrak{X}$ contains a cyclic group of order p. Note also that by [Har71, Lemma 2.1] and [GHT71, Theorem 2.10], every subgroup of a periodic soluble nilpotent-by-finite group has a conjugate into which a given Sylow subgroup reduces. This implies the following necessary and sufficient condition for an $\mathfrak{H}$-maximal subgroup to be factorized or prefactorized.

Corollary. *Let $\mathfrak{H}$ be a Schunck class of nilpotent-by-finite groups of characteristic π and suppose that the periodic soluble nilpotent-by-finite group G is the product of two locally nilpotent subgroups A and B. If H is an $\mathfrak{H}$-maximal subgroup of G, then:*

(a) *If π contains $\pi(A) \cap \pi(B)$, then H is prefactorized if and only if the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H. Thus an $\mathfrak{H}$-maximal subgroup of G is prefactorized if and only if it is factorized.*

(b) *If H is a π-group, then H is prefactorized if and only if the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

Similar results hold for the classes of all periodic locally soluble FC- and CC-groups and for the class of all periodic locally soluble groups satisfying min-p for all primes p, since the groups belonging to these classes have sufficiently many nilpotent-by-finite factor groups (see [KW73, Theorem 3.17]). However, our results have to be formulated in terms of local formations, because the theory of Schunck classes of finite groups has not yet been extended to such groups. Our theorems 4.1.10, 4.2.2 and 4.3.1 can be summarized as follows.

Theorem. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of periodic locally soluble groups and assume that $\mathfrak{X}$ is a class of nilpotent-by-finite groups, of CC-groups, or of groups satisfying min-p for every prime p. Further, suppose that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation of characteristic π and that the $\mathfrak{X}$-group G is the product of two locally nilpotent groups A and B. If H is an $\mathfrak{F}$-subgroup of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, then H is contained in a prefactorized $\mathfrak{F}$-subgroup of G. If $\pi(A) \cap \pi(B)$ is contained in π, then H is even contained in a factorized $\mathfrak{F}$-subgroup of G.*

As in the nilpotent-by-finite case, these statements yield necessary and sufficient conditions for an $\mathfrak{F}$-maximal subgroup to be prefactorized or factorized; see Corollary 4.2.3 and Corollary 4.3.2.

Projectors

If the product G of two locally nilpotent subgroups A and B possesses $\mathfrak{F}$-projectors, then the above results about $\mathfrak{F}$-maximal subgroups can be used to prove the existence of a unique prefactorized $\mathfrak{F}$-projector. This is for instance the case for periodic locally soluble FC-groups and certain groups satisfying min-p for every prime p; see Corollary 4.2.8 and Corollary 4.3.6.

It seems to be an open question whether the factorizers of certain $\mathfrak{F}$-subgroups of a $\mathfrak{U}$-group G are $\mathfrak{F}$-groups, and in particular, whether every $\mathfrak{F}$-maximal subgroup of G has a prefactorized conjugate. However, the following theorem (see Theorem 5.1.5) shows that a soluble $\mathfrak{U}$-group G has a unique prefactorized $\mathfrak{F}$-projector. Thus our result holds in particular for all periodic locally soluble linear groups.

Theorem. *Let $\mathfrak{X}$ be a* QS*-closed class of $\mathfrak{U}$-groups and suppose that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation of characteristic π. Moreover, let the $\mathfrak{X}$-group G be the product of two locally nilpotent subgroups A and B. If G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and N has a hypoabelian Sylow π-subgroup, then G has a unique prefactorized $\mathfrak{F}$-projector, and this $\mathfrak{F}$-projector contains $A_\pi \cap B_\pi$. Thus if the characteristic π of $\mathfrak{F}$ contains $\pi(A) \cap \pi(B)$, then this $\mathfrak{F}$-projector is factorized.*

Recall that a group is *hypoabelian* if it has a descending normal series with abelian factors. Without the assumption that G is hypoabelian in the preceding theorem, we can prove the existence of a unique prefactorized $\mathfrak{F}$-injector only in a very special case, namely when $\mathfrak{F}$ is the class of all periodic locally nilpotent groups; see Theorem 5.2.2.

Our results about $\mathfrak{F}$-maximal subgroups and $\mathfrak{F}$-projectors widely generalize a result of Heineken [Hei90] which states that if $\mathfrak{F}$ is a local formation of finite groups, then every product of two finite nilpotent subgroups possesses a prefactorized $\mathfrak{F}$-projector.

Trifactorized groups

The above results about factorized and prefactorized $\mathfrak{H}$- and $\mathfrak{F}$-subgroups can also be used to prove theorems concerning trifactorized groups. Here, a group G is called *trifactorized* if it has subgroups A, B and C such that $G = AB = AC = BC$. In particular, we obtain the following result; see Corollary 4.1.9, Theorem 4.2.9 and Theorem 4.3.3.

Theorem. *Let $\mathfrak{X} =$ QS$\mathfrak{X}$ be a class of periodic locally soluble groups and assume that $\mathfrak{X}$ is a class of nilpotent-by-finite groups, of CC-groups or of groups satisfying min-p for every prime p. Let $\mathfrak{F}$ be a local $\mathfrak{X}$-formation of characteristic π and suppose that the $\mathfrak{X}$-group G has subgroups A, B and C such that $G = AB = AC = BC$. If A and B are locally nilpotent, C is an $\mathfrak{F}$-group and $\pi(A) \cap \pi(B)$ is contained in π, then $G \in \mathfrak{F}$.*

Example 4.4.1 shows that a finite trifactorized group $G = AB = AC = BC$ need not be supersoluble if A and B are normal supersoluble subgroups of G and C is nilpotent. On the other hand, in Theorems 4.4.2, 4.4.3, 4.4.4 and 4.4.6, we prove the following.

Theorem. *Let $\mathfrak{X} = \text{QS}\mathfrak{X}$ be a class of periodic locally soluble groups and assume that $\mathfrak{X}$ is a class of nilpotent-by-finite groups, of CC-groups, of groups satisfying min-p for every prime p, or of $\mathfrak{U}$-groups. Further, suppose that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation of characteristic π. If the $\mathfrak{X}$-group G has $\mathfrak{F}$-subgroups A and B and a normal locally nilpotent π-subgroup R such that $G = AB = AR = BR$, then G is an $\mathfrak{F}$-group.*

Note that trifactorized groups $G = AB = AC = BC$ in which one of the subgroups A, B or C is even normal in G occur for instance as factorizers of normal subgroups; see [AFG92, Lemma 1.1.4].

Injectors

In Section 5.3 and Section 5.4, we investigate injectors and radicals of FC- and CC-groups. Observe that there exist Fitting classes $\mathfrak{F}$ and products of two finite nilpotent subgroups which do not have a prefactorized $\mathfrak{F}$-injector or a prefactorized $\mathfrak{F}$-radical; see e.g. [AH94, Example 2]. However, as in the finite case [AH94, Theorem C*], a relation between prefactorized or factorized $\mathfrak{F}$-injectors and $\mathfrak{F}$-radicals can be established. For FC-groups, we obtain the following statement (see Theorem 5.3.8).

Theorem. *Suppose that the periodic FC-group is the product of two locally nilpotent subgroups A and B and let $\mathcal{F}$ be a Fitting set of G. Then the following statements are equivalent:*

(a) *For every prefactorized subgroup S of G, there exists a unique $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(b) *For every prefactorized subgroup S of G, the $\mathcal{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

(c) *For every finite prefactorized subgroup S of G, there exists an $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(d) *For every finite prefactorized subgroup S of G, the $\mathcal{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

This shows that, in order to decide whether an FC-group has a prefactorized $\mathcal{F}$-injector or an $\mathcal{F}$-radical, it suffices to consider its finite prefactorized subgroups. Note that the preceding theorem holds in particular for Fitting classes $\mathfrak{F}$. If we consider central-by-finite prefactorized subgroups instead of finite subgroups, it is also possible to obtain results concerning Fitting sets and Fitting classes of FC-groups which are not necessarily periodic (see Theorem 5.3.12 and Corollary 5.3.13).

The above theorem about FC-groups can also be applied to obtain the following result for CC-groups (see Theorem 5.4.7).

Theorem. *Let the CC-group G be the product of two locally nilpotent subgroups A and B and suppose that $\mathfrak{F}$ is a Fitting class of CC-groups. If G is periodic or $\mathfrak{F}$ contains an infinite cyclic group, then the following statements are equivalent:*

(a) *For every prefactorized subgroup S of G, there exists a unique $\mathfrak{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(b) *For every prefactorized subgroup S of G, the $\mathfrak{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

(c) *For every central-by-finite prefactorized subgroup S of G, there exists an $\mathfrak{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(d) *For every central-by-finite prefactorized subgroup S of G, the $\mathfrak{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

Subgroups of products of two finite nilpotent groups

In Section 6.1, we study groups which can be embedded into a product of two nilpotent groups. In the case of finite groups, this leads to very satisfactory results. Let $\mathfrak{F}$ be the class of all finite groups which occur as subgroups of a product of two finite nilpotent subgroups. By Example 6.1.6, the class $\mathfrak{F}$ is strictly larger than the class $\mathfrak{G}$ of all products of two finite nilpotent subgroups. Unlike $\mathfrak{G}$, the class $\mathfrak{F}$ has a number of surprising properties which do not hold for the class of all groups which are the product of two finite nilpotent subgroups. For instance, Theorem 6.1.5 below shows that the class $\mathfrak{F}$ is a local formation and a Fitting class.

Theorem. *Let $\mathfrak{F}$ denote the class of all subgroups of products of two finite nilpotent subgroups. Then the class $\mathfrak{F}$ has the following properties:*

(a) *$\mathfrak{F}$ is the class of all finite groups such that $G/O_\pi(G)$ has a nilpotent Hall π-subgroup for every set π of primes.*

(b) *$\mathfrak{F}$ is a class of finite soluble groups; hence if $G \in \mathfrak{F}$, then every Hall π-subgroup of $G/O_\pi(G)$ is nilpotent.*

(c) *$\mathfrak{F}$ is the smallest formation of soluble groups which contains all products of two finite nilpotent subgroups.*

(d) *$\mathfrak{F}$ is the smallest local formation which contains every product of two locally nilpotent groups. Moreover, $\mathfrak{F}$ can be locally defined by the formation function f, where for every prime p, $f(p)$ is the class of all finite soluble groups having a nilpotent Hall p'-subgroup.*

(e) *$\mathfrak{F}$ is the smallest subgroup-closed Fitting class of soluble groups which contains all products of two finite nilpotent groups.*

(f) *$\mathfrak{F}$ is the smallest Schunck class of finite soluble groups which contains all products of two finite nilpotent subgroups.*

Products of more than two nilpotent subgroups

In Section 6.2, we discuss briefly which results about products of two locally nilpotent subgroups can be extended to products of more than two locally nilpotent subgroups. (For a definition of such products and how to extend the notion of prefactorized and factorized subgroups, see Section 1.1.) For instance, the theorem of Kegel [Keg61] and Wielandt [Wie58] states that a finite product G of finitely many nilpotent subgroups is soluble; moreover by [Wie51], such a product G has a prefactorized Sylow basis. It shows that a product of finitely many finite nilpotent subgroups has prefactorized Sylow basis.

On the other hand, if the group G is the product of two finite nilpotent subgroups, then by [Amb73] or [Pen73], the Fitting subgroup $F(G)$ is factorized. However, Proposition 6.2.1

shows that this is not the case for a product of three pairwise permutable finite nilpotent subgroups. Moreover, by Proposition 6.2.2 in general no term of the Fitting series of a product G of three pairwise permutable nilpotent subgroups is prefactorized.

Notation

Basic definitions and some elementary results connected with them are collected in Chapter 1. In particular, in Section 1.1, we introduce prefactorized and factorized subgroups of products in full generality; Section 1.2 and Section 1.5 contain some fundamental results about Sylow subgroups, Sylow bases and formations of locally finite groups, some of which seem not to have been proved in such generality. Our notation is mostly standard and follows [AFG92], [DH92], [KW73], [Rob72] and [Rob82]. For an overview, see also the list of symbols in the appendix.

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Chapter 1

Basic concepts

1.1. Prefactorized and factorized subgroups of products

A group G is the *product of its subgroups A and B* if G equals the set

$$AB = \{ab \mid a \in A, b \in B\}.$$

A subgroup S of $G = AB$ is called *prefactorized* if S is the product of a subgroup of A and a subgroup of B. Thus S is prefactorized if and only if $S = (S \cap A)(S \cap B)$, or equivalently, if every $s \in S$ can be written $s = ab$ with $a \in A \cap S$ and $b \in B \cap S$.

A subgroup S of G is called *factorized* if, whenever $s = ab$ with $a \in A$ and $b \in B$, then $a \in S$ (and $b \in S$). Since every $g \in G$, and thus every element g of S, can be written $g = ab$ with $a \in A$ and $b \in B$, every factorized subgroup of G is prefactorized. It is also clear that every subgroup of G containing A or B is factorized. By [Wie58, *Hilfssatz* 1], a subgroup S of G is factorized if it is prefactorized and contains $A \cap B$. See also Lemma 1.1.1 below.

In order to handle Sylow bases of a group G, to be introduced in Section 1.2, more efficiently, we introduce the following more general concept of a product of groups. Let $\mathcal{S}$ be a set of subgroups of the group G. Then G is the *product of its subgroups $S \in \mathcal{S}$* if $G = <S \in \mathcal{S}>$ and $UV = VU$ for all U, $V \in \mathcal{S}$. Observe also that for every normal subgroup N of G, the factor group G/N is the product of the subgroups in $\{SN/N \mid S \in \mathcal{S}\}$.

A subgroup U of G is *prefactorized in G* if U is the product of its subgroups $U \cap S$, where $S \in \mathcal{S}$. A subgroup U of G is *factorized in G* if U is prefactorized and contains $S \cap <T \mid T \in \mathcal{S}, T \neq S>$ for every $S \in \mathcal{S}$. Factorized subgroups can also be characterized as follows:

1.1.1 Lemma. *Suppose that the group G is the product of its subgroups $S \in \mathcal{S}$. Then the following statements about a subgroup U of G are equivalent:*

(a) *U is factorized.*

(b) *Whenever an element $g \in U$ can be written as $g = s_1 \ldots s_n$ with $s_i \in S_i$ for $i \in \{1, \ldots, n\}$, where $S_1, \ldots, S_n$ are pairwise distinct subgroups contained in $\mathcal{S}$, then the elements $s_1, \ldots, s_n$ belong to U.*

Proof. Assume first that U is factorized and let $g \in U$. Now suppose that $n \in \mathbb{N}$, that $S_1, \ldots, S_n \in \mathcal{S}$ are pairwise distinct subgroups of G, and that $s_i \in S_i$ for $i \in \{1, \ldots, n\}$ such that $g = s_1 \ldots s_n$. Since U is in particular prefactorized, there exists an integer $m \in \mathbb{N}$ and subgroups $S_{n+1}, \ldots, S_m \in \mathcal{S}$ such that $S_1, \ldots, S_m$ are pairwise distinct and $g = u_1 \ldots u_m$ with $u_i \in U \cap S_i$ for every $i \in \{1, \ldots, m\}$. Then $u_1^{-1} s_1 = u_2 \ldots u_m s_n^{-1} \ldots s_2^{-1}$ is contained in $S_1 \cap <S \in \mathcal{S} \mid S \neq S_1>$. Since U is factorized, this shows that $u_1^{-1} s_1 \in U$, and consequently $s_1 \in U$. Now $s_1^{-1} g = s_2 \dots s_n \in U$, and so by induction on n, it follows that also $s_2, \dots, s_n \in U$.

Conversely, suppose that (b) holds, then clearly U is prefactorized. Now let $S \in \mathcal{S}$ and suppose that $s \in S \cap <T \mid T \in \mathcal{S}, T \neq S>$. Since the subgroups in $\mathcal{S}$ permute, there exists an integer $n \in \mathbb{N}$, pairwise distinct subgroups $S_1, \dots, S_n \in \mathcal{S}$ and elements $s_1, \dots, s_n$ of G with $S_i \neq S$ and $s_i \in S_i$ for every $i \in \{1, \dots, n\}$ such that $s = s_1 \dots s_n$. Then we have $s \in U$ by hypothesis, and so U is factorized. □

The next proposition studies the behaviour of factorized (prefactorized) subgroups in the subgroup lattice and the factor groups of a factorized group G.

1.1.2 Proposition. *Let the group G be the product of its subgroups $S \in \mathcal{S}$.*

(a) *If U is prefactorized (factorized) in G, then $V \leq U$ is a prefactorized (factorized) subgroup of U (regarded as a product of its subgroups $U \cap S$, where $S \in \mathcal{S}$) if and only if V is prefactorized (factorized) in G.*

(b) *If U is a prefactorized subgroup of G and V is a factorized subgroup of G, then $U \cap V$ is a factorized subgroup of U (regarded as a product of its subgroups $U \cap S$, where $S \in \mathcal{S}$), hence is a prefactorized subgroup of G.*

(c) *The intersection of any family of factorized subgroups of G is factorized.*

(d) *If $\mathcal{T}$ is a set of prefactorized subgroups of G whose union U is a subgroup of G, then U is a prefactorized subgroup of G. It is factorized, provided that one of the subgroups $T \in \mathcal{T}$ is factorized.*

(e) *If N is a normal subgroup of G and U is a prefactorized subgroup of G, then UN/N is a prefactorized subgroup of G/N, where G/N is regarded as a product of the subgroups $\{SN/N \mid S \in \mathcal{S}\}$.*

(f) *If N is a normal subgroup of G and U is subgroup of G which contains N, then U is a factorized subgroup of G if and only if U/N is a factorized subgroup of G/N.*

Proof. (a) The statement concerning prefactorized subgroups follows directly from the definition of a prefactorized subgroup. It is also clear that a factorized subgroup V of G which is contained in U is a factorized subgroup of U. Now suppose that V is a factorized subgroup of U and that U is a factorized subgroup of G. Let $g \in V$ and assume that n is an integer and $S_1, \dots, S_n$ are pairwise distinct subgroups of G contained in $\mathcal{S}$ such that g can be written $g = s_1 \dots s_n$ with $s_i \in S_i$. Since U is a factorized subgroup of G and $g \in U$, we have $s_i \in U \cap S_i$ for every $i \in \{1, \dots, n\}$ by Lemma 1.1.1. Since V is factorized in U, it follows that $s_i \in V \cap S_i$ for every $i \in \{1, \dots, n\}$, and so V is factorized in G, as required.

(b) Let $g \in U \cap V$. Since U is prefactorized and $g \in U$, there exists an $n \in \mathbb{N}$ such that g can be written as $g = s_1 \dots s_n$ with $s_i \in U \cap S_i$ for pairwise distinct subgroups $S_1, \dots, S_n \in \mathcal{S}$. Now V is factorized and $g \in V$, and so we have $s_i \in U \cap V \cap S_i$ for every $i \in \{1, \dots, n\}$. In view of Lemma 1.1.1, this shows that $U \cap V$ is a factorized subgroup of U. Hence $U \cap V$ is a prefactorized subgroup of G by (a).

(c) Let $\mathcal{T}$ be a set of factorized subgroups of G and let U denote the intersection of all $T \in \mathcal{T}$. Let $g \in U$, then there exists an integer $n \in \mathbb{N}$ and elements $s_1, \dots, s_n$ with $s_i \in S_i$ for $i \in \{1, \dots, n\}$, where $S_1, \dots, S_n$ are pairwise distinct subgroups of $\mathcal{S}$. If $V \in \mathcal{T}$, then $g \in V$

and so $s_i \in V$ for every $i \in \{1, \ldots, n\}$ by Lemma 1.1.1. This shows that $s_i \in U$ for every $i \in \{1, \ldots, n\}$, and so U is factorized by Lemma 1.1.1.

(d) Let $u \in U$, then $u \in T$ for some $T \in \mathcal{T}$. Since T is prefactorized, there exists an integer n and pairwise distinct subgroups $S_1, \ldots, S_n \in \mathcal{S}$ such that u can be written $u = s_1 \ldots s_n$ with $s_i \in S_i \cap T$. In particular, $s_i \in S_i \cap U$ for every $i \in \{1, \ldots, n\}$, and so U is prefactorized. Moreover, if one of the subgroups $T \in \mathcal{T}$ is factorized, it contains $S \cap <V \in \mathcal{S} \mid V \neq S>$ for every $S \in \mathcal{S}$, and so U is factorized.

(e) is obvious.

(f) Suppose first that U is factorized and let $uN \in U/N$. If there exists an integer n and pairwise distinct subgroups $S_1, \ldots, S_n \in \mathcal{S}$ such that $uN = s_1 N \ldots s_n N$, where $s_i \in S_i$ for every integer $i \in \{1, \ldots, n\}$, then $u = s_1 \ldots s_n x$ for some $x \in N$, and since $N \leq U$, we have $s_1 \ldots s_n = ux^{-1} \in U$. Since U is factorized, it follows that $s_i \in S_i \cap U$ for every $i \in \{1, \ldots, n\}$, and so $s_i N \in (U \cap S_i N)/N$ for every i. Therefore U/N is factorized.

Conversely, assume that U/N is factorized and let $u \in U$. If there exists an integer n and pairwise distinct subgroups $S_1, \ldots, S_n \in \mathcal{S}$ such that $u = s_1 \ldots s_n$ such that $s_i \in S_i$ for every integer $i \in \{1, \ldots, n\}$, then $uN = s_1 N \ldots s_n N$, and since U/N is factorized, we have $s_i N \in (U \cap S_i N)/N$ for every $i \in \{1, \ldots, n\}$. In particular, $s_i \in U \cap S_i$ for every i, and so U is factorized. □

If the group G is the product of two subgroups, then also a number of additional statements hold. The statements about factorized subgroups can also be found in Chapter 1 of [AFG92].

1.1.3 Proposition. *Let the group G be the product of its subgroups A and B.*

(a) *If U is prefactorized (factorized) in G, then $V \leq U$ is prefactorized (factorized) with respect to the factorization $U = (U \cap A)(U \cap B)$ of U if and only if V is prefactorized (factorized) in $G = AB$.*

(b) *If U is a prefactorized subgroup of G and V is a factorized subgroup of G, then $U \cap V$ is a factorized subgroup of $U = (U \cap A)(U \cap B)$, hence is prefactorized in $G = AB$.*

(c) *The intersection of any family of factorized subgroups of G is factorized.*

(d) *If $\mathcal{S}$ is a set of prefactorized subgroups of G whose union U is a subgroup of G, then U is a prefactorized subgroup of G. It is factorized, provided that one of the subgroups $S \in \mathcal{S}$ is factorized.*

(e) *The product of two prefactorized subgroups one of which is normalized by the other is prefactorized. It is factorized, provided that one of the subgroups is factorized.*

(f) *The product of any number of prefactorized normal subgroups is prefactorized. It is factorized if one of the normal subgroups is factorized.*

(g) *If N is a normal subgroup of G and S is a prefactorized (factorized) subgroup of $G = AB$, then SN/N is a prefactorized (factorized) subgroup of $G/N = (AN/N)(BN/N)$.*

(h) *If N is a prefactorized normal subgroup of G and S is subgroup of G which contains N, then S is a prefactorized subgroup of $G = AB$ if and only if S/N is a prefactorized subgroup of $G/N = (AN/N)(BN/N)$.*

(i) *If N is a normal subgroup of G and S is a subgroup of G which contains N, then S is a factorized subgroup of $G = AB$ if and only if S/N is a factorized subgroup of $G/N = (AN/N)(BN/N)$.*

Proof. (a), (b), (c) and (d) follow from the respective statements in Proposition 1.1.2.

(e) Let N and P be prefactorized subgroups of G with $N \trianglelefteq PN$. Then

$$\begin{aligned} PN &= (P \cap A)(P \cap B)N = (P \cap A)N(P \cap B) \\ &= (P \cap A)(N \cap A)(N \cap B)(P \cap B) \\ &\leq (PN \cap A)(PN \cap B), \end{aligned}$$

which shows that PN is prefactorized. The statement about factorized subgroups now follows as in (d).

(f) Let $\mathcal{N}$ be a set of prefactorized normal subgroups of G. By (e), the product of two prefactorized normal subgroups is prefactorized, and since it is clearly normal, the statement is true for every finite subset of $\mathcal{N}$. Now the product of all $N \in \mathcal{N}$ is the union of all products of a finite number of the $N \in \mathcal{N}$, and so the full result follows from (d).

(g) If $S = (S \cap A)(S \cap B)$ is prefactorized, then $SN = (S \cap A)(S \cap B)N$ which is contained in $(SN \cap AN)(SN \cap BN)$ by the modular law. Therefore $(SN \cap AN)(SN \cap BN) = SN$ and so $SN/N = (SN/N \cap AN/N)(SN/N \cap BN/N)$.

(h) If S is prefactorized in G, then S/N is prefactorized in G/N by (g). Conversely, suppose that S/N is prefactorized, then $S = (S \cap AN)(S \cap BN)$. Moreover, $N = (A \cap N)(B \cap N)$, and since $N \leq S$, it follows from the modular law that

$$\begin{aligned} S &= (S \cap A(B \cap N))(S \cap B(A \cap N)) \\ &= (S \cap A)(B \cap N)(A \cap N)(S \cap B) \\ &= (S \cap A)(A \cap N)(B \cap N)(S \cap B) \\ &= (S \cap A)(S \cap B). \end{aligned}$$

(i) has been proved in Proposition 1.1.2 (f) and can also be found in [AFG92, Lemma 1.1.2]. □

Suppose that the group G is the product of its subgroups $S \in \mathcal{S}$ and let U be a subgroup of G. Then by Proposition 1.1.3 (c), the intersection X of all factorized subgroups of G which contain U is itself factorized. The subgroup X is called the *factorizer of U in G*, and evidently X is the unique smallest factorized subgroup containing U. If G is the product of its subgroups A and B and N is a normal subgroup of G, then $AN \cap BN$ is the factorizer of N in G; see [AFG92, Lemma 1.1.4].

We give an example which shows that the intersection of a descending chain of prefactorized subgroups or the intersection of two prefactorized subgroups need not be prefactorized.

1.1.4 Example. Let G be the direct product of countably many isomorphic cyclic subgroups $C_i = <c_i>$ $(i \in \mathbb{N})$, and let

$$A = <c_1 c_n^{-1} \mid n \in \mathbb{N}, n \geq 2>$$

and

$$B = <c_n \mid n \in \mathbb{N}, n \geq 2>,$$

then clearly $G = AB$. For every $n \in \mathbb{N}$, let $S_n = <c_1, c_k \mid k \geq n>$. Then

$$c_1 = (c_1 c_n^{-1}) \cdot c_n \in (S_n \cap A)(S_n \cap B)$$

and so every S_n is prefactorized. Moreover, the S_n form a descending chain of subgroups of G. Clearly, $S = <c_1>$ equals the intersection of all S_n. But $S \cap A = 1 = S \cap B$ and so S is not prefactorized.

To see that the intersection of two prefactorized subgroups is not prefactorized in general, let $H_1 = <c_1, c_3>$ and $H_2 = <c_1, c_4>$, then H_1 and H_2 are prefactorized subgroups of G. But $S = H_1 \cap H_2$ is not prefactorized. (Of course, for the second part, it would have been possible to replace G by its prefactorized subgroup $<c_1, c_2, c_3, c_4>$.)

The following lemma gives a criterion for the intersection of a descending chain of prefactorized subgroups to be prefactorized.

1.1.5 Lemma. *Let the group G be the product of two subgroups A and B and suppose that $\mathcal{S}$ is a totally ordered set of prefactorized subgroups of G. If $A \cap B \cap S$ is finite for some $S \in \mathcal{S}$, then the intersection of all $S \in \mathcal{S}$ is prefactorized.*

Proof. Let U denote the intersection of all $S \in \mathcal{S}$ and let $T \in \mathcal{S}$ such that $A \cap B \cap T$ is finite. If $\mathcal{T} = \{S \in \mathcal{S} \mid S \leq T\}$, then U also equals the intersection of all $S \in \mathcal{T}$. Therefore it suffices to show that U is a prefactorized subgroup of T, and so we may suppose without loss of generality that $G = T$ and $\mathcal{S} = \mathcal{T}$, and so $A \cap B$ is finite.

Let $g \in U$. For every $S \in \mathcal{S}$, fix elements $a_S \in A \cap S$ and $b_S \in B \cap S$ such that $g = a_S b_S$. If S, $T \in \mathcal{S}$, then $a_S b_S = a_T b_T$ and so $a_T^{-1} a_S = b_T b_S^{-1} \in A \cap B$. This shows that the set $A_0 = \{a_S \mid S \in \mathcal{S}\}$ is finite. For every $a \in A_0$, let $\mathcal{S}_a = \{S \in \mathcal{S} \mid a_S = a\}$, then the $\mathcal{S}_a$ form a partition of $\mathcal{S}$.

If, for every $a \in A_0$, there exists $S_a \in \mathcal{S}$ which is contained in every $S \in \mathcal{S}_a$, then the totally ordered set $\mathcal{S}$ has a least element S, since A_0 is finite. Therefore $U = S$ is prefactorized.

Therefore assume that there exists $a \in A_0$ such that $\mathcal{S}_a$ does not have a least element. It follows that U equals the intersection of all $S \in \mathcal{S}_a$, and so $a \in U \cap A$. Moreover, $b = a^{-1} g = b_S \in S$ for every $S \in \mathcal{S}_a$ and so $b \in U \cap B$. Therefore $g = ab \in (U \cap A)(U \cap B)$. □

The next lemma shows in particular that a finite subset of a product G of two subgroups is contained in a countable prefactorized subgroup of G. It is also useful as a (weak) substitute of Proposition 1.1.3 (c), since it ensures that certain intersections of prefactorized subgroups are prefactorized.

1.1.6 Lemma. *Suppose that the group G is the product of two subgroups A and B. Moreover, let $\mathcal{S}$ be a set of prefactorized subgroups of G which is closed with respect to arbitrary intersections of its members. Then every subset X of G is contained in a prefactorized subgroup H of cardinality not exceeding* $\max(\aleph_0, |X|)$, *such that $H \cap S$ is a prefactorized subgroup of H for every $S \in \mathcal{S}$.*

Proof. Suppose without loss of generality that $G \in \mathcal{S}$. For every $x \in G$, let G_x denote the intersection of all $S \in \mathcal{S}$ such that $x \in S$, then by hypothesis $G_x \in \mathcal{S}$ for every $x \in G$, and in particular G_x is prefactorized. Now define functions $a_1 \colon G \to A$, $b_1 \colon G \to B$, $a_2 \colon G \to A$ and $b_2 \colon G \to B$ as follows: For each $x \in G$, choose elements a, $a' \in A \cap G_x$ and b, $b' \in B \cap G_x$ such that $x = ab = b'a'$ and put $a_1(x) = a$, $b_1(x) = b$, $a_2(x) = a'$ and $b_2(x) = b'$.

Let $X_0 = X$ and $A_0 = B_0 = \varnothing$. By induction, we construct from the set X_i the sets A_{i+1}, B_{i+1} and X_{i+1} containing A_i, B_i and X_i respectively: If i is even, let $A_{i+1} = <A_i, a_1(x) \mid x \in X_i>$, $B_{i+1} = <B_i, b_1(x) \mid x \in X_i>$ and $X_{i+1} = A_{i+1} B_{i+1}$. If i is odd, put $A_{i+1} = <A_i, a_2(x) \mid x \in X_i>$, $B_{i+1} = <B_i, b_2(x) \mid x \in X_i>$ and $X_{i+1} = B_{i+1} A_{i+1}$. Then in both cases, X_i is contained in X_{i+1}, and the cardinalities of A_{i+1}, B_{i+1} and X_{i+1} do not exceed $\max(\aleph_0, |X|)$.

Now let $A_\infty = \bigcup_{i\in\mathbb{N}} A_i$ and $B_\infty = \bigcup_{i\in\mathbb{N}} B_i$, then it is easy to verify that $A_\infty B_\infty \subseteq B_\infty A_\infty \subseteq A_\infty B_\infty$ so that $H = A_\infty B_\infty$ is a prefactorized subgroup of G whose cardinality does not exceed $\max(\aleph_0, |X|)$.

Let $S \in \mathcal{S}$, then it remains to show that $H \cap S$ is prefactorized. Choose $x \in H \cap S$, then $x \in X_i \subseteq X_{i+1}$ for some integer i so that i may be assumed odd. Then $X_i = A_i B_i$ and $x \in X_i$ is the product of $a = a_1(x) \in A$ and $b = b_1(x) \in B$. By construction, a and b belong to G_x and $G_x \leq S$ because $x \in S$. This shows that $x = ab \in (A \cap H \cap S)(B \cap H \cap S)$ and $H \cap S$ is prefactorized. □

For some statements concerning groups which are the product of two of their subgroups, the following generalization of Dedekind's modular law will be useful:

1.1.7 Lemma. *Let G be a group and suppose that X, Y and U are subsets of G such that $u^{-1} \in U$ for every $u \in U$.*

(a) *If $XU \subseteq X$, then $(X \cap Y)U = X \cap YU$.*

(b) *If $UX \subseteq X$, then $U(X \cap Y) = X \cap UY$.*

Proof. Suppose that $XU \subseteq X$. Clearly, $(X \cap Y)U \subseteq XU \cap YU$ which is contained in $X \cap YU$ by hypothesis. Conversely, let $x \in X \cap YU$ and write $x = yu$ with $y \in Y$ and $u \in U$. Then xu^{-1} is contained in $XU \cap Y \subseteq X \cap Y$ and therefore $x = (xu^{-1})u$ is contained in the set $(X \cap Y)U$. The proof of the second statement is similar. □

1.2. Sylow subgroups and Sylow bases

Let G be a group and π be a set of primes. We define a *Sylow π-subgroup of G* to be a maximal π-subgroup of G. This terminology differs from that of [KW73], although for the group classes considered in the sequel, namely $\mathfrak{U}$-groups, periodic FC- and CC-groups, periodic locally soluble groups with the minimal condition on p-subgroups, our definition coincides with that of [KW73]. If G is a finite group and π is a set of primes, a subgroup H of G is a *Hall π-subgroup of G* if H is a π-group whose index is a π'-number. In particular, every Hall π-subgroup of G is a Sylow π-subgroup of G. The Sylow subgroup G_π *reduces into a subgroup H of G* if $G_\pi \cap H$ is a Sylow π-subgroup of H. Two subgroups U and V of G are *conjugate* (*locally conjugate*, *conjugate via an automorphism α*) if there exists an element $g \in G$ (a locally inner automorphism α, an automorphism α) such that $U^g = V$ ($U^\alpha = V$). An automorphism α of G is called *locally inner* if for every finite subset $\{x_1, \dots, x_n\}$ of G, there exists an element $g \in G$ such that $x_i^\alpha = x_i^g$ for every $i \in \{1, \dots, n\}$.

Our first simple lemma is a weak version of the Schur-Zassenhaus theorem, which, nevertheless, holds for arbitrary locally finite groups.

1.2.1 Lemma. *Suppose that G is a locally finite group and π a set of primes such that G/N is a π-group for some subgroup $N \leq Z(G)$. Then G has a unique Sylow π-subgroup G_π and a unique Sylow π'-subgroup $G_{\pi'}$ such that $G = G_\pi G_{\pi'}$.*

Proof. Let $N_{\pi'}$ be the unique Sylow π'-subgroup of N, then also $G/N_{\pi'}$ is a π-group. Therefore assume without loss of generality that N is a π'-group. Let S be the set of all π-elements of G. If g, $h \in S$, then $F = \langle g, h\rangle$ is a finite group. Hence by the theorem of Schur and Zassenhaus, $F = F_\pi F_{\pi'}$, where F_π is a Hall π-subgroup of F and $F_{\pi'} = F \cap N$ is the unique Hall π'-subgroup of F. Since $F_{\pi'} \leq Z(F)$, the subgroup F_π is the unique Hall π-subgroup of F and so $gh \in F_\pi$ is a π-element. Hence $gh \in S$ and S is a π-subgroup of G.

Now every element $g \in G$ can be expressed as the product of a π-element s and a π'-element x. Since G/N is a π-group, we have $x \in N$; moreover, $s \in S$ by the definition of S. Therefore $G = SN$, as required. □

We will call a set $\{G_p \mid p \in \mathbb{P}\}$ of subgroups of an arbitrary group G a *Sylow basis* of G if it satisfies the following conditions.

(SS1) For every set π of primes, the group $\langle G_p \mid p \in \pi\rangle$ is a Sylow π-subgroup of G.

(SS2) $G_pG_q = G_qG_p$ for all primes p and q.

Observe that if $\{G_p \mid p \in \mathbb{P}\}$ is a Sylow basis of the group G, then G is the product of its subgroups G_p, where $p \in \mathbb{P}$.

Note that our definition of a Sylow basis differs from that in [Bae70] and [Dix82]. There, a set of subgroups $\{G_p \mid p \in \mathbb{P}\}$ is called a Sylow basis if the G_p are Sylow p-subgroups of G satisfying (SS2), and a set of subgroups $\{G_p \mid p \in \mathbb{P}\}$ satisfying (SS1) and (SS2) is called a Sylow generating basis. For an example of a group where these concepts differ, see e.g. Baer [Bae70, *Satz* 5.3]). Since in our context only Sylow bases satisfying (SS1) are relevant, we do not make this distinction between Sylow bases and Sylow generating bases.

As in the case of Sylow subgroups, one is often interested whether the Sylow bases of a group G satisfy some form of conjugacy. Let $\{G_p \mid p \in \mathbb{P}\}$ and $\{G^*_p \mid p \in \mathbb{P}\}$ be Sylow bases of G, then $\{G_p \mid p \in \mathbb{P}\}$ and $\{G^*_p \mid p \in \mathbb{P}\}$ are *conjugate* (*locally conjugate*, *conjugate via an automorphism* α) if there exists an element $g \in G$ (a locally inner automorphism α, an automorphism α) such that $G^g_p = G^*_p$ ($G^\alpha_p = G^*_p$) for every prime $p \in \mathbb{P}$.

The next lemma states some equivalent definitions of a Sylow basis.

1.2.2 Lemma. *Let G be a group and suppose that $\{G_p \mid p \in \mathbb{P}\}$ is a set of subgroups of G and for every set of primes π, set $G_\pi = \langle G_p \mid p \in \pi\rangle$. Then the following statements are equivalent:*

(a) *$\{G_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

(b) *For every set π of primes, G_π is a π-group, $G = \langle G_p \mid p \in \mathbb{P}\rangle$ and $G_pG_q = G_qG_p$ for all primes p and q.*

(c) *For every set π of primes, G_π is a π-group and $G = G_\pi G_{\pi'}$.*

(d) *For every prime p, G_p is a p-group and $G_{p'}$ is a p'-group; moreover, $G = G_pG_{p'}$ for every prime p.*

Proof. (a) ⇒ (b) is obvious.

(b) ⇒ (c). Let $g \in G = \langle G_p \mid p \in \mathbb{P}\rangle$, then there exist an integer n and primes $p_1, \ldots, p_n$ such that $g \in \langle G_{p_1}, \ldots, G_{p_n}\rangle$. Since $\langle G_{p_1}, \ldots, G_{p_n}\rangle = G_{p_1} \cdot \ldots \cdot G_{p_n}$ and $G_pG_q = G_qG_p$ for all primes p, q, we may assume without loss of generality that $p_1, \ldots, p_m \in \pi$ and $p_{m+1}, \ldots, p_n \in \pi'$ for some $m \in \mathbb{N}$. This shows that $g \in G_{p_1} \cdot \ldots \cdot G_{p_n} \leq G_\pi G_{\pi'}$ and so $G = G_\pi G_{\pi'}$.

(c) ⇒ (d) is trivial.

(d) $\Rightarrow$ (a). In order to show that G_pG_q is a subgroup of G for all primes p and q with $p \neq q$, observe that

$$G_q = \bigcap_{r \in \mathbb{P} \setminus \{q\}} G_{r'},$$

so that

$$G_pG_q = G_p\Big(\bigcap_{r \in \mathbb{P} \setminus \{q\}} G_{r'} \Big) = G_p\Big(G_{p'} \cap \bigcap_{r \in \mathbb{P} \setminus \{p,q\}} G_{r'} \Big),$$

and by Dedekind's modular law, we obtain

$$G_pG_q = G_pG_{p'} \cap \Big(\bigcap_{r \in \mathbb{P} \setminus \{p,q\}} G_{r'} \Big) = \bigcap_{r \in \mathbb{P} \setminus \{p,q\}} G_{r'},$$

whence G_pG_q is a subgroup of G. Now let π be a set of primes, then clearly

$$G_\pi \leq \bigcap_{p \in \pi'} G_{p'}.$$

This shows that G_π is a π-group. Therefore G satisfies (b), and since we have already proved that (b) implies (c), it follows that $G = G_\pi G_{\pi'}$. Thus if G_π is contained in a π-group P, then we have $P = P \cap G_\pi G_{\pi'} = G_\pi(P \cap G_{\pi'}) = G_\pi$ by Dedekind's modular law. This shows that G_π is a Sylow π-subgroup of G, as required. □

Let G be a group possessing a Sylow basis $\{G_p \mid p \in \mathbb{P}\}$. The Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ of G *reduces into a subgroup* H *of* G if $\{G_p \cap H \mid p \in \mathbb{P}\}$ is a Sylow basis of H. Thus, if we consider G as a product of its subgroups G_p, where $p \in \mathbb{P}$, then in view of Lemma 1.2.2 (b), the Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ reduces into H if and only if H is a prefactorized subgroup of G. Note that, since $<G_q \mid q \in \mathbb{P}, q \neq p>$ is a p'-group, every prefactorized subgroup of G is actually factorized. This is extremely useful for proving the following statements.

1.2.3 Lemma. *Let G be a group and suppose that G possesses a Sylow basis $\{G_p \mid p \in \mathbb{P}\}$.*

(a) *If $N \trianglelefteq G$, then $\{G_pN/N \mid p \in \mathbb{P}\}$ is a Sylow basis of GN/N.*

(b) *If $N \trianglelefteq G$ and U is a subgroup of G into which $\{G_p \mid p \in \mathbb{P}\}$ reduces, then the Sylow basis $\{G_pN/N \mid p \in \mathbb{P}\}$ of G/N reduces into UN/N.*

(c) *If $\mathcal{S}$ is a set of subgroups of G such that $\{G_p \mid p \in \mathbb{P}\}$ reduces into every $U \in \mathcal{S}$, then $\{G_p \mid p \in \mathbb{P}\}$ also reduces into the intersection S of all $U \in \mathcal{S}$.*

(d) *Let U be a subgroup of G such that $\{G_p \mid p \in \mathbb{P}\}$ reduces into U and assume that N is a locally nilpotent normal subgroup of G. Then $\{G_p \mid p \in \mathbb{P}\}$ also reduces into UN.*

(e) *Suppose that N is a normal subgroup of G. If the Sylow basis $\{G_pN/N \mid p \in \mathbb{P}\}$ reduces into the subgroup U/N of G/N, then $\{G_p \mid p \in \mathbb{P}\}$ reduces into U.*

Proof. (a) follows directly from Lemma 1.2.2 (b). (b) and (c) are consequences of Proposition 1.1.2 (e) and Proposition 1.1.2 (c), respectively.

(d) For every set π of primes, let $G_\pi = <G_p \mid p \in \pi>$, $U_\pi = U \cap G_\pi$ and $N_\pi = O_\pi(N)$. Put $U_p = U_{\{p\}}$ and $N_p = N_{\{p\}}$ for every prime p, then by hypothesis, $\{U_p \mid p \in \mathbb{P}\}$ and $\{N_p \mid p \in \mathbb{P}\}$ are Sylow bases of U and N, respectively. Let π be a set of primes. Since $N_\pi G_\pi$ is a π-group, we have $N_\pi \leq G_\pi$ and so $U_\pi N_\pi \leq G_\pi$ is a π-group. Moreover, $U_\pi N_\pi U_{\pi'} N_{\pi'} =$

$U_\pi U_{\pi'} N_\pi N_{\pi'} = UN$, and so $\{U_p N_p \mid p \in \mathbb{P}\}$ satisfies condition (c) of Lemma 1.2.2, hence it is a Sylow basis of UN.

(e) This follows from Proposition 1.1.2 (f). □

1.3. Series, chains and major subgroups

Let G be a group, Γ a set of endomorphisms of G and Ω a totally ordered set. A set $\mathcal{S} = \{U_\sigma, V_\sigma \mid \sigma \in \Omega\}$ of Γ-invariant subgroups of G is called a *Γ-series of G* if:

(S1) V_σ is a normal subgroup of U_σ for every $\sigma \in \Omega$.

(S2) $U_\sigma \leq V_\tau$ for all σ, $\tau \in \Omega$ with $\sigma < \tau$.

(S3) For every $g \in G$ with $g \neq 1$, there exists a $\sigma \in \Omega$ such that $g \in U_\sigma \setminus V_\sigma$.

For an equivalent formulation, see also [Rob82, Section 12.4]. The subgroups U_σ and V_σ are called *terms* of $\mathcal{S}$ and the factor groups U_σ/V_σ are called the *factors* of $\mathcal{S}$. The order type of $\mathcal{S}$ is defined to be the order type of the index set Ω. The series is called *finite* (*finite of length* $n \in \mathbb{N}$) if Ω is finite ($|\Omega| = n$). The series is called *ascendant* if Ω is well-ordered and *descendant* if Ω is well-ordered with respect to its inverse ordering. A group is *radical* if it has an ascending series with locally nilpotent factors; it is *hypoabelian* if it has a descending normal series with abelian factors.

If Γ is the set of all inner automorphisms (all automorphisms, all endomorphisms) of G, then the series is called *normal* (*characteristic*, *fully invariant*). A $\varnothing$-series of a group G is just called a *series of G*.

Let $\mathcal{S}$ and $\mathcal{T}$ be Γ-series of the group G. If $\mathcal{S}$ is contained in $\mathcal{T}$, then $\mathcal{T}$ is called a *refinement of $\mathcal{S}$*. The Γ-series $\mathcal{S}$ is a *Γ-composition series of G* if $\mathcal{S}$ does not have a proper refinement. A $\varnothing$-composition series of G is called just a *composition series of G*, and if Γ consists of all inner automorphisms, a Γ-composition series is called a *chief series of G* or a *principal series of G*. The factors occurring in a composition series are called *composition factors of G*; those of a chief series are called *chief factors* or *principal factors*.

A subgroup S of the group G is called *serial* if it is a term in some $\varnothing$-series of G. The subgroup S is called *subnormal* (*ascendant*, *descendant*) if S is a term of a finite (ascendant, descendant) series of G.

A set $\mathcal{C}$ of subgroups of the group G is called a *chain of subgroups of G* if it is totally ordered (with respect to inclusion). If the set $\mathcal{C}$ is well-ordered (well-ordered with respect to the inverse ordering), the set $\mathcal{C}$ is an *ascending chain* (*descending chain*). Let $\mathfrak{X}$ be a class of groups. Then the group G has the *minimal (maximal) condition for $\mathfrak{X}$-groups* if every descending (ascending) chain of G whose members are $\mathfrak{X}$-groups is finite.

Let G be a group and $\mathcal{C}$ a chain of subgroups of G. If $\mathcal{C}$ has a minimal element U and a maximal element V, then $\mathcal{C}$ is called *a chain from U to V*.

Following Tomkinson [Tom75], we define major subgroups of a group G as follows. For every subgroup U of G, let $m(U)$ denote the least upper bound of the lengths of all ascending chains from U to G. A subgroup M of G is a *major subgroup of G* if $m(M) = m(V)$ for every subgroup V of G with $M \leq V$.

1.4. Classes and closure operations

A *class* $\mathfrak{X}$ *of groups* (or *group class*) is a class whose members are groups and such that if $G \in \mathfrak{X}$, then $\mathfrak{X}$ contains every group isomorphic with G. If $G \in \mathfrak{X}$, then the group G will be called an $\mathfrak{X}$-group.

Since this dissertation is concerned with locally finite groups only, we denote with $\mathfrak{S}$ the class of all locally finite-soluble groups. Moreover, $\mathfrak{A}$ and $\mathfrak{N}$ are the classes of all periodic abelian and of all periodic nilpotent groups, respectively. A group G is an FC-group (a CC-group) if, for every $g \in G$, the factor group $G/C_G(g^G)$ is finite (a Černikov group). A Černikov group is a finite extension of a periodic radicable abelian group of finite rank, and hence it satisfies the minimal condition on subgroups. Let p be a prime, then the group G satisfies the *minimal condition on p-subgroups*, also called *min-p*, if every p-subgroup of G satisfies the minimal condition on subgroups.

The class $\mathfrak{U}$ can be characterized as follows. A periodic locally soluble group G belongs to the class $\mathfrak{U}$ if, for every subgroup H and every set π of primes, the Sylow π-subgroups of H are conjugate in H. Note that by a result of Hartley [Har72a, Theorem E], every $\mathfrak{U}$-group has a finite series with locally nilpotent factors.

If $\mathfrak{X}$ is a class of groups, then $\mathfrak{X}^*$ denotes the class of all finite $\mathfrak{X}$-groups, and if π is a set of primes, then $\mathfrak{X}_\pi$ is the class of all π-groups contained in $\mathfrak{X}$.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups, then $\mathfrak{X}\mathfrak{Y}$ is the class of all groups G which possess a normal $\mathfrak{X}$-subgroup N such that G/N is an $\mathfrak{Y}$-group. If $\mathfrak{Z}$ is another class of groups, we define $\mathfrak{X}\mathfrak{Y}\mathfrak{Z} = (\mathfrak{X}\mathfrak{Y})\mathfrak{Z}$.

The set of all primes p such that the group class $\mathfrak{X}$ contains a cyclic group of order p is called the *characteristic of* $\mathfrak{X}$.

A map $\textsc{c}$: {group classes} $\to$ {group classes} is called a *closure operation* if, for any two group classes $\mathfrak{X}$ and $\mathfrak{Y}$, we have

(C1) $\mathfrak{X} \subseteq \textsc{c}\mathfrak{X}$.

(C2) $\textsc{c}\mathfrak{X} = \textsc{c}^2\mathfrak{X}$.

(C3) If $\mathfrak{X} \subseteq \mathfrak{Y}$, then $\textsc{c}\mathfrak{X} \subseteq \textsc{c}\mathfrak{Y}$.

We introduce the following closure operations: $\textsc{q}$, $\textsc{s}$, $\textsc{l}$, $\textsc{s}_n$, $\textsc{n}$, $\textsc{d}$ and $\textsc{r}$. If $\mathfrak{X}$ is a class of groups, then $\textsc{q}\mathfrak{X}$ and $\textsc{s}\mathfrak{X}$ are the classes of all factor groups of $\mathfrak{X}$-groups and the class of all subgroups of $\mathfrak{X}$-groups, respectively. $\textsc{l}\mathfrak{X}$ is the class of all groups G such that every finite subset of G is contained in an $\mathfrak{X}$-subgroups of G, and $\textsc{s}_n\mathfrak{X}$ is the class of all subnormal subgroups of $\mathfrak{X}$-groups. $\textsc{n}\mathfrak{X}$ is the class of all groups which are generated by their serial $\mathfrak{X}$-subgroups and $\textsc{d}\mathfrak{X}$ is the class of all groups which are the direct product of an arbitrary number of their normal $\mathfrak{X}$-subgroups. $\textsc{r}\mathfrak{X}$ is the class of all groups G which possess a set $\mathcal{N}$ of normal subgroups such that $\bigcap_{N \in \mathcal{N}} N = 1$ and $G/N \in \mathfrak{X}$ for every $N \in \mathcal{N}$.

Let $\textsc{c}$ be a closure operation. A group class $\mathfrak{X}$ is called *$\textsc{c}$-closed* if $\mathfrak{X} = \textsc{c}\mathfrak{X}$. We follow [DH92] in defining $\textsc{c}\varnothing = \varnothing$ for every closure operation $\textsc{c}$.

If $\textsc{c}_1$ and $\textsc{c}_2$ are closure operations, then $\textsc{c}_1\textsc{c}_2$ and $\langle\textsc{c}_1,\textsc{c}_2\rangle$ are defined as follows. If $\mathfrak{X}$ is a group class, then $\textsc{c}_1\textsc{c}_2\mathfrak{X} = \textsc{c}_1(\textsc{c}_2\mathfrak{X})$ and $\langle\textsc{c}_1,\textsc{c}_2\rangle\mathfrak{X}$ is the intersection of all group classes $\mathfrak{Y}$ which contain $\mathfrak{X}$ and are both $\textsc{c}_1$- and $\textsc{c}_2$-closed. By [DH92, II, Lemma 1.14], $\langle\textsc{c}_1,\textsc{c}_2\rangle\mathfrak{X}$ is the unique smallest class which contains $\mathfrak{X}$ and is both $\textsc{c}_1$- and $\textsc{c}_2$-closed.

Sometimes, it is useful to restrict closure operations c to a certain universe $\mathfrak{Y}$ of groups. The next lemma shows that such restricted closure operations are again closure operations.

1.4.1 Lemma. *Let $\mathfrak{Y}$ be a class of groups and c a closure operation. Then $c_{\mathfrak{Y}}$, defined by $c_{\mathfrak{Y}}\mathfrak{X} = c\mathfrak{X} \cap (\mathfrak{X} \cup \mathfrak{Y})$ for every group class $\mathfrak{X}$, is a closure operation.*

Proof. Let $\mathfrak{X}$ be a group class. Then $\mathfrak{X} \subseteq c\mathfrak{X} \cap (\mathfrak{X} \cup \mathfrak{Y}) = c_{\mathfrak{Y}}\mathfrak{X}$ and if $\mathfrak{X}$ is contained in the group class $\mathfrak{X}_1$, then $c_{\mathfrak{Y}}\mathfrak{X} = c\mathfrak{X} \cap (\mathfrak{X} \cup \mathfrak{Y}) \subseteq c\mathfrak{X}_1 \cap (\mathfrak{X}_1 \cup \mathfrak{Y}) = c_{\mathfrak{Y}}\mathfrak{X}_1$. Moreover,

$$(c_{\mathfrak{Y}})^2\mathfrak{X} = c(c_{\mathfrak{Y}}\mathfrak{X}) \cap (c_{\mathfrak{Y}}\mathfrak{X} \cup \mathfrak{Y}) = c(c\mathfrak{X} \cap (\mathfrak{X} \cup \mathfrak{Y})) \cap (c_{\mathfrak{Y}}\mathfrak{X} \cup \mathfrak{Y})$$
$$\subseteq c^2\mathfrak{X} \cap (\mathfrak{X} \cup \mathfrak{Y}) \cap (c_{\mathfrak{Y}}\mathfrak{X} \cup \mathfrak{Y}) = c\mathfrak{X} \cap (\mathfrak{X} \cap \mathfrak{Y}) = c_{\mathfrak{Y}}\mathfrak{X}.$$

This shows that $(c_{\mathfrak{Y}})^2 = c_{\mathfrak{Y}}$. □

1.5. Local formations

In order to introduce the concept of a local formation, we follow the approach of [GHT71]. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes of groups. For every $G \in \mathfrak{X}$, we define $C_G(\mathfrak{Y}, p)$ to be the intersection of the centralizers of all p-principal factors U/V of G such that $G/C_G(U/V) \in \mathfrak{Y}$ and $C_G(\mathfrak{Y}, p) = G$ if no such chief factors exist. The class $\mathfrak{Y}$ is an $(\mathfrak{X}, p)$-*preformation* if $\mathfrak{Y}$ is empty or satisfies:

(PF1) $\mathfrak{Y} = Q\mathfrak{Y}$.

(PF2) For every group $G \in \mathfrak{X}$, we have $G/C_G(\mathfrak{Y}, p) \in \mathfrak{Y}$.

$\mathfrak{X}$-formations are important examples of group classes which form $(\mathfrak{X}, p)$-preformations for every prime p. Here the group class $\mathfrak{F}$ is called an $\mathfrak{X}$-formation if:

(F1) $\mathfrak{F} = Q\mathfrak{F}$.

(F2) $\mathfrak{F} = R\mathfrak{F} \cap \mathfrak{X}$.

Note that the second condition implies that $\mathfrak{F}$ is a subclass of $\mathfrak{X}$.

Let $\mathfrak{X} = Q\mathfrak{X}$ be a class of periodic locally soluble groups. A function f assigning to every prime p a (possibly empty) $(\mathfrak{X}, p)$-preformation is called an *$\mathfrak{X}$-preformation function*. The *support π of f* is the set of primes p such that $f(p)$ is nonempty. Now let $\mathfrak{F}$ be the class of all $\mathfrak{X}$-groups G such that $G/C_G(U/V) \in f(p)$ for every prime p and every p-principal factor U/V of G. The class $\mathfrak{F}$ is called the *local $\mathfrak{X}$-formation defined by the $\mathfrak{X}$-preformation function f*. A class $\mathfrak{F}$ is a *local $\mathfrak{X}$-formation* or *local formation of $\mathfrak{X}$-groups* if it is a local $\mathfrak{X}$-formation for some $\mathfrak{X}$-preformation function.

We give some useful alternative descriptions of local formations of periodic locally soluble groups.

1.5.1 Lemma. *Let $\mathfrak{X} = Q\mathfrak{X}$ be a class of periodic locally soluble groups and suppose that $\mathfrak{F}$ a local $\mathfrak{X}$-formation defined by the $\mathfrak{X}$-preformation function f and let π denote the support of G. Then the following statements about the $\mathfrak{X}$-group G are equivalent:*

(a) *$G \in \mathfrak{F}$.*

(b) *Let $\mathcal{S}$ be a chief series of G. If U/V is a p-factor of $\mathcal{S}$ for the prime p, then $G/C_G(U/V) \in f(p)$.*

(c) *G is a π-group and $G/O_{p',p}(G) \in f(p)$ for every prime $p \in \pi$.*

(d) *G is a π-group and $G \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$ for every prime $p \in \pi$.*

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are obvious.

(b) $\Rightarrow$ (c) Let $\mathcal{S}$ be a chief series of G and p a prime. If $p \notin \pi$, then $f(p)$ is empty and so G does not have p-chief factors. Since every principal factor of G is an elementary abelian p-group for some prime p (see e.g. [KW73, Corollary 1.B.4] or [Rob82, 12.5.1]), the group G is a π-group. Now let $p \in \pi$. By [GHT71, Theorem 3.8], $O_{p',p}(G)$ equals the intersection of all centralizers of the p-principal factors in $\mathcal{S}$. Thus $G/O_{p',p}(G) \in f(p)$ by (PF2).

(d) $\Rightarrow$ (a) Let $p \in \pi$ and U/V a p-principal factor of G. Then G possesses a normal subgroup $N \in \mathfrak{S}_{p'}\mathfrak{S}_p$; moreover, $G/N \in f(p)$ and N is contained in $O_{p',p}(G)$. Since $O_{p',p}(G) \leq C_G(U/V)$ by [GHT71, Theorem 3.8], $G/C_G(U/V)$ is a factor group of G/N, hence belongs to $f(p)$. □

The following lemma shows that a local $\mathfrak{X}$-formation is indeed an $\mathfrak{X}$-formation.

1.5.2 Lemma. *Let $\mathfrak{X} = Q\mathfrak{X}$ be a class of periodic locally soluble groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation. Then $\mathfrak{F} = Q\mathfrak{F} = R\mathfrak{F} \cap \mathfrak{X}$, and so $\mathfrak{F}$ is an $\mathfrak{X}$-formation.*

Proof. Let $G \in \mathfrak{F}$ and $N \trianglelefteq G$. If $p \in \pi$, then $G/O_{p',p}(G) \in f(p)$ and since $O_{p',p}(G)N/N \leq O_{p',p}(G/N)$, the factor group $(G/N)/O_{p',p}(G/N)$ is also an $f(p)$-group. Therefore $G/N \in \mathfrak{F}$ by Lemma 1.5.1.

Now assume that $\mathcal{N}$ is a set of normal subgroups of the $\mathfrak{X}$-group G such that $G/N \in \mathfrak{F}$ for every $N \in \mathcal{N}$ and let π denote the characteristic of $\mathfrak{F}$. Then G/N is a π-group for every $N \in \mathcal{N}$, and since G is periodic, G is likewise is a π-group. Let $p \in \pi$ and put $C = C_G(f(p), p)$. Now set $L_N/N = O_{p',p}(G/N)$ for every $N \in \mathcal{N}$. Since $G/L_N \in f(p)$ for every $N \in \mathcal{N}$, by [GHT71, Theorem 3.8], we have $G/C_G(U/V) \in f(p)$ for every p-principal factor U/V of G with $N \leq V$. This shows that $C \leq L_N$ for every $N \in \mathcal{N}$. Since G is periodic, we have $\bigcap_{N \in \mathcal{N}} L_n = O_{p',p}(G)$ and so $C \leq O_{p',p}(G)$. Moreover, by the definition of C, we have $G/C \in f(p)$ and so $G/O_{p',p}(G) \in f(p)$. Since this holds for every $p \in \pi$, we have $G \in \mathfrak{F}$ by Lemma 1.5.1. □

We mention a number of elementary yet useful properties of local formations.

1.5.3 Lemma. *Let $\mathfrak{X} = Q\mathfrak{X}$ be a class of periodic locally soluble groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. If G is an $\mathfrak{X}$-group such that $G/Z(G) \in \mathfrak{F}$ and G is a π-group, then $G \in \mathfrak{F}$.*

Proof. Suppose that f is an $\mathfrak{X}$-preformation function defining $\mathfrak{F}$. Refine the series $1 \trianglelefteq Z(G) \trianglelefteq G$ to a chief series $\mathcal{S}$ of G and let U/V be a p-principal factor of $\mathcal{S}$ for some $p \in \pi$. If $U \leq Z(G)$, then $C_G(U/V) = G$ and so $G/C_G(U/V) \in f(p)$. Otherwise, $Z(G) \leq V$, and since $C_{G/Z(G)}(U/V) = C_G(U/V)/Z(G)$ and $G/Z(G) \in \mathfrak{F}$, we have $G/C_G(U/V) \in f(p)$ by an isomorphism theorem. Therefore $G \in \mathfrak{F}$ by Lemma 1.5.1. □

The next elementary lemma will be needed later.

1.5.4 Lemma. *Let $\mathfrak{X} = Q\mathfrak{X}$ be a class of periodic locally soluble group and assume that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation of characteristic π. Further, let G be an $\mathfrak{X}$-group and N a normal p-subgroup of G such that $G/N \in \mathfrak{F}$. If $p \in \pi$ and $O_{p'}(G/N) = 1$, then $G \in \mathfrak{F}$.*

Proof. Since N and G/N are π-groups, also G is a π-group. If $q \in \pi$ and $q \neq p$, then $N \leq O_{q'}(G)$ and so $O_{q'}(G/N) = O_{q'}(G)/N$. Thus $G \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$. Moreover, $O_{p'}(G)N/N \leq$

$O_{p'}(G/N) = 1$ and so $O_{p'}(G) = 1$. It follows that $O_{p',p}(G) = O_p(G)$ and so $O_{p',p}(G)/N = O_{p',p}(G/N)$. Consequently, $G \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$ and hence G belongs to $\mathfrak{F}$. □

Sometimes it will be necessary to restrict the universe $\mathfrak{X}$ of a local $\mathfrak{X}$-formation $\mathfrak{F}$. The restriction of $\mathfrak{F}$ to that universe is again a local formation:

1.5.5 Lemma. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two* QS*-closed classes of groups such that $\mathfrak{Y} \subseteq \mathfrak{X}$. If $\mathfrak{F}$ is a local $\mathfrak{X}$-formation of characteristic π defined by the preformation function f, then $\mathfrak{G} = \mathfrak{F} \cap \mathfrak{Y}$ is a local $\mathfrak{Y}$-formation. Moreover, the function g defined by $g(p) = f(p) \cap \mathfrak{Y}$ for all primes is a local definition of $\mathfrak{G}$.*

Proof. Since is easy to see that $g(p)$ is a preformation function for every $p \in \pi$, we only have to show that $\mathfrak{G}$ is locally defined by g. Let $G \in \mathfrak{G}$ and $p \in \pi$, then $G/O_{p',p}(G) \in f(p) \cap Q\mathfrak{Y} = g(p)$ and G belongs to the local $\mathfrak{Y}$-formation defined by g. Conversely, suppose that the $\mathfrak{Y}$-group G satisfies $G/O_{p',p}(G) \in g(p)$ for every $p \in \pi$, then $G/O_{p',p}(G) \in f(p)$ and so $G \in \mathfrak{F} \cap \mathfrak{Y} = \mathfrak{G}$ by Lemma 1.5.1. □

Let $\mathfrak{F}$ be a local $\mathfrak{X}$-formation, then in view of Zorn's lemma, the following proposition shows that every $\mathfrak{F}$-group supplementing the Hirsch-Plotkin radical of a group G is contained in a maximal $\mathfrak{F}$-subgroup of G.

1.5.6 Proposition. *Let $\mathfrak{X}$ be a* QS*-closed class of periodic locally finite groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation. Suppose that the $\mathfrak{X}$-group G is the union of an ascending chain $\mathcal{C}$ of $\mathfrak{F}$-subgroups. If, for every prime p, there exists $S \in \mathcal{C}$ which supplements $O_{p',p}(G)$, then G is an $\mathfrak{F}$-group.*

Proof. Suppose that f is an $\mathfrak{X}$-preformation function for $\mathfrak{F}$ and let p be a prime and U/V a p-principal factor of G. We have to show that $G/C \in f(p)$, where $C = C_G(U/V)$. Hence we may assume without loss of generality that $SO_{p',p}(G) = G$ for every $S \in \mathcal{C}$.

By [GHT71, Theorem 3.8], U/V is centralized by $O_{p',p}(G)$. Let $S \in \mathcal{C}$, then $(U \cap S)V/V$ is normalized by S and centralized by $O_{p',p}(G)$, hence is normal in $SO_{p',p}(G) = G$. Since $U = \bigcup_{S\in\mathcal{C}}(U \cap S)$ and U/V is a principal factor of G, there exists an $S \in \mathcal{C}$ such that $U = (U \cap S)V$. Let K be a normal subgroup of S with $V \cap S \leq K < U \cap S$, then also $KV/V < U/V$ is centralized by $O_{p',p}(G)$, hence is normal in $G = SO_{p',p}(G)$ and so $K = S \cap V$. This shows that $(U \cap S)/(V \cap S)$ is a principal factor of S. Therefore $O_{p',p}(S)$ centralizes $(U \cap S)/(V \cap S)$. Since U/V is S-isomorphic with $(U \cap S)/(V \cap S)$, the subgroup $O_{p',p}(S)$ is contained in $C = C_G(U/V)$. By an isomorphism theorem, we have $G/C \cong S/S \cap C$, and since $S/O_{p',p}(S) \in f(p)$ by hypothesis, we also have $G/C \in f(p)$, as required. □

1.6. Projectors and injectors

Let $\mathfrak{X}$ be a class of groups and G any group. An $\mathfrak{X}$-subgroup X of G is called *$\mathfrak{X}$-maximal (in G)* if, whenever Y is an $\mathfrak{X}$-subgroup of G containing X, then $X = Y$; in other words, X is $\mathfrak{X}$-maximal if it is an $\mathfrak{X}$-group and X is not properly contained in any $\mathfrak{X}$-subgroup of G.

Following [DH92], a subgroup X of G is called *$\mathfrak{X}$-projector of G* if XN/N is an $\mathfrak{X}$-maximal subgroup of G/N for every normal subgroup N of G. A subgroup X of G is called an *$\mathfrak{X}$-covering subgroup of G* if X is an $\mathfrak{X}$-projector of H for every subgroup H of G which contains X. Note that our terminology differs from that used in [Dix82], [GHT71] or [Kli75], whose definition of an $\mathfrak{X}$-projector coincides with our definition of an $\mathfrak{X}$-covering subgroup. However, since the $\mathfrak{X}$-projectors which will occur in the sequel are in fact $\mathfrak{X}$-covering subgroups, there should be no danger of confusion.

Let $\mathfrak{X} = \mathrm{QS}\mathfrak{X}$, $G \in \mathfrak{X}$ and $\mathfrak{F}$ a local $\mathfrak{X}$-formation. If $\mathcal{N}$ a set of normal subgroups of G, the following well-known lemma shows in particular that

$$H\Big(\bigcap_{N\in\mathcal{N}} N\Big) = \bigcap_{N\in\mathcal{N}} HN$$

for every $\mathfrak{F}$-projector H of G.

1.6.1 Lemma. *Let $\mathfrak{X}$ be a* QS*-closed class of periodic locally soluble groups and assume that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation. Further, let H be an $\mathfrak{F}$-maximal subgroup of the $\mathfrak{X}$-group G and assume that $\mathcal{N}$ is a set of normal subgroups of G such that the intersection of all $N \in \mathcal{N}$ is trivial. Then*

$$H = \bigcap_{N\in\mathcal{N}} HN.$$

Proof. Let $L = \bigcap_{N\in\mathcal{N}} HN$, then $LN = HN$ for every $N \in \mathcal{N}$. This shows that $L/L\cap N \cong LN/N = HN/N \in \mathfrak{F}$. Therefore by Lemma 1.5.2, we have $L \in \mathfrak{F}$. Since H is contained in L and H is $\mathfrak{F}$-maximal, we have $H = L$, as required. □

Let G be a group. A nonempty set $\mathcal{F}$ of subgroups of G is called a *Fitting set of* G if it satisfies the following conditions:

(FS1) If $U \in \mathcal{F}$ and V is a serial subgroup of U, then $V \in \mathcal{F}$.

(FS2) If $\mathcal{S}$ is a set of $\mathcal{F}$-subgroups and every $V \in \mathcal{S}$ is serial in the subgroup U generated by all $V \in \mathcal{S}$, then U belongs to $\mathcal{F}$.

(FS3) If $U \in \mathcal{F}$ and $g \in G$, then $U^g \in \mathcal{F}$.

If $\mathcal{F}$ is a Fitting set of the group G, then the subgroup $G_{\mathcal{F}}$ of G generated by all serial $\mathcal{F}$-subgroups of G is called the *$\mathcal{F}$-radical of* G; note that by (FS2), $G_{\mathcal{F}}$ is an $\mathcal{F}$-subgroup of G. A subgroup I of G such that $I \cap S$ is an $\mathcal{F}$-maximal subgroup of S for every serial subgroup of G is called an *$\mathcal{F}$-injector of* G.

Let $\mathcal{F}$ be a Fitting set of the group G. If H is a subgroup of G, then the set

$$\mathcal{F}_H = \{U \mid U \le H, U \in \mathcal{F}\}$$

is a Fitting set of H. If there is no ambiguity, we will usually omit the reference to H and call the $\mathcal{F}_H$-injectors of H simply $\mathcal{F}$-injectors of H. The $\mathcal{F}_H$-radical of H is then just called the $\mathcal{F}_H$-radical of H and will be denoted with $H_{\mathcal{F}}$.

If G is a finite soluble group, these definitions evidently coincide with those introduced by Anderson in [And75]. Moreover, if G is a locally soluble FC-group, then it is easy to see that our definition of a Fitting set agrees with that of Beidleman and Karbe in [BK86]; (see [Ens90], *Bemerkung* 2.2 and *Bemerkung* 2.9).

Our first lemma shows in particular that for every Fitting set of a group G, the $\mathcal{F}$-radical is contained in every $\mathcal{F}$-injector of G.

1.6.2 Lemma. *Let $\mathcal{F}$ be a Fitting set of the group G and suppose that I is an $\mathcal{F}$-injector of G. Then $G_{\mathcal{F}}$ equals the core of I in G.*

Proof. Let N denote the core of I in G. Then $N \in \mathcal{F}$ by (FS1). Therefore $N \leq G_{\mathcal{F}}$ by (FS2). On the other hand, $I \cap G_{\mathcal{F}}$ is an $\mathcal{F}$-maximal subgroup of the $\mathcal{F}$-group $G_{\mathcal{F}}$ and so $G_{\mathcal{F}} \leq I$, as required. □

The next lemma shows how injectors and the radical of a group G are related to the corresponding injectors and radicals of certain subgroups of G.

1.6.3 Lemma. *Let G be a group and $\mathcal{F}$ a Fitting set of G. Suppose that X is a subgroup X of G which possesses a set $\mathcal{S}$ of subgroups which are serial in G and whose union equals X. Then the $\mathcal{F}$-radical $X_{\mathcal{F}}$ equals the union of the subgroups $S_{\mathcal{F}}$, where $S \in \mathcal{S}$, and $X_{\mathcal{F}} = X \cap G_{\mathcal{F}}$. Moreover, if G possesses an $\mathcal{F}$-injector I, then $X \cap I$ is the union of the $\mathcal{F}$-injectors $S \cap I$ of S, where $S \in \mathcal{S}$, and $X \cap I$ is an $\mathcal{F}$-injector of X.*

Proof. If S is a serial subgroup of G contained in X, then S is also serial in X. Therefore $S \cap X_{\mathcal{F}} = S_{\mathcal{F}} = S \cap G_{\mathcal{F}} = S \cap X \cap G_{\mathcal{F}}$. Since every $g \in X$ is contained in such a subgroup S of X, it follows that $X_{\mathcal{F}} = X \cap G_{\mathcal{F}}$. A similar argument can be used to prove the statement about $\mathcal{F}$-injectors. □

Let $\mathfrak{X}$ and $\mathfrak{F}$ be classes of groups. $\mathfrak{F}$ is called an $\mathfrak{X}$-*Fitting class* (or *Fitting class of* $\mathfrak{X}$-*groups*) if, for every $G \in \mathfrak{X}$, the set $\mathcal{F}$ of all $\mathfrak{F}$-subgroups of G forms a Fitting set of G. In this case, an $\mathfrak{F}$-injector of G is simply an $\mathcal{F}$-injector and the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ equals $G_{\mathcal{F}}$.

Chapter 2

Prefactorized Sylow subgroups and Sylow bases of products

2.1. Prefactorized Sylow subgroups

The following section is concerned with finding prefactorized Sylow π-subgroups of a group G which is the product of two subgroups A and B which have normal Sylow π- and π'-subgroups A_π, $A_{\pi'}$, B_π and $B_{\pi'}$, respectively. The finite case suggests that the sets $A_\pi B_\pi$ and $A_{\pi'} B_{\pi'}$ are natural candidates for prefactorized Sylow π- and π'-subgroups of G, although even if $<A_\pi, B_\pi>$ is not a π-group, then there may nevertheless be subsets A_0 and B_0 of A_π and B_π, respectively, such that $A_0 B_0$ is a Sylow π-subgroup of G. For instance, in Example 2.1.8 below, the subgroups A_p and B_p themselves are Sylow p-subgroups of G, while $G = <A_p, B_p>$ is not a p-group. However, in the sequel, we will only investigate the question under which hypotheses the product of the π-components of A and B is a Sylow π-subgroup of G.

If a product G of two subgroups is, in addition, the direct product of a π-group and a π'-group, the existence of a prefactorized Sylow π-subgroup can be proved using the following elementary lemma.

2.1.1 Lemma. *Suppose that the group $G = M \times N$ is the product of two subgroups A and B. If $A = (A \cap M)(A \cap N)$ and $B = (B \cap M)(B \cap N)$, then $M = (M \cap A)(M \cap B)$.*

Proof. Clearly, $G = AB = (A \cap M)(B \cap M)N$. Therefore $M = M \cap (A \cap M)(B \cap M)N$ and so by Lemma 1.1.7, $M = (A \cap M)(B \cap M)(M \cap N) = (A \cap M)(B \cap M)$ as required. □

In particular, if $\pi(M)$ and $\pi(N)$ are disjoint, this leads to:

2.1.2 Corollary. *Let G be a group and suppose that G is the direct product of a normal Sylow π-subgroup G_π and a normal Sylow π'-subgroup $G_{\pi'}$. If $G = AB$ for two subgroups A and B, then $G_\pi = A_\pi B_\pi$ and $G_{\pi'} = A_{\pi'} B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are normal Sylow π- and Sylow π'-subgroups of A and B, respectively.*

Thus we obtain a first result about Sylow bases of periodic locally nilpotent products.

2.1.3 Corollary. *Suppose that the periodic locally nilpotent group G is the product of two subgroups A and B. If π is a set of primes, then the set $\{A_p B_p \mid p \in \mathbb{P}\}$ is the unique Sylow basis of G.*

The next proposition states some criteria for a periodic product of two subgroups to have prefactorized Sylow subgroups.

2.1.4 Proposition. *Suppose that the periodic group G is the product of two subgroups A and B and that $A = A_\pi A_{\pi'}$ and $B = B_\pi B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively.*

(a) *(N. S. Černikov [Cer82, Lemma 2]) If $<A_\pi, B_\pi>$ is a π-group and $<A_{\pi'}, B_{\pi'}>$ is a π'-group, then $A_\pi B_\pi N/N = B_\pi A_\pi N/N$ is a Sylow π-subgroup of G/N for every normal subgroup N of G (and $A_{\pi'} B_{\pi'} N/N$ is a Sylow π'-subgroup of G/N).*

(b) *If $\mathcal{N}$ is a set of normal subgroups of G such that $\bigcap_{N\in\mathcal{N}} N = 1$ and for every $N \in \mathcal{N}$, the subgroups $<A_\pi, B_\pi>N/N$ and $<A_{\pi'}, B_{\pi'}>N/N$ are a π-and a π'-subgroup of G/N, respectively, then $A_\pi B_\pi$ and $A_{\pi'} B_{\pi'}$ are a Sylow π- and a π'-subgroup of G.*

(c) *If G is locally finite, $N \le Z(G)$ and $<A_\pi, B_\pi>N/N$ and $<A_{\pi'}, B_{\pi'}>N/N$ are a π- and a π'-subgroups of G/N, respectively, then $A_\pi B_\pi$ and $A_{\pi'} B_{\pi'}$ are a Sylow π- and a π'-subgroup of G.*

Proof. (a) Since the hypotheses are inherited by every factor group G/N of G, it clearly suffices to consider the case when $N = 1$. Now assume that the π-group $<A_\pi, B_\pi>$ is contained in a π-group P of G and let $g \in P$. Since $G = AB = A_\pi A_{\pi'} B_{\pi'} B_\pi$, the element g can be written as $g = a_\pi a_{\pi'} b_{\pi'} b_\pi$, where $a_\pi \in A_\pi$, $a_{\pi'} \in A_{\pi'}$, $b_\pi \in B_\pi$ and $b_{\pi'} \in B_{\pi'}$. Therefore $a_{\pi'} b_{\pi'} = a_\pi^{-1} g b_\pi^{-1}$ is contained in $P \cap <A_{\pi'}, B_{\pi'}> = 1$. Hence $g = a_\pi b_\pi$ is contained in the set $A_\pi B_\pi$ and so $<A_\pi, B_\pi> = A_\pi B_\pi$ is a Sylow π-subgroup of G. A similar argument shows that $A_{\pi'} B_{\pi'}$ is a Sylow π'-subgroup of G.

(b) Let $S = <A_\pi, B_\pi>$, then by hypothesis, $S/S \cap N \cong SN/N$ is a π-group for every $N \in \mathcal{N}$. Since G is periodic, this shows that S is a π-group. Similarly, $<A_{\pi'}, B_{\pi'}>$ is a π'-group. Now the result follows from (a).

(c) Let $H = <A_\pi, B_\pi>N$, then H has a normal Sylow π-subgroup H_π by Lemma 1.2.1. Since $A_\pi H_\pi$ and $B_\pi H_\pi$ are π-subgroups of H, it follows that $<A_\pi, B_\pi> \le H_\pi$ is a π-group. Similarly, $<A_{\pi'}, B_{\pi'}>$ is a π'-group, and the desired result follows from (a). □

Remark. Note that in Proposition 2.1.4 (a), we do *not* claim that the Sylow subgroups $A_\pi B_\pi$ and $A_{\pi'} B_{\pi'}$ of $G = AB$ permute. See Theorem 2.2.5 for an additional hypothesis which ensures that $G = (A_\pi B_\pi)(A_{\pi'} B_{\pi'})$.

Next, we investigate the question under which hypotheses the π-radical $O_\pi(G)$ of a product of two subgroups is prefactorized.

2.1.5 Lemma. *Let π be a set of primes and suppose that the group G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π-and π'-subgroups of A and B, respectively. Further, assume that $A_\pi B_\pi$ is a π-subgroup of G. If N is a normal π-subgroup of G contained in $A_\pi B_\pi$, then the factorizer of N is the direct product of its maximal π-subgroup $A_\pi N \cap B_\pi N$ and its maximal π'-subgroup $A_{\pi'} \cap B_{\pi'}$.*

Proof. Let $X = AN \cap BN$ denote the factorizer of N and put $P = A_\pi N \cap B_\pi N$. Then P is a subgroup of $A_\pi B_\pi$, hence is a factorized subgroup of $A_\pi B_\pi$. Therefore $A_{\pi'} \cap B_{\pi'}$ centralizes $P = (P \cap A_\pi)(P \cap B_\pi)$ and so $Y = (A_{\pi'} \cap B_{\pi'}) \times P = (A_{\pi'} \cap B_{\pi'})(P \cap A)(P \cap B)$ is a prefactorized subgroup of G. Since $A = A_\pi \times A_{\pi'}$, we also have $A \cap B = (A_{\pi'} \cap B_{\pi'})(A_\pi \cap B_\pi)$ and so Y contains $A \cap B$. Since $N \le Y \le X$ and X is the smallest factorized subgroup of G that contains N, we have $Y = X$. □

Observe that the following corollary holds in particular if $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N for every normal subgroup N of G.

2.1.6 Corollary. *Let π be a set of primes and suppose that the group G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively. If the set $A_\pi B_\pi$ is a π-subgroup of G which contains $O_\pi(G)$, then the factorizer $X = AO_\pi(G) \cap BO_\pi(G)$ of $O_\pi(G)$ is the direct product of its maximal π-subgroup $A_\pi O_\pi(G) \cap B_\pi O_\pi(G)$ and its maximal π'-subgroup $A_{\pi'} \cap B_{\pi'}$. Moreover, if $O_{\pi',\pi}(G)$ is contained in $A_\pi B_\pi O_{\pi'}(G)$, then the factorizer of $O_{\pi',\pi}(G)$ is an extension of a π'-group by a π-group.*

Proof. Put $N = O_\pi(G)$, then the first statement follows directly from Lemma 2.1.5. Now let X denote the factorizer of $O_{\pi',\pi}(G)$, then by Proposition 1.1.3, $X/O_{\pi'}(G)$ is the factorizer of $O_{\pi',\pi}(G)/O_{\pi'}(G) = O_\pi(G/O_{\pi'}(G))$ in $G/O_{\pi'}(G)$, hence is an extension of a π'-group by a π-group, as required. □

We mention one important special case when a product of two subgroups possesses prefactorized Sylow π-subgroups. Note that Proposition 2.1.7 holds in particular if the Sylow π- and π'-subgroups of G are conjugate.

2.1.7 Proposition. *Let π be a set of primes and suppose that the periodic group G is the product of two subgroups A and B which are the product of their Sylow π- and π'-groups A_π, $A_{\pi'}$, B_π and $B_{\pi'}$. If A_π and B_π are contained in conjugate Sylow π-subgroups of G and $A_{\pi'}$ and $B_{\pi'}$ are contained in conjugate Sylow π'-subgroups of G, then there exist $a \in A$ and $b \in B$ such that $A = A_\pi^a A_{\pi'}^a$, $B = B_\pi^b B_{\pi'}^b$, and furthermore, $A_\pi^a B_\pi^b$ is a Sylow π-subgroup of G and $A_{\pi'}^a B_{\pi'}^b$ is a Sylow π'-subgroup of G.*

Proof. By hypothesis, there exists a Sylow π-subgroup G_π of G and an element $g \in G$ such that G_π and G_π^g contain A_π and B_π, respectively. Since $G = AB$, there exist $a_1 \in A$ and $b_1 \in B$ such that $g = a_1 b_1^{-1}$. Consequently, $A_\pi^{a_1}$ is contained in $G_\pi^{a_1}$, and also $B_\pi^{b_1}$ is a subgroup of $G_\pi^{gb_1} = G_\pi^{a_1}$. Therefore $<A_\pi^{a_1}, B_\pi^{b_1}>$ is a π-group. Since $A = A_\pi A_{\pi'}$ and $B = B_\pi B_{\pi'}$, the elements a_1 and b_1 may clearly be chosen from $A_{\pi'}$ and $B_{\pi'}$, respectively.

As $A_{\pi'}$ and $B_{\pi'}$ are contained in conjugate Sylow π'-subgroups of G, the same also holds for $A_{\pi'}^{a_1}$ and $B_{\pi'}^{b_1}$. Now a similar argument, applied to the π'-subgroups $A_{\pi'}^{a_1}$ and $B_{\pi'}^{b_1}$ of $A = A_\pi^{a_1} A_{\pi'}^{a_1}$ and $B = B_\pi^{b_1} B_{\pi'}^{b_1}$, respectively, yields that there exist $a_2 \in A_\pi^{a_1}$ and $b_2 \in B_\pi^{b_1}$, such that $<A_{\pi'}^{a_1 a_2}, B_{\pi'}^{b_1 b_2}>$ is a π'-group. Observing that $A_\pi^{a_1} = A_\pi^{a_1 a_2}$ and $B_\pi^{b_1} = B_\pi^{b_1 b_2}$, it is now clear that $a = a_1 a_2$ and $b = b_1 b_2$ are the required elements of A and B, respectively.

The rest of the proposition now follows from Proposition 2.1.4 (a). □

The following example shows that in Proposition 2.1.7, it does not suffice to assume that A_π and B_π are contained in locally conjugate Sylow π-subgroups of G.

2.1.8 Example. Let p be a prime. By [Sys95, Corollary 1], there exists a locally finite group $G = AB = A \ltimes M = B \ltimes M$, where A and B are residually finite p-groups and M is an elementary abelian q-group for a prime $q \neq p$. In particular, G is a periodic radical group.

We show that A and B are locally conjugate. Let $\delta : A \to M$ be the surjective derivation constructed in the proof of [Sys95, Theorem 3A], written multiplicatively. Then $(a_1 a_2)^\delta = (a_1^\delta)^{a_2} a_2^\delta$ for all a_1, $a_2 \in A$, and by construction, there exist elements $v_1, v_2, \ldots$ of M such that for every $a \in A$, there exists an integer $n \in \mathbb{N}$ such that for every integer $m \geq n$, we have $a^\delta = [v_m, a]$.

Let $g \in G$, then g can be written in a unique way as $g = am$, where $a \in A$ and $m \in M$. We define a map $\phi : G \to G$ by $g^\phi = aa^\delta m$. If $g^\phi = 1$, then $a = 1$ and $a^\delta m = 1$, and since $(a \cdot 1)^\delta = (a^\delta)^1 1^\delta$, it follows that $m = 1$. To see that ϕ is a homomorphism, let $g_1 = a_1 m_1$ and $g_2 = a_2 m_2$ be elements of G, where a_1, $a_2 \in A$ and m_1, $m_2 \in M$. Then

$$\begin{aligned}(a_1 m_1 a_2 m_2)^\phi &= (a_1 a_2 m_1^{a_2} m_2)^\phi = a_1 a_2 (a_1 a_2)^\delta m_1^{a_2} m_2 = a_1 a_2 (a_1^\delta)^{a_2} a_2^\delta m_1^{a_2} m_2 \\ &= a_1 a_2 (a_1^\delta)^{a_2} m_1^{a_2} a_2^\delta m_2 = a_1 a_1^\delta m_1 a_2 a_2^\delta m_2 = (a_1 m_1)^\phi (a_2 m_2)^\phi.\end{aligned}$$

Now let g be an element of G. Since $G = BM$, there exists $b \in B$ and $m \in M$ such that $g = bm$. Since $B = \{aa^\delta \mid a \in A\}$, there exists $a \in A$ such that $b = aa^\delta$. Thus we have $g = (am)^\phi$, and so ϕ is an automorphism of G. We show that ϕ is locally inner. Let $g_1, \ldots, g_n$ be elements of G and write $g_i = a_i m_i$, where $a_i \in A$ and $m_i \in M$. By construction, we have $a_i^\delta = [v_{n_i}, a_i]$ for suitable integers $n_i \in \mathbb{N}$. Let n denote the maximum of the n_i, then also $a_i^\delta = [v_n, a_i] = v_n^{a_i} v_n^{-1}$ and so

$$= v_n a_i v_n^{-1} m_i = a_i^{v_n^{-1}} m_i^{v_n^{-1}} = g_i^{v_n^{-1}},$$

as required.

Next, we show that a prefactorized subgroup of a periodic product of two locally nilpotent groups having prefactorized Sylow π- and π'-subgroups likewise has prefactorized Sylow π- and π'-subgroups.

2.1.9 Proposition. *Suppose that the group G is the product of its subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are Sylow π- and Sylow π'-subgroups of A and B, respectively. Further, assume that $<A_\pi, B_\pi>$ and $<A_{\pi'}, B_{\pi'}>$ are a π- and a π'-subgroup of G. If S is a prefactorized subgroup of G, then $(S \cap A_\pi)(S \cap B_\pi) = S \cap A_\pi B_\pi$ and $(S \cap A_{\pi'})(S \cap B_{\pi'}) = S \cap A_{\pi'} B_{\pi'}$ are a Sylow π- and a Sylow π'-subgroup of S. Hence the Sylow subgroups $A_\pi B_\pi$ and $A_{\pi'} B_{\pi'}$ reduce into S.*

Proof. Clearly, $S \cap A = (S \cap A_\pi)(S \cap A_{\pi'})$ and $S \cap B = (S \cap B_\pi)(S \cap B_{\pi'})$. Since $<S \cap A_\pi, S \cap B_\pi> \leq A_\pi B_\pi$ is a π-group and $<S \cap A_{\pi'}, S \cap B_{\pi'}>$ is a π'-group, the subgroup $S = (S \cap A)(S \cap B)$ satisfies the hypotheses of Proposition 2.1.4 (a). Therefore $(S \cap A_\pi)(S \cap B_\pi)$ is a Sylow π-subgroup of S. Since this Sylow subgroup is contained in the π-subgroup $S \cap A_\pi B_\pi$ of G, it follows that $(S \cap A_\pi)(S \cap B_\pi) = S \cap A_\pi B_\pi$. The corresponding result about $A_{\pi'} B_{\pi'}$ follows by exchanging π and π'. □

In particular, Proposition 2.1.9 can be used to prove the uniqueness of a prefactorized Sylow π-subgroup of the form $A_\pi B_\pi$.

2.1.10 Corollary. *Suppose that the group G is the product of its subgroups A and B. Further, assume that $A = A_\pi \times A_{\pi'}$, $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are Sylow π- and Sylow π'-subgroups of A and B, respectively, and that $<A_\pi, B_\pi>$ and $<A_{\pi'}, B_{\pi'}>$ are a π- and a π'-subgroup of G. Then G possesses a unique prefactorized Sylow π-subgroup and a unique prefactorized Sylow π'-subgroup, namely $A_\pi B_\pi$ and $A_{\pi'} B_{\pi'}$.*

Proof. Let S be a prefactorized Sylow π-subgroup of G, then by Proposition 2.1.9, the subgroup $S \cap A_\pi B_\pi$ is a Sylow π-subgroup of S. Thus $S \leq A_\pi B_\pi$ and $S = A_\pi B_\pi$, since $A_\pi B_\pi$

is a π-group by Proposition 2.1.4 (a). Therefore $A_\pi B_\pi$ is the unique prefactorized Sylow π-subgroup of G. The statement about $A_{\pi'} B_{\pi'}$ follows by exchanging π and π'. □

2.2. Permutable Sylow subgroups of π-separable groups

Suppose that the group G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π-and π'-subgroups of A and B, respectively. In the preceding section, we have established certain conditions for the sets $A_\pi B_\pi$ and $A_{\pi'} B_{\pi'}$ to be Sylow π- and Sylow π'-subgroups of G. In this section, we will show that under the additional hypothesis that G has an ascending series whose factors are π- and π'-groups, the group G is the product of its Sylow π-subgroup $A_\pi B_\pi$ and its Sylow π'-subgroup $A_{\pi'} B_{\pi'}$. It seems to be an open question whether this is true in general.

The next lemma is probably known. It generalizes a result about π-soluble finite groups; see [HH56, Lemma 1.2.3]. Recall that a group is π-soluble if it has a finite series whose factors are π'-groups or soluble π-groups.

2.2.1 Lemma. *Let π be a set of primes such that the locally finite group G has an ascending series whose factors are either π-groups or π'-groups. If every finite subgroup of G is either π-soluble or π'-soluble, then $C_G(O_{\pi',\pi}(G)) \leq O_{\pi',\pi}(G)$.*

Proof. Clearly, we may assume without loss of generality that $O_{\pi'}(G) = 1$. Let $C = C_G(O_\pi(G))$ and define $P/O_\pi(C) = O_{\pi'}(C/O_\pi(C))$, then C, and hence P, are normal subgroups of G. Let g, $h \in P$ be π'-elements and put $F = <g, h>$, then F is finite. Now $F/F \cap O_\pi(C) \cong FO_\pi(C)/O_\pi(C) \leq P/O_\pi(C)$ is a π'-group. By the Schur-Zassenhaus theorem, g and h are contained in conjugate Hall π'-subgroups $F_{\pi'}$ and $F_{\pi'}^x$ of F, where $x \in F$. Since $F = F_{\pi'}(F \cap O_\pi(C))$, we may assume that $x \in O_\pi(C)$, and because $h \in P \leq C_G(O_\pi(C))$, we have $h^x = h$. So $<g, h>$ is contained in the π'-group $F_{\pi'}$. This shows that the subgroup Q generated by the π'-elements of P is a π'-subgroup of P, hence is a characteristic π'-subgroup of P. It follows that $Q \leq O_{\pi'}(G) = 1$, and so we have $O_{\pi'}(C/O_\pi(C)) = 1$. On the other hand, G, and hence C, possesses an ascending series whose factors are either π- or π'-groups. Since also $O_\pi(C/O_\pi(C)) = 1$, we must have $C = O_\pi(C)$, and so C is contained in $O_\pi(G)$. □

From this, we can derive a criterion for the characteristic subgroups $O_\pi(G)$ and $O_{\pi'}(G)$ of a product G of two groups to be prefactorized.

2.2.2 Proposition. *Let π be a set of primes and suppose that G is a locally finite group which has an ascending series whose factors are π-groups or π'-groups. Further, assume that G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively. If $<A_\pi, B_\pi>$ and $<A_{\pi'}, B_{\pi'}>$ are a π-group and a π'-group, respectively, then $O_{\pi',\pi}(G)$ is factorized. Moreover, $O_\pi(G)$ is a factorized subgroup of the Sylow π-subgroup $A_\pi B_\pi$. Hence $O_\pi(G)$ is a prefactorized subgroup of $G = AB$.*

Proof. First, we show that $O_{\pi',\pi}(G)$ is factorized. By Proposition 2.1.4, $A_\pi B_\pi N/N$ and $A_{\pi'} B_{\pi'} N/N$ are a maximal π-subgroup and a maximal π'-subgroup of G/N for every normal subgroup N of G. Thus it suffices to consider the case when $O_{\pi'}(G) = 1$. Now by Corollary 2.1.6, $A_{\pi'} \cap B_{\pi'}$ centralizes $O_\pi(G)$ and so by Lemma 2.2.1, $A_{\pi'} \cap B_{\pi'} = 1$. Therefore the factorizer X of $O_\pi(G)$ is a π-group. Exchanging the roles of π and π', it follows by the same arguments that $Y/O_\pi(G)$ is a π'-group, where Y is the factorizer of $O_{\pi,\pi'}(G)$. Since the factorized subgroup Y contains $O_\pi(G)$, we have $X \leq Y$, and so the π-group X must be contained in $O_\pi(G)$. Hence $O_\pi(G)$ is factorized.

To prove the second statement, observe that, exchanging the roles of π and π', it follows from the first part that, applied to $G/O_\pi(G)$, and Proposition 1.1.3 that $O_{\pi,\pi'}(G)$ is factorized. Therefore $O_\pi(G) = A_\pi B_\pi \cap O_{\pi,\pi'}(G)$ is a factorized subgroup of $A_\pi B_\pi$, hence is a prefactorized subgroup of G by Proposition 1.1.3. □

Now suppose that the locally finite group G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Further, assume that the subgroups $O_\pi(G)$ and $O_{\pi'}(G)$ are prefactorized. In the sequel, we will examine in how far this can be used to show that G possesses prefactorized Sylow π- and π'-subgroups. In doing this, we generalize an approach used in [FGS94]. The proof of the following lemma is derived from that of [FGS94, Lemma 2.2].

2.2.3 Lemma. *Let π be a set of primes and suppose that the locally finite group G possesses an ascending series whose factors are π-groups or π'-groups. Further, assume that G is the product of a π-group A and a π'-group B. If Γ is a finite group of automorphisms of G and X is a finite subset of G, then there exists a finite factorized Γ-invariant subgroup of G containing X.*

Proof. Observe first that $A \cap B = 1$, so that every prefactorized subgroup of G is factorized. By hypothesis, there exists an ordinal β such that G possesses an ascending series

$$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_\beta = G$$

whose factors are π- or π'-groups, and clearly the G_i may be assumed characteristic in G. Since AN/N and BN/N are maximal π- and π'-subgroups of G/N for every normal subgroup N of G, we show by induction on α that every G_α is factorized: if α is a limit ordinal, we have

$$G_\alpha = \bigcup_{\beta<\alpha} G_\beta = \bigcup_{\beta<\alpha} (A \cap G_\beta)(B \cap G_\beta)$$

which is clearly contained in $(A \cap G_\alpha)(B \cap G_\alpha)$. Therefore suppose that $\alpha - 1$ exists. If $G_\alpha/G_{\alpha-1}$ is a π-group, then $G_\alpha \leq AG_{\alpha-1}$. Therefore $G_\alpha = G_\alpha \cap AG_{\alpha-1} = (G_\alpha \cap A)G_{\alpha-1}$, and since $G_{\alpha-1}$ is factorized by hypothesis, G_α is contained in $(A \cap G_\alpha)(B \cap G_\alpha)$. Thus G_α is factorized. Otherwise, the π'-group $G_\alpha/G_{\alpha-1}$ is contained in $G_{\alpha-1}B$, and a similar argument shows that G_α is factorized also in that last case.

Now let α be the least ordinal such that G possesses a finite Γ-invariant subgroup K containing X such that KG_α is factorized, and assume that $\alpha > 0$. By the modular law, we have

$$A \cap KG_\alpha = A \cap K(B \cap G_\alpha)(A \cap G_\alpha) = (A \cap K(B \cap G_\alpha))(A \cap G_\alpha).$$

Let $A_0 = A \cap KB$ and $B_0 = B \cap KA$, then $A \cap KG_\alpha$ is contained in $A_0(A \cap G_\alpha)$ and similarly, $B \cap KG_\alpha \leq B_0(B \cap G_\alpha)$. Now the sets A_0 and B_0 are contained in the factorizer X of K by [AFG92, Lemma 1.1.3]. Since KG_α is factorized, it contains X, and so we have $A_0 \leq A \cap KG_\alpha$ and $B_0 \leq B \cap KG_\alpha$. This shows that $A \cap KG_\alpha = A_0(A \cap G_\alpha)$ and $B \cap KG_\alpha = B_0(B \cap G_\alpha)$.

Moreover, K is obviously contained in the set A_0B_0. Since KG_α is Γ-invariant and $<A_0, B_0>$ is finite, there exists a Γ-invariant finite subgroup F of G such that $<A_0, B_0> \leq F \leq KG_\alpha$. Thus, applying the modular law twice, we obtain

$$F = F \cap KG_\alpha = F \cap A_0G_\alpha B_0 = A_0(F \cap G_\alpha B_0) = A_0(F \cap G_\alpha)B_0.$$

Assume that α is a limit ordinal, then $F \cap G_\alpha = F \cap G_\beta$ for some $\beta < \alpha$ and so $FG_\beta = A_0G_\beta B_0 = A_0(A \cap G_\beta)(B \cap G_\beta)B_0$. Therefore FG_β is factorized, contradicting the choice of α.

Therefore $\alpha - 1$ exists, and we may assume without loss of generality that $\alpha = 1$ and that G_1 is a π'-group. Then $F \cap G_1$ is a subgroup of B and $F = A_0(F \cap B)B_0$ is factorized. This final contradiction proves the lemma. □

Our next lemma is a slight extension of [FGS94, Lemma 2.3].

2.2.4 Lemma. *Let π be a set of primes and suppose that the countable locally finite group G has an ascending series whose factors are π- or π'-groups. If N is a normal subgroup of G such that G/N is a π-group and N is the product of a π-group A_0 and a π'-group B, then there exists a π-subgroup A of G such that $G = AB$.*

Proof. Since G is countable, G is the union of an ascending chain of finite subgroups $G_1 \leq G_2 \leq \ldots$ of type ω. We define an ascending chain $\{K_i \mid i \in \mathbb{N}\}$ of finite subgroups of N as follows: Put $K_1 = 1$. If $i > 1$, then by Lemma 2.2.3, there exists a finite G_i-invariant subgroup K_i of $N = A_0B$ which contains $G_i \cap N$ and K_{i-1} and satisfies $K_i = (A_0 \cap K_i)(B \cap K_i)$.

Suppose now that $G_{i-1} = A_{i-1}(B \cap G_{i-1})$ for a π-subgroup A_{i-1} of G_{i-1}. Since K_i is G_i-invariant, G_iK_i is a finite subgroup of G, hence is π-separable. Therefore A_{i-1} is contained in a Hall π-subgroup A_i of G_iK_i. Since $G_i \cap N \leq K_i$, the factor group $G_iK_i/K_i \cong G_i/G_i \cap K_i$ is a π-group and $B \cap K_i$ is a Hall π'-subgroup of G_iK_i so that $G_iK_i = A_i(B \cap K_i)$. Thus $A = \bigcup_{i \in \mathbb{N}} A_i$ is the required π-subgroup of G. □

We can now formulate the relation between the existence of prefactorized Sylow subgroups and of certain prefactorized characteristic subgroups of a group G which is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$.

2.2.5 Theorem. *Let π be a set of primes and suppose that the locally finite group G has an ascending series whose factors are either π-groups or π'-groups. Further, assume that G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π-and π'-subgroups of A and B, respectively. Then the following statements are equivalent:*

(a) *$<A_\pi, B_\pi>$ is a π-group and $<A_{\pi'}, B_{\pi'}>$ is a π'-group.*

(b) *For every normal subgroup N of G, $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N and $A_{\pi'}B_{\pi'}N/N$ is a Sylow π'-subgroup of G; moreover, $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$.*

(c) *$O_{\pi',\pi}(G/N)$ and $O_{\pi,\pi'}(G/N)$ are factorized for every normal subgroup N of G.*

(d) *$O_\pi(G/N)$ and $O_{\pi'}(G/N)$ are prefactorized for every normal subgroup N of G.*

(e) *The group G possesses an ascending series of prefactorized subgroups whose factors are either π- or π'-groups.*

Proof. The implication (a) ⇒ (c) has been proved in Proposition 2.2.2.

(c) $\Rightarrow$ (d). Since $O_{\pi',\pi}(G/N) \cap O_{\pi,\pi'}(G/N) = O_\pi(G/N) \times O_{\pi'}(G/N)$ is factorized by Proposition 1.1.3 (c), this follows from Corollary 2.1.2.

Since the implications (d) $\Rightarrow$ (e) and (b) $\Rightarrow$ (a) are trivial, it remains to show that (e) $\Rightarrow$ (b).

In view of Proposition 2.1.4, it clearly suffices to consider the case when $N = 1$. Let $\{N_\alpha\}_{\alpha \le \beta}$ be an ascending series of prefactorized subgroups such that $N_{\alpha+1}/N_\alpha$ is a π-group or a π'-group for every $\alpha < \beta$. By transfinite induction on β, the sets $(A_\pi \cap N_\alpha)(B_\pi \cap N_\alpha)$ and $(A_{\pi'} \cap N_\alpha)(B_{\pi'} \cap N_\alpha)$ are Sylow π- and π'-subgroups of N_α such that

$$N_\alpha = (A_\pi \cap N_\alpha)(B_\pi \cap N_\alpha)(A_{\pi'} \cap N_\alpha)(B_{\pi'} \cap N_\alpha)$$

for every $\alpha < \beta$. Thus if β is a limit ordinal, then

$$A_\pi B_\pi = \Big(\bigcup_{\alpha<\beta} A_\pi \cap N_\alpha\Big) \cdot \Big(\bigcup_{\alpha<\beta} B_\pi \cap N_\alpha\Big) = \bigcup_{\alpha<\beta} (A_\pi \cap N_\alpha)(B_\pi \cap N_\alpha)$$

and so $A_\pi B_\pi$ is a π-group. Similarly, $A_{\pi'}B_{\pi'}$ is a π'-group and $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$.

Therefore assume that β possesses a predecessor $\beta - 1$ and set $N = N_{\beta-1}$. Exchanging π and π' if necessary, we may also assume that G/N is a π-group, so that $<A_{\pi'}, B_{\pi'}> = (A_{\pi'} \cap N_{\beta-1})(B_{\pi'} \cap N_{\beta-1}) = A_{\pi'}B_{\pi'}$ is a Sylow π'-group of G. Now suppose that $<A_\pi, B_\pi>$ is a π-group. Then it follows from Proposition 2.1.4 that $A_\pi B_\pi$ is a Sylow π-subgroup of G and that $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N. Hence

$$G = A_\pi B_\pi N = (A_\pi B_\pi)(A_{\pi'} \cap N_{\beta-1})(B_{\pi'} \cap N_{\beta-1}) = (A_\pi B_\pi)(A_{\pi'}B_{\pi'}),$$

as required.

Thus it remains to show that $<A_\pi, B_\pi>$ is a π-group. Let A_0 and B_0 be arbitrary finite subsets of A_π and B_π, respectively, then it clearly suffices to show that $<A_0, B_0>$ is a π-group. By Lemma 1.1.6, there exists a countable prefactorized subgroup H of G containing A_0 and B_0 such that $H \cap N_\alpha$ is prefactorized for every $\alpha \le \beta$. Therefore we may assume without loss of generality that $G = H$ and so G is countable. Then by Lemma 2.2.4, there exists a maximal π-subgroup P of G containing $(A_\pi \cap N)(B_\pi \cap N)$ such that $G = PA_{\pi'}B_{\pi'}$. By Lemma 2.2.3, $<A_0, B_0>$ is contained in a finite subgroup F satisfying $F = (F \cap P)(F \cap A_{\pi'}B_{\pi'})$. Let Q be a Hall π-subgroup of F containing A_0, then $B_0 \le Q^g$ for some $g \in F$, since F is π-separable. Since $F = Q(F \cap A_{\pi'}B_{\pi'})$, we may clearly assume that $g \in F \cap A_{\pi'}B_{\pi'}$. Write $g = ab^{-1}$ with $a \in A_{\pi'}$ and $b \in B_{\pi'}$, then $A_0 = A_0^a$ is contained in Q^a and also $B_0 = B_0^b$ is contained in $Q^{gb} = Q^a$. Therefore $<A_0, B_0>$ is a π-group, as required. □

In order to examine whether, under the hypotheses of the preceding theorem, one of the groups $O_\pi(G)$, $O_{\pi'}(G)$, $O_{\pi',\pi}(G)$ or $O_{\pi,\pi'}(G)$ is prefactorized, it often suffices to investigate whether $O_{\pi',\pi}(G)$ and $O_{\pi,\pi'}(G)$ are factorized, which is in most cases a much easier task.This can be expressed as follows.

2.2.6 Corollary. *Let π be a set of primes and G a locally finite group which is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ where A_π, B_π, $A_{\pi'}$ and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively. If G possesses an ascending series whose factors are either π- or π'-groups, then the following statements are equivalent:*

(a) *For every $N \trianglelefteq G$, the groups $O_\pi(G/N)$ and $O_{\pi'}(G/N)$ are prefactorized.*

(b) *For every $N \trianglelefteq G$, the groups $O_{\pi',\pi}(G/N)$ and $O_{\pi,\pi'}(G/N)$ are prefactorized.*

(c) *For every $N \trianglelefteq G$, the groups $O_{\pi',\pi}(G/N)$ and $O_{\pi,\pi'}(G/N)$ are factorized.*

In view of Theorem 2.2.5, we can also strengthen the statement of Proposition 2.1.7.

2.2.7 Proposition. *Let π be a set of primes and G a group whose Sylow π-subgroups and Sylow π'-subgroups of G are conjugate. If G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively, then $A_\pi B_\pi$ is a Sylow π-subgroup of G and $A_{\pi'}B_{\pi'}$ is a Sylow π'-subgroup of G such that $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$. Moreover, the subgroups $O_{\pi',\pi}(G)$ and $O_{\pi,\pi'}(G)$ are factorized subgroups of G and $O_\pi(G)$ and $O_{\pi'}(G)$ are factorized subgroups of $A_\pi B_\pi$ and $A_{\pi'}B_{\pi'}$, respectively. Hence $O_\pi(G)$ and $O_{\pi'}(G)$ are prefactorized subgroups of G.*

Proof. By Proposition 2.1.7, the sets $A_\pi B_\pi$ and $A_{\pi'}B_{\pi'}$ are a Sylow π- and a Sylow π'-subgroup of G. Since by [Har72a, Theorem D], the group G possesses a finite series whose factors are π- or π'-groups, Theorem 2.2.5 shows that $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$, as required. □

2.3. Sylow bases of radical groups

The results obtained so far are of special interest when G is a periodic radical group which is the product of two locally nilpotent subgroups A and B. We will show that if $A_\pi B_\pi$ is a Sylow π-group of G for every set of primes π, then the set $\{A_pB_p \mid p \in \mathbb{P}\}$ even forms a Sylow basis of G. First, we study the factorizer of the Hirsch-Plotkin radical of G.

2.3.1 Lemma. *Suppose that the periodic group G is the product of two locally nilpotent subgroups A and B. If p is a prime such that the set A_pB_p is a p-group containing $O_p(G)$, then the factorizer $X = AR \cap BR$ of the Hirsch-Plotkin radical R of G is an extension of a p'-group by a p-group.*

Proof. Let $R_p = O_p(G)$ and $R_{p'}$ be the Sylow p- and p'-subgroups of R, respectively, and denote with X the factorizer of R. Then $R/R_{p'}$ is contained in $A_pB_pR_{p'}/R_{p'}$. Now by Lemma 2.1.5, the factorizer $X/R_{p'}$ of $R/R_{p'}$ is an extension of a p'-group by a p-group. Therefore also X is an extension of a p'-group by a p-group. □

In particular, this result can be applied to locally finite groups which are the product of two locally nilpotent subgroups.

2.3.2 Corollary. *Suppose that the locally finite group is the product of two locally nilpotent subgroups A and B. If the set A_pB_p is a Sylow p-subgroup of G for every prime p, then the factorizer of the Hirsch-Plotkin radical of G is locally nilpotent.*

Proof. Let X denote the factorizer of the Hirsch-Plotkin radical of G. By Lemma 2.3.1, $X/O_{p'}(X)$ is a p-group for every $p \in \mathbb{P}$. Since

$$O_p(X) = \bigcap_{q \in \mathbb{P}\setminus\{p\}} O_{q'}(X),$$

it follows that $X/O_p(X)$ is a p'-group for every prime p, and so X is the direct product of its Sylow subgroups. Since X is locally finite, it is locally nilpotent. □

In the following theorem, we collect the main properties of a periodic radical group which is the product of two locally nilpotent subgroups. Note that, despite the similarities with Theorem 2.2.5, the results for radical groups are slightly stronger, mainly because it suffices to consider p-groups in (d) below.

2.3.3 Theorem. *Let the periodic radical group G be the product of two locally nilpotent subgroups A and B. Then the following statements are equivalent:*

(a) *$\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

(b) *$<A_{p'}, B_{p'}>$ is a p'-group for every prime p.*

(c) *For every set of primes π and every normal subgroup N of G, the set $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N.*

(d) *For every prime p and every normal subgroup of G, the set A_pB_pN/N is a Sylow p-subgroup of G/N.*

(e) *For every normal subgroup N of G, the Hirsch-Plotkin radical $R(G/N)$ of G/N is factorized.*

(f) *Every term of the Hirsch-Plotkin series of G is factorized.*

(g) *The group G possesses an ascending series of prefactorized subgroups with locally nilpotent factors.*

Proof. (a) $\Rightarrow$ (b) follows directly from the definition of a Sylow basis.

(b) $\Rightarrow$ (c) Clearly, A and B are the direct product of their Sylow π- and π'-subgroups and $<A_\pi, B_\pi>$ is obviously contained in the π-group

$$\bigcap_{q \in \mathbb{P}\setminus\pi} <A_{q'}, B_{q'}>.$$

Therefore $<A_\pi, B_\pi>$ and $<A_{\pi'}, B_{\pi'}>$ are a π- and a π'-subgroup, and so (c) follows from Proposition 2.1.4 (a).

The implication (c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (e). Let $R_0 = N$ and for every ordinal α, define $R_{\alpha+1}/R_\alpha = R(G/R_\alpha)$; moreover, put $R_\alpha = \bigcup_{\gamma<\alpha} R_\gamma$ if α is a limit ordinal. Then $G = R_\beta$ for some ordinal β since G is radical. For every ordinal α, let X_α denote the factorizer of R_α. Then for every α, the factor group $X_{\alpha+1}/R_\alpha$ is locally nilpotent by Corollary 2.3.2. Therefore by [Rob72, II, p. 10], the subgroup X_α/R_α is a serial subgroup of $X_{\alpha+1}/R_\alpha$ for every α hence of G/R_α for every $\alpha \leq \beta$. Now suppose that $\alpha \leq \beta$ has a predecessor $\alpha - 1$. Since G is locally finite, it follows from [Har72b, Lemma 3] that the serial locally nilpotent subgroup $X_\alpha/R_{\alpha-1}$ is contained in $R_\alpha/R_{\alpha-1}$, and thus we have $X_\alpha = R_\alpha$ and so R_α is factorized for every α that is not a limit ordinal. For limit ordinals α, the same statement follows from Proposition 1.1.3 (d) and the fact that

$$R_\alpha = \bigcup_{\gamma<\alpha} R_\gamma = \bigcup_{\gamma<\alpha} X_\gamma.$$

Obviously, (e) implies (f) and (f) implies (g).

Suppose now that (g) holds. For every set σ of primes, let A_σ and B_σ denote the (unique) Sylow σ-subgroup of A and B, respectively. Let π be any set of primes, then an ascending series of prefactorized subgroups with locally nilpotent factors can be refined to a series whose factors are π- or π'-groups by Corollary 2.1.2. Therefore it follows from Theorem 2.2.5 that $A_\pi B_\pi$ and $A_{\pi'}B_{\pi'}$ are a Sylow π- and a Sylow π'-subgroup of G such that $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$. Since this holds for arbitrary sets π of primes, $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G by Lemma 1.2.2. This proves (a). □

We mention a number of particularly useful consequences of the preceding theorem. The first shows that in most cases, the question whether the Hirsch-Plotkin radical of a periodic radical product of two locally nilpotent subgroups is prefactorized can be reduced to the easier question whether it is factorized.

2.3.4 Corollary. *Suppose that the periodic radical group G is the product of its locally nilpotent subgroups A and B. If the Hirsch-Plotkin radical $R(G/N)$ of G/N is prefactorized for every normal subgroup N of G, then $R(G/N)$ is factorized for all $N \trianglelefteq G$.*

We also mention a useful criterion for the Hirsch-Plotkin radical of a product G of two locally nilpotent subgroups (and, indeed, of every factor group of G) to be factorized.

2.3.5 Corollary. *Suppose that the periodic radical group G is the product of its locally nilpotent subgroups A and B. If every factor group of G possesses a prefactorized locally nilpotent normal subgroup, then the Hirsch-Plotkin radical of G is factorized.*

In view of Theorem 2.2.5 (b) and Theorem 2.3.3, we also obtain:

2.3.6 Corollary. *Suppose that the periodic radical group G is the product of its locally nilpotent subgroups A and B. For every set of primes, let A_π and B_π denote the (unique) Sylow π-subgroups of A and B, respectively, and suppose that $<A_\pi, B_\pi>$ is a π-group for every set of primes π. Then $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G, and in particular $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$ for every set π of primes.*

The next theorem is a direct consequence of Proposition 2.1.9 and Theorem 2.3.3; however, it will be of great importance in the sequel.

2.3.7 Theorem. *Let the periodic radical group G be the product of its locally nilpotent subgroups A and B, and suppose that the set $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G. If S is a prefactorized subgroup of G, then $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into S.*

Proof. By Proposition 2.1.9, $(S \cap A_\pi)(S \cap B_\pi) = S \cap A_\pi B_\pi$ is a (maximal) π-subgroup of S for every set of primes π. Therefore S satisfies Theorem 2.3.3 (b) and so $\{A_pB_p \cap S \mid p \in \mathbb{P}\}$ is a Sylow basis of S. □

An argument similar to Corollary 2.1.10 can now be used to show that a periodic radical product G of two locally nilpotent subgroups has at most one Sylow basis consisting of prefactorized Sylow subgroups of G.

2.3.8 Corollary. *Let the periodic radical group G be the product of its locally nilpotent subgroups A and B, and suppose that the set $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G. Then $\{A_pB_p \mid p \in \mathbb{P}\}$ is the unique Sylow basis of G which consists of prefactorized subgroups of G.*

Next, we mention one important case when such a Sylow basis of prefactorized subgroups exists. Observe that Example 2.1.8 shows that it does not suffice to assume that the π-components of A and B are contained in locally conjugate Sylow π-subgroups of G.

2.3.9 Theorem. *Suppose that the periodic radical group G is the product of two locally nilpotent subgroups A and B. If the π-components of A and B are contained in conjugate Sylow π-subgroups of G for every set of primes π, then $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

Proof. By Proposition 2.1.7, the subgroup $A_{p'}B_{p'}$ is a Sylow p'-subgroups of G for every prime p. Therefore the result follows from Theorem 2.3.3. □

The following theorem restates the results of Proposition 2.1.4 for Sylow bases of periodic radical products of two locally nilpotent subgroups.

2.3.10 Theorem. *Suppose that the periodic radical group G is the product of two locally nilpotent subgroups A and B.*

(a) *If the group $<A_{\pi}, B_{\pi}>$ is a π-group for every set π of primes, then $\{A_pB_pN/N \mid p \in \mathbb{P}\}$ is a Sylow basis of G/N for every normal subgroup N of G.*

(b) *If $\mathcal{N}$ is a set of normal subgroups of G such that $\bigcap_{N \in \mathcal{N}} N = 1$ and for every $N \in \mathcal{N}$, the set $\{A_pB_pN/N \mid p \in \mathbb{P}\}$ is a Sylow basis of G/N, then $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

(c) *If $N \leq Z(G)$ and $\{A_pB_pN/N \mid p \in \mathbb{P}\}$ is a Sylow basis of G/N, then $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

Proof. (a) follows directly from Theorem 2.3.3. In view of the equivalence of statements (a) and (b) of Theorem 2.2.5, the statements (b) and (c) follow from (a) and Proposition 2.1.4. □

2.4. Existence of prefactorized Sylow bases

In this section, we collect the consequences of the results obtained so far for the classes of periodic locally soluble groups that we will consider in the following chapters.

2.4.1 Theorem. *Suppose that the $\mathfrak{U}$-group G is the product of its locally nilpotent subgroups A and B. Then:*

(a) *The set $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

(b) *For every set π of primes, $O_{\pi}(G)$ is a factorized subgroup of $A_{\pi}B_{\pi}$, hence is a prefactorized subgroup of G.*

(c) *For every set π of primes, $O_{\pi',\pi}(G)$ is a factorized subgroup of G.*

(d) *The Hirsch-Plotkin radical of G is factorized.*

Proof. By Theorem 2.3.9, the set $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G. Therefore the remaining statements follow from Theorem 2.3.3. □

Černikov [Cer82, Lemma 5] has shown that, as in the finite case, the existence of a Sylow basis of an $\mathfrak{U}$-group $G = AB$ which consists entirely of prefactorized Sylow subgroups of $\mathfrak{U}$-groups can be proved without the assumption that A and B be locally nilpotent. We restate Černikov's result for the convenience of the reader.

2.4.2 Proposition. *Suppose that the $\mathfrak{U}$-group G is the product of two subgroups A and B. Then there are Sylow bases $\{A_p \mid p \in \mathbb{P}\}$ and $\{B_p \mid p \in \mathbb{P}\}$ of A and B, respectively, such that $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

Proof. Let $\{A_p \mid p \in \mathbb{P}\}$ and $\{B_p \mid p \in \mathbb{P}\}$ be Sylow bases of A and B respectively. By [Har71, Lemma 2.1], these Sylow bases can be extended to Sylow bases $\{G_p \mid p \in \mathbb{P}\}$ and $\{G_p^* \mid p \in \mathbb{P}\}$ of G, which are conjugate by [GHT71, Theorem 2.10]. So there exists an element $g = ab^{-1} \in G$ such that $G_p^g = G_p^*$ for all p. Then $A_p^a \leq G_p^a$ and $B_p^b \leq G_p^{gb} = G_p^a$. For all sets π of primes, $S = <A_p^a, B_p^b \mid p \in \pi>$ is contained in $<G_p^a \mid p \in \pi>$, which is a π-group. Hence by Proposition 2.1.4 (a), we have $G_p^a = A_p^aB_p^b$ for every prime p. Thus $\{A_p^a \mid p \in \mathbb{P}\}$ and $\{B_p^b \mid p \in \mathbb{P}\}$ are the required Sylow bases of A and B, respectively. □

In order to prove a result similar to Theorem 2.4.1 for periodic CC-groups, we need the following auxiliary results on CC-groups, which is also mentioned in the introduction of [Dix88]. For the corresponding result about FC-groups, see e.g. [Rob72, Theorem 4.32] or [Tom84, Theorem 1.4].

2.4.3 Lemma. *Let G be a CC-group. Then G has a local system of normal subgroups which are central-by-Černikov. Moreover G has a local system of central-by-finite subgroups.*

Proof. For every $x \in G$, the normal subgroup $[G, x]$ of G is a Černikov group ([Pol64]; see also [Rob72, Theorem 4.36]). So if $X = \{x_1, \ldots, x_n\}$ is a finite subset of G, then also $[G, X] = [G, x_1] \cdot \ldots \cdot [G, x_n]$ is Černikov and so $N = X^G = [G, X]X$ is Černikov-by-(free abelian of finite rank). Since N is likewise a CC-group and $Z(N) = \bigcap_{x \in X} C_N(x^N)$, also $N/Z(G)$ is Černikov and N is central-by-Černikov. In particular, every finitely generated subgroup is central-by-finite. □

Also the next proposition is known for FC-groups; see e.g. [Tom84, Theorem 1.18] and [AO87, Lemma 1].

2.4.4 Proposition. *Assume that every finite image of the CC-group G is soluble. Then G is locally soluble and G has a descending series of type $\leq \omega + 1$ whose factors are abelian.*

Proof. Let $x \in G$, then $G/C_G(x^G)$ is a Černikov group, hence abelian-by-finite. Since every finite image of G is soluble, the factor group $G/C_G(x^G)$ is soluble. Since $\bigcap_{x \in G} C_G(x^G) = Z(G)$, the group $G/Z(G)$ has a descending series of type $\leq \omega$ whose factors are abelian. Consequently G has such a descending normal series of type $\leq \omega + 1$. Now let X be a finite subset of G, then $N = X^G$ is central-by-Černikov by Lemma 2.4.3. Therefore $<X>Z(N)/Z(N)$ is finite, hence soluble, and so also $<X>$ is soluble and central-by-finite. □

Since every FC-group is a CC-group, the following theorem holds in particular, if G is a periodic FC-group.

2.4.5 Theorem. *Suppose that the periodic CC-group G is the product of its locally nilpotent subgroups A and B. Then:*

(a) *G is periodic and locally soluble; moreover, it has a descending series of length $\leq \omega + 1$ whose factors are abelian.*

(b) *The set $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

(c) *For every set π of primes, $O_\pi(G)$ is a factorized subgroup of $A_\pi B_\pi$, hence is a prefactorized subgroup of G.*

(d) *For every set π of primes, $O_{\pi',\pi}(G)$ is a factorized subgroup of G.*

(e) *The Hirsch-Plotkin radical of G is factorized.*

Proof. (a) Clearly, every finite image G/N of G is the product of two nilpotent subgroups AN/N and BN/N, hence is soluble by the theorem of Kegel and Wielandt [Keg61], [Wie58]. Therefore by Proposition 2.4.4, the group G has a descending series of length $\leq \omega+1$ with abelian factors and is locally soluble.

(b) Since $G/C_G(x^G)$ is an $\mathfrak{U}$-group for every $x \in G$ and $\bigcap_{x\in G} C_G(x^G) = Z(G)$, it follows from Theorem 2.4.1 and Theorem 2.3.10 (b) that $\{A_pB_pZ(G)/Z(G) \mid p \in \mathbb{P}\}$ is a Sylow basis of $G/Z(G)$. Thus $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G by Theorem 2.3.10 (c). The remaining statements now follow from Theorem 2.3.3. □

It may also be of interest that Theorem 2.4.5 (e) remains true for general CC-groups. To prove this, we need the following results.

2.4.6 Proposition. *The class of locally nilpotent CC-groups is* R*-closed.*

Proof. Let G be a CC-group and $\mathcal{N}$ a set of subgroups of G such that G/N is locally nilpotent for every $N \in \mathcal{N}$. Then every finitely generated subgroup U of G is central-by-finite by Lemma 2.4.3. Moreover, for every $N \in \mathcal{N}$, the finitely generated group $U/U \cap N$ is nilpotent of class at most $|U : Z(U)|$. Therefore U is nilpotent and G is locally nilpotent. □

The following elementary lemma will also be needed later.

2.4.7 Lemma. *Let G be a CC-group. If $R_x/C_G(x^G)$ denotes the Hirsch-Plotkin radical of $G/C_G(x^G)$ and R equals the intersection of all R_x, then R is the Hirsch-Plotkin radical of G. Moreover, G/R is a periodic FC-group.*

Proof. Clearly, the Hirsch-Plotkin radical of G is contained in every R_x and hence in R. Since $R/C_R(x^G)$ is locally nilpotent for every $x \in G$, it follows from Proposition 2.4.6 that $R/Z(G)$ is locally nilpotent. Therefore also R is locally nilpotent and so the normal subgroup R of G equals the Hirsch-Plotkin radical of G. Now let X be a finite subset of G, then X^G is central-by-Černikov by Lemma 2.4.3. Therefore the Hirsch-Plotkin radical $R \cap X^G$ of X^G has finite index in X^G and so X^GR/R is finite. Thus every finite subgroup of G/R is contained in a finite normal subgroup of G/R, and so G/R is a periodic FC-group. □

From this, we deduce the following result about the Hirsch-Plotkin radical of a CC-group which is the product of two locally nilpotent subgroups.

2.4.8 Theorem. *Let the CC-group G be the product of its locally nilpotent subgroups A and B. Then the Hirsch-Plotkin radical of G is factorized.*

Proof. For every $x \in G$, let $R_x/C_G(x^G)$ denote the Hirsch-Plotkin radical of $G/C_G(x^G)$, then by Lemma 2.4.7, the intersection $R = \bigcap_{x\in G} R_x$ equals the Hirsch-Plotkin radical of G. By Theorem 2.4.5 (e), the subgroups R_x of G are factorized for every $x \in G$, and so R is factorized by Proposition 1.1.3 (c). □

Results like Theorem 2.4.1 and Theorem 2.4.5 can also be proved for periodic locally soluble groups satisfying the minimal condition on p-subgroups for every prime p. However, different arguments are required because such groups need not be radical; see e.g. [Bae70, *Folgerungen* 4.5 and 5.4]. Note that what we call a Sylow basis is referred to as a Sylow generating basis in [Dix82].

2.4.9 Theorem. *Let G be a periodic locally soluble group which satisfies min-p for every prime p. Suppose that G is the product of its locally nilpotent subgroups A and B. Then:*

(a) *G is countable and has a descending series of length $\leq \omega$ whose factors are abelian.*

(b) *The set $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.*

(c) *For every set π of primes, $O_{\pi',\pi}(G)$ is a factorized subgroup of G.*

(d) *For every set π of primes, $O_\pi(G)$ is a factorized subgroup of $A_\pi B_\pi$, hence is a prefactorized subgroup of G.*

(e) *The Hirsch-Plotkin radical of G is factorized.*

(f) *If U is a prefactorized subgroup of G, then the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into U.*

Proof. (a) Since the p-components of A and B are locally soluble and satisfy the minimal condition on subgroups, the p-components of A and B are Černikov groups (see e.g. [KW73, Theorem 1.E.6]), hence are countable. Therefore also A and B are countable, and so G is countable.

Moreover, since G is locally soluble, for every prime p, the factor group $G/O_{p'}(G)$ is a Černikov group by [KW73, Theorem 3.17]. Hence these factor groups are soluble by the theorem of Kegel and Wielandt. Since $\bigcap_{p\in\mathbb{P}} O_{p'}(G) = 1$, it follows that G has a descending series of length $\leq \omega$ whose factors are abelian.

(b) Since $G/O_{p'}(G)$ is a soluble Černikov group and thus an $\mathfrak{U}$-group, it follows from Theorem 2.4.1 that for every prime p, $\{A_qB_qO_{p'}(G)/O_{p'}(G) \mid q \in \mathbb{P}\}$ is a Sylow basis of $G/O_{p'}(G)$. Therefore $G/O_{p'}(G) = (A_pB_pO_{p'}(G)/O_{p'}(G)) \cdot (A_{p'}B_{p'}/O_{p'}(G))$ and so $G = (A_pB_p)(A_{p'}B_{p'})$. Moreover, by Lemma 1.2.2, $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G.

(c) Let π be a set of primes, and for every prime p, set $P_p/O_{p'}(G) = O_{\pi',\pi}(G/O_{p'}(G))$. Then $O_{\pi',\pi}(G) = \bigcap_{p\in\mathbb{P}} P_p$ since G is periodic. By Theorem 2.4.1, the subgroups $P_p/O_{p'}(G)$ are factorized for every $p \in \mathbb{P}$, and so every P_p is factorized. Therefore by Proposition 1.1.3 (c), also their intersection $O_{\pi',\pi}(G)$ is factorized.

(d) By (c), $O_{\pi,\pi'}(G)$ is factorized. Therefore by Proposition 1.1.3 (b), the subgroup $O_\pi(G) = O_{\pi,\pi'}(G) \cap A_\pi B_\pi$ is factorized in $A_\pi B_\pi$, hence is a prefactorized subgroup of G.

(e) Let $R(G)$ denote the Hirsch-Plotkin radical of G. Clearly, $R(G) = \bigcap_{p\in\mathbb{P}} O_{p',p}(G)$ and so $R(G)$ is the intersection of factorized subgroups, hence is factorized by Proposition 1.1.3 (c).

(f) Since U likewise satisfies min-p for every prime p, $\{(U \cap A_p)(U \cap B_p) \mid p \in \mathbb{P}\}$ is a Sylow basis of U by (b). Since obviously $(U \cap A_\pi)(U \cap B_\pi) \leq U \cap A_\pi B_\pi$, it follows that $(U \cap A_\pi)(U \cap B_\pi) = U \cap A_\pi B_\pi$ for every set π of primes, as required. □

Chapter 3

Projectors of nilpotent-by-finite groups

3.1. Schunck classes of periodic soluble nilpotent-by-finite groups

Recall that $\mathfrak{AS}^*$ and $\mathfrak{NS}^*$ denote the classes of all periodic soluble abelian-by-finite groups and of all of all periodic soluble nilpotent-by-finite groups, respectively. Let $\mathfrak{H}$ be a class of $\mathfrak{NS}^*$-groups. In this section, we will show that every $\mathfrak{NS}^*$-group possesses an $\mathfrak{H}$-projector if and only if $\mathfrak{H}$ is a $\mathfrak{NS}^*$-Schunck class. Moreover, in this case, the $\mathfrak{H}$-projectors of an $\mathfrak{NS}^*$-group are conjugate. Here a subclass $\mathfrak{H}$ of $\mathfrak{NS}^*$ is called an $\mathfrak{NS}^*$-*Schunck class* if a group G belongs to $\mathfrak{H}$ if and only if

(SC1) every finite primitive image G/N of G is an $\mathfrak{H}$-group and

(SC2) every semiprimitive image G/K of G is the union of an ascending chain $X_1/K \leq X_2/K \leq \ldots$ of finite $\mathfrak{H}$-groups X_i/K.

A finite group G is *primitive* if it has a maximal subgroup with trivial core. A group G is *semiprimitive* if it is a semidirect product $M \ltimes D$ of a nontrivial radicable abelian group D of finite rank with a finite soluble group M such that $M_G = 1$ and every proper M-invariant subgroup of D is finite. In particular, D is a p-group for some prime p.

While it is well-known that condition (SC1) is necessary (and sufficient) to guarantee the existence of $\mathfrak{H}$-projectors in every finite soluble group (see e.g. [DH92, III, Theorem 3.10]), the following proposition shows that the second condition is also necessary for the existence of projectors in every $\mathfrak{NS}^*$-group; cf. also the example given in [Tom95, Section 3].

3.1.1 Proposition. *Assume that $\mathfrak{H}$ is a class of groups such that every $\mathfrak{AS}^*$-group has an $\mathfrak{H}$-projector. Then a semiprimitive Černikov group is an $\mathfrak{H}$-group if and only if it is the union of an ascending chain of finite $\mathfrak{H}$-groups.*

Proof. If the semiprimitive Černikov group G is the union of finite $\mathfrak{H}$-groups, then G is an $\mathfrak{H}$-group by [Tom95, Lemma 3.1].

Conversely, suppose that $G = M \ltimes D \in \mathfrak{H}$ is an infinite semiprimitive Černikov group, where M is finite with trivial core and D is a radicable abelian p-group for the prime p. Then also $M \cong G/D \in \mathfrak{H}$. Put $X_0 = M$ and for every positive integer n, put $D_n = D[p^n]$ and let X_n be an $\mathfrak{H}$-maximal supplement of D_n in MD_n which contains X_{n-1}. Then the X_n form an ascending chain of finite $\mathfrak{H}$-subgroups of G. Moreover, for every integer n, the X_n are $\mathfrak{H}$-projectors of MD_n by [DH92, III, Lemma 3.14].

Put $X = \bigcup_{n\in\mathbb{N}} X_n$. Since $M \leq X$, we have $X = X \cap MD = M(X \cap D)$ and so $X \cap D$ is a normal subgroup of G. Assume first that X is finite. Then we have $X_n = X_{n+1} = \ldots = X$ for an integer n, and moreover, $D \cap X \leq D_m$ for some integer $m \in \mathbb{N}$. Now by [Tom95, Proposition 2.3 (ii)], there exists an isomorphism $\alpha : G \to G/D_m$ which maps MD_n to MD_{m+n}/D_m and M to $MD_m/D_m = X_{m+n}D_m/D_m$. This shows that the subgroup M is an $\mathfrak{H}$-projector of MD_n. Since the $\mathfrak{H}$-projectors of MD_n are conjugate by [DH92, III, Theorem 3.13], we have $M \cong X_n$, and since $M \leq X_n$, it follows that $M = X_n = X$.

Now let $N = \bigtimes_{n\in\mathbb{N}} D_n$ be the (external) direct product of the D_n and set $H = M \ltimes N$, where M acts on the components of N in the natural way. Then H is an $\mathfrak{AS}^*$-group, hence possesses an $\mathfrak{H}$-projector Y. For every integer n, put

$$K_n = \bigtimes_{\substack{i\in\mathbb{N}\\ i\neq n}} D_i,$$

then K_n is a normal subgroup of H contained in N and H/K_n is isomorphic with MD_n. Therefore MK_n/K_n and YK_n/K_n are $\mathfrak{H}$-projectors of the finite group H/K_n. By [DH92, III, Theorem 3.21], there exists $g \in H$ such that $YK_n = M^gK_n$, and so we have $YK_n \cap N = M^gK_n \cap N = K_n(M^g \cap N) = K_n$ for every integer n. This shows that $Y \cap N \leq \bigcap_{n\in\mathbb{N}} K_n = 1$. and Y complements N in H. Let r denote the rank of D and for every $n \in \mathbb{N}$, fix generators $d_{n,1}, \ldots, d_{n,r}$ of $D[p^n]$. Let

$$K = <d_{n,i}^{-1}d_{n+1,i}^{p} \mid n \in \mathbb{N}, i \in \{1, \ldots, r\}>,$$

then $H/K \cong G$ and so YK/K is an $\mathfrak{H}$-projector of $H/K \cong G$. Since YK/K is finite, this proves that $G \notin \mathfrak{H}$. This contradiction shows that X must be infinite, and so also $X \cap D$ is infinite. As G is semiprimitive, we have $X \cap D = D$ and so $G = X$ is the union of the chain $\{X_n\}_{n\in\mathbb{N}}$ of finite $\mathfrak{H}$-groups. □

The next proposition shows that every local $\mathfrak{NS}^*$-formation is an $\mathfrak{NS}^*$-Schunck class.

3.1.2 Proposition. *Let $\mathfrak{X}$ be a* QS*-closed class of $\mathfrak{NS}^*$-groups. Then every local $\mathfrak{X}$-formation is an $\mathfrak{NS}^*$-Schunck class.*

Proof. Let $\mathfrak{F}$ be a local $\mathfrak{X}$-formation and $G \in \mathfrak{X}$. Further, assume that $G/N \in \mathfrak{F}$ for every finite primitive and every infinite semiprimitive factor group G/N of G. By [GHT71, Theorem 5.4], the $\mathfrak{U}$-group G possesses an $\mathfrak{F}$-projector H. If $H < G$, then by [Tom75, Lemma 2.3], H is contained in a major subgroup M of G. Now G/M_G is a finite primitive or infinite semiprimitive group by [Tom92], hence is an $\mathfrak{F}$-group. Since H is an $\mathfrak{F}$-projector of G, we have $G = HM_G \leq M$. This contradiction shows that $G = H \in \mathfrak{F}$. □

The proof of the second statement of the next lemma is similar to that of [Tom95, Lemma 3.3].

3.1.3 Lemma. *Let $\mathfrak{H}$ be an $\mathfrak{NS}^*$-Schunck class. Then:*

(a) $Q\mathfrak{H} = \mathfrak{H}$.

(b) $L\mathfrak{H} \cap \mathfrak{NS}^* = \mathfrak{H}$.

(c) *Every $\mathfrak{H}$-subgroup of an $\mathfrak{NS}^*$-group G is contained in an $\mathfrak{H}$-maximal subgroup of G.*

Proof. (a) Let $G \in \mathfrak{H}$ and $N \trianglelefteq G$. Since every factor group of G/N is isomorphic with a factor group of G, every finite primitive and every infinite semiprimitive factor group of G/N belongs to $\mathfrak{H}$. Therefore $G/N \in \mathfrak{H}$.

(b) Let $G \in L\mathfrak{H} \cap \mathfrak{N}\mathfrak{S}^*$. Since $QL\mathfrak{X} \leq LQ\mathfrak{X}$ for every group class $\mathfrak{X}$ and $\mathfrak{H} = Q\mathfrak{H}$ by (a), the class $L\mathfrak{H}$ is Q-closed. Therefore every factor group of G belongs to the class $L\mathfrak{H}$. Let G/N be a finite primitive image of G. Then $G = FN$ for some finite $\mathfrak{H}$-group F, and so $G/N \cong F/F \cap N$ belongs to $\mathfrak{H}$. If G/N is an infinite semiprimitive image of G and D/N is the finite residual of G/N, then G/N is the union of an ascending chain of finite subgroups L_i/N, and without loss of generality, $L_iD = G$ for every integer i. As in the finite case, it is possible to find an ascending chain $\{F_i/N\}$ of $\mathfrak{H}$-subgroups of G/N satisfying $L_i/N \leq F_i/N$ and $F_i \leq F_i + 1$ for every $i \in \mathbb{N}$. Therefore G/N is the union of the finite $\mathfrak{H}$-groups F_i/N, hence is an $\mathfrak{H}$-group. Thus G is an $\mathfrak{H}$-group by the definition of an $\mathfrak{N}\mathfrak{S}^*$-Schunck class.

(c) follows at once from (b). □

Since the definition of a $\mathfrak{N}\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ depends only on the finite $\mathfrak{H}$-groups, it is no surprise that there is a one-one correspondence between the Schunck classes of finite soluble groups and the $\mathfrak{N}\mathfrak{S}^*$-Schunck classes.

3.1.4 Proposition. *Let $\mathfrak{H}_0$ be a Schunck class of finite soluble groups. Then the class $\mathfrak{H}$ consisting of all $\mathfrak{N}\mathfrak{S}^*$-groups whose finite primitive factor groups are $\mathfrak{H}_0$-groups and whose infinite semiprimitive groups are unions of chains of $\mathfrak{H}_0$-groups is the smallest Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups containing $\mathfrak{H}_0$, and the class $\mathfrak{H}^*$ of all finite $\mathfrak{H}$-groups coincides with $\mathfrak{H}_0$. Therefore there is a one-one correspondence between the Schunck classes of finite soluble groups and the $\mathfrak{N}\mathfrak{S}^*$-Schunck classes.*

Proof. Clearly, $\mathfrak{H}$ is a Schunck class containing $\mathfrak{H}_0$ so that in particular $\mathfrak{H}_0 \subseteq \mathfrak{H}^*$. If G is a finite $\mathfrak{H}$-group, then every primitive image of G is an $\mathfrak{H}_0$-group, and so by the definition of a Schunck class of finite groups, G is an $\mathfrak{H}_0$-group. This shows that $\mathfrak{H}^* \subseteq \mathfrak{H}_0$ and so $\mathfrak{H}_0 = \mathfrak{H}^*$. Thus the map defined by $\mathfrak{H} \mapsto \mathfrak{H}^*$ for every $\mathfrak{N}\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ is a bijection between the $\mathfrak{N}\mathfrak{S}^*$-Schunck classes and the Schunck classes of finite soluble groups. □

The following proposition shows that not only semiprimitive Černikov $\mathfrak{H}$-groups are the union of an ascending chain of $\mathfrak{H}$-groups.

3.1.5 Proposition. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups. Then a Černikov group G is an $\mathfrak{H}$-group if and only if it is the union of an ascending chain $\{G_i \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups.*

Proof. First, suppose that G is the union of an ascending chain $\{G_i \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups. If G/N is a finite primitive image of G, then $G = NG_i$ for some i and so $G/N \in \mathfrak{H}$. Moreover, if G/N is an infinite semiprimitive Černikov group, then G/N is the union of an ascending chain $\{G_iN/N\}$ of $\mathfrak{H}$-groups, hence belongs to $\mathfrak{H}$ by the definition of a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups. Therefore every finite primitive and every infinite semiprimitive image belongs to $\mathfrak{H}$, and consequently G is an $\mathfrak{H}$-group.

Conversely, suppose that the Černikov group G belongs to the class $\mathfrak{H}$ and let D be the maximal radicable abelian normal subgroup of G and H a finite supplement of D in G. Let L be an $\mathfrak{H}$-projector of H, then $H = L(D \cap H)$ because $H/H \cap D \in \mathfrak{H}$. Therefore $G = LD$ and we may assume without loss of generality that $H \in \mathfrak{H}$.

Assume first that D does not have infinite G-invariant subgroups. Then D is a p-group for a prime p. Let $N = C_H(D)$, then $N \trianglelefteq HD = G$ and $H \cap D \leq N$. If $N = 1$, then G is semiprimitive and thus possesses an ascending chain of $\mathfrak{H}$-groups by the definition of $\mathfrak{H}$.

If $N \neq 1$, then by induction on $|G : D|$, $G/N = (H/N)(DN/N)$ possesses an ascending chain $\{G_i/N \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups, and since H is finite, we may assume without loss of generality that $H \leq G_i$ for every i. Hence $G_i = HD \cap G_i = H(D \cap G_i)$ by the modular law. Since N is finite, it suffices to show that every G_i is an $\mathfrak{H}$-group.

Fix an $i \in \mathbb{N}$ and let G_i/K be a finite primitive image of G_i with unique minimal normal subgroup $L/K = F(G_i/K)$. If $N \leq K$, we have $G_i/K \in \mathfrak{H}$, as required. Therefore assume that $L \leq NK$. Then $L = L \cap NK = (L \cap N)K$ by the modular law. Moreover, the abelian normal subgroup $(DK \cap G_i)/K$ of G_i/K is contained in $F(G_i/K) = L/K$. It follows that $G_i = HL = H(L \cap N)K$. Since N is contained in H, we even have $G_i = HK$ and so $G_i/K \cong H/H \cap K \in \mathfrak{H}$. This shows that every primitive image of G_i is an $\mathfrak{H}$-group, and so $G_i \in \mathfrak{H}$ by the definition of a Schunck class.

Therefore G is the union of the finite $\mathfrak{H}$-groups $\{G_i \mid i \in \mathbb{N}\}$. This completes the proof when D does not have infinite G-invariant subgroups.

Finally, suppose that D has a proper infinite G-invariant subgroup E. By induction on the rank of a maximal radicable abelian normal subgroup of G, the factor group G/E possesses an ascending chain $\{G_i/E \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups. Since the G_i are Černikov groups, by induction on the rank of a maximal radicable abelian normal subgroup of G_i, each G_i possesses an ascending chain $\{G_{i,j} \mid j \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups. We define an ascending chain $\{G_i^* \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups satisfying $G_i^* \leq G_i$ for every positive integer i: firstly, let $G_1^* = G_{1,1}$. Now let $n > 1$. Since G_n is the union of its subgroups $\{G_{n,j} \mid j \in \mathbb{N}\}$, there exists an integer m such that the $\mathfrak{H}$-group $G_{n,m} = G_n^*$ contains the (finite) subgroups $G_{1,n-1}, G_{2,n-2}, \dots, G_{n-2,2}, G_{n-1,1}$ and G_{n-1}^* of G_n. By construction, $\{G_n^*\}$ is an ascending chain of $\mathfrak{H}$-groups and $G_{i,j} \leq G_{i+j}^*$ for every i, $j \in \mathbb{N}$. Therefore G is the union of the chain $\{G_n^* \mid p \in \mathbb{P}\}$ of finite $\mathfrak{H}$-groups, as required. □

3.2. Existence of projectors in periodic soluble nilpotent-by-finite groups

Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class. We will now prove the existence and conjugacy of $\mathfrak{H}$-projectors in $\mathfrak{N}\mathfrak{S}^*$-groups. This generalizes a theorem of Tomkinson [Tom95] who established the existence and conjugacy of $\mathfrak{H}$-projectors for Schunck classes of $\mathfrak{A}\mathfrak{S}^*$-groups. Except for some auxiliary results, our proofs are formally independent of the results in [Tom95].

As a first step, we show that the existence and conjugacy of $\mathfrak{H}$-maximal supplements of a nilpotent normal subgroup of a semiprimitive group can be deduced from the finite soluble case (cf. [DH92, III, Theorem 3.14]). Note that the next lemma can also be deduced from the results about $\mathfrak{A}\mathfrak{S}^*$-groups in [Tom95].

3.2.1 Lemma. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class and $G = M \ltimes D$ a semiprimitive Černikov group, where $M \in \mathfrak{H}$ is finite and soluble and D is a radicable abelian p-group. If $G \notin \mathfrak{H}$, then:*

(a) *G possesses $\mathfrak{H}$-maximal subgroups which supplement D.*

(b) *If U and V are $\mathfrak{H}$-maximal supplements of D, then there exists a finite nilpotent normal subgroup N of G such that $UN = VN$.*

(c) *Any two $\mathfrak{H}$-maximal supplements U and V of G are conjugate, and every Sylow basis of G reduces into a unique $\mathfrak{H}$-maximal supplement of D.*

(d) *Every $\mathfrak{H}$-maximal supplement of D is an $\mathfrak{H}$-projector of G.*

(e) *M is an $\mathfrak{H}$-projector of G. Hence every $\mathfrak{H}$-maximal supplement of D is conjugate to M, and every $\mathfrak{H}$-projector of G is a complement of D.*

(f) *Every supplement H of D contains an $\mathfrak{H}$-maximal supplement of G.*

Proof. (a) Since $\mathfrak{H}$ is closed with respect to unions of ascending chains by Lemma 3.1.3 (c), there exists an $\mathfrak{H}$-maximal subgroup of G containing M.

(b) Let U and V be $\mathfrak{H}$-maximal supplements of D in G. Then $U \cap D$ and $V \cap D$ are normal subgroups of G, and since $G \notin \mathfrak{H}$ and G is semiprimitive, the normal subgroups $U \cap D$ and $V \cap D$ are finite. Thus U and V are finite. Therefore there exists an integer n such that $U \leq VD[p^n]$ and $V \leq UD[p^n]$. This shows that $UD[p^n] = VD[p^n]$ and U and V are $\mathfrak{H}$-maximal supplements of $D[p^n]$ in $UD[p^n]$.

(c) Let N be a finite nilpotent normal subgroup of G such that $UN = VN$, then by [DH92, III, Lemma 3.14], U and V are $\mathfrak{H}$-projectors of N, hence are conjugate. Let $\{G_p \mid p \in \mathbb{P}\}$ be a Sylow basis of G reducing into U and V. Then by Lemma 1.2.3 (d), $\{G_p \mid p \in \mathbb{P}\}$ also reduces into $UN = VN$. Therefore the statement follows from [DH92, I, Theorem 6.6] and the fact that $\mathfrak{H}$-projectors of finite soluble groups are pronormal.

(d) Let N be a normal subgroup of G and assume that H is an $\mathfrak{H}$-maximal supplement of D in G. Moreover, let Y/N be an $\mathfrak{H}$-subgroup of G which contains HN/N. In order to show that H is an $\mathfrak{H}$-projector of G, we have to prove that $HN = Y$.

Observe that $Y \cap D \trianglelefteq YD = G$, and so $Y \cap D$ is finite. Thus Y is finite, and so by [DH92, III, Theorem 3.21], Y contains an $\mathfrak{Y}$-projector Y_0. Hence we have $Y = Y_0(Y \cap D)$. On the other hand, we obtain $Y = Y \cap HD = H(Y \cap D)$ by the modular law. Therefore H is an $\mathfrak{H}$-projector of Y by [DH92, III, Lemma 3.14]. In particular HN/N is an $\mathfrak{H}$-maximal subgroup of Y/N, and so $HN = Y$.

(e) Let H be an $\mathfrak{H}$-projector of G. Since $H \cap D \trianglelefteq HD = G$ and $G \notin \mathfrak{H}$, the intersection $H \cap D$ is finite. Therefore there exists an integer n such that $H \cap D \leq D[p^n]$ and $HD[p^n]/D[p^n]$ is an $\mathfrak{H}$-maximal subgroup of $G/D[p^n]$. Since $H = H \cap MD = M(H \cap D)$, we have $HD[p^n]/D[p^n] \cong M$. Moreover, by [Tom95, Proposition 2.3 (ii)], the factor group $G/D[p^n]$ is isomorphic with G and so M is $\mathfrak{H}$-maximal in G.

(f) Since $H \cap D$ is a normal subgroup of G, the subgroup H is finite or equals G, and in the last case, the statement is trivial. Therefore assume that H is finite and let H_0 be an $\mathfrak{H}$-projector of H. Then $H = H_0(H \cap D)$ and so $G = H_0D$. Let L be an $\mathfrak{H}$-maximal subgroup of G containing H, then L is an $\mathfrak{H}$-projector of G by (d). Therefore L complements D by (e), and so $L = L \cap H_0D = H_0(L \cap D) = H_0$, as required. □

The conjugacy of the $\mathfrak{H}$-projectors of a periodic soluble nilpotent-by-finite group will be a consequence of the next proposition.

3.2.2 Proposition. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups. Suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G has a nilpotent subgroup N of finite index such that $G/N \in \mathfrak{H}$. Then there exist $\mathfrak{H}$-maximal supplements of N in G, and any two are conjugate.*

Proof. Let H be a finite supplement of N in G, then by [DH92, III, Theorem 3.21], the subgroup H possesses an $\mathfrak{H}$-projector H_0. Since $H/H \cap N \cong G/N \in \mathfrak{H}$, we have $H = H_0(H \cap N)$ and hence $G = H_0 N$. Therefore by Lemma 3.1.3 (c), there exists an $\mathfrak{H}$-maximal subgroup U of G containing H_0. Clearly, U is the required $\mathfrak{H}$-maximal supplement of N in G.

Now suppose that U and V are $\mathfrak{H}$-maximal supplements of N in G. Since the Sylow bases of G are conjugate by [GHT71, Theorem 2.10], we may assume without loss of generality that the Sylow basis $\{S_p \mid p \in \mathbb{P}\}$ of G reduces into U and V. We show that if $G \notin \mathfrak{H}$, then G possesses a proper subgroup H containing U and V such that the Sylow basis $\{S_p \mid p \in \mathbb{P}\}$ of G reduces into H.

By the definition of an $\mathfrak{N}\mathfrak{S}^*$-Schunck class, there exists a finite primitive or an infinite semiprimitive image G/K of G which is not an $\mathfrak{H}$-group. In both cases, we have $NK/K \leq F(G/K)$, and so UK/K and VK/K are $\mathfrak{H}$-groups supplementing $F(G/K)$. If G/K is finite and primitive, UK/K and VK/K are conjugate maximal subgroups of G_β/K by [DH92, A, Theorem 15.2] and [DH92, A, Theorem 16.1]. Hence UK/K and VK/K are pronormal subgroups of G/K into which the Sylow basis $\{S_pK/K \mid p \in \mathbb{P}\}$ of G/K reduces, and it follows from [DH92, I, Theorem 6.6] that $UK = VK$. If G/K is an infinite semiprimitive group, we have $UK = VK$ by Lemma 3.2.1 (c). Since the Sylow basis $\{S_p \mid p \in \mathbb{P}\}$ of G reduces into UK and VK by Lemma 1.2.3 (e), we put $H = UK = VK$.

Now the hypotheses of the proposition are inherited by the subgroup H. Put $G_0 = G$ and $G_1 = H$. If $H \notin \mathfrak{H}$, we can find a subgroup G_2 which is properly contained in G_1 and contains U and V. Continuing like this, we obtain a descending chain

$$G = G_0 > G_1 > \ldots > G_\alpha \geq \langle U, V \rangle$$

of subgroups G_β of G which can be continued transfinitely, since by Lemma 1.2.3 (c), the Sylow basis $\{S_p \mid p \in \mathbb{P}\}$ of G also reduces into $\bigcap_{\beta<\lambda} G_\beta = G_\lambda$ for every limit ordinal λ. This process must terminate since the cardinality of α cannot exceed that of G, and so we have $G_\alpha \in \mathfrak{H}$ for some α. But then we find that $G_\alpha = U = V$ because U and V are $\mathfrak{H}$-maximal subgroups of G. □

Although not needed in the sequel, we mention the following generalization of [Tom95, Lemma 4.1] to the class of all periodic soluble nilpotent-by-finite groups.

3.2.3 Lemma. *Let N be a normal nilpotent subgroup of the periodic soluble nilpotent-by-finite group G and assume that $X \in \mathfrak{H}$ is a subgroup of G such that $G = XC_G(N)$ and $G/X \cap N \in \mathfrak{H}$. Then $G \in \mathfrak{H}$.*

Proof. Let $C = C_G(N)$ and observe that $X \cap N$ is indeed a normal subgroup of $XC = G$. Now let G/K be an image of G. If $X \cap N \leq K$, then obviously $G/K \in \mathfrak{H}$. Moreover, if $(X \cap N)K = CK$, then $XK = XCK = G$, and so $G/K \cong X/X \cap K \in \mathfrak{H}$. Since $\mathfrak{H}$ is an $\mathfrak{N}\mathfrak{S}^*$-Schunck class, it suffices to show that every finite primitive and every infinite semiprimitive factor group of G belongs to $\mathfrak{H}$.

First, let G/K be a finite primitive image of G. By our preliminary observations, we may assume that $K < (X \cap N)K$. As $(X \cap N)K/K$ is nilpotent and $F(G/K)$ is the unique minimal normal subgroup of G/K, we have $(X \cap N)K/K = F(G/K)$. Since $F(G/K) = C_{G/K}(F(G/K))$ by [DH92, A, Theorem 15.6], it follows that $CK = (X \cap N)K$, and so $G/K \in \mathfrak{H}$.

Now let G/K be an infinite semiprimitive image of G. Then $G/K = (M/K)(D/K)$, where M/K is finite and $D/K = C_{G/K}(D/K)$ is a radicable abelian p-group. If $(X \cap N)K$ is finite, it is contained in $D_n/K = (D/K)[p^n]$ for some $n \in \mathbb{N}$. Since $G/K \cong G/D_n$ by [Tom95, Proposition 2.3 (ii)] and $G/D_n \in \mathfrak{H}$ because D_n contains $X \cap N$, the factor group G/K is an $\mathfrak{H}$-group. Therefore assume that $(X \cap N)K$ is infinite. Since $D/K = F(G/K) = C_G(F(G/K))$ by [Tom95, Proposition 2.3 (i)] and $(X \cap N)K/K$ is nilpotent, we have $(X \cap N)K = D$ and $CK = DK = (X \cap N)K$. Therefore G/K is an $\mathfrak{H}$-group by our introductory remarks. □

In order to prove the main theorem of this section, we first consider the following special case:

3.2.4 Proposition. *Suppose that G is an $\mathfrak{N}\mathfrak{S}^*$-group and that $\mathfrak{H}$ is an $\mathfrak{N}\mathfrak{S}^*$-Schunck class. Let X be an $\mathfrak{H}$-maximal subgroup of G supplementing a nilpotent normal subgroup N of G. Let $K \leq N$ be a normal subgroup of G such that $G/K \in \mathfrak{H}$. Then $G = XK$.*

Proof. Without loss of generality, we may suppose that N has finite index in G. Assume that $XK < G$, then XK is contained in a major subgroup M of G, and so K is contained in $L = M_G$. Now $G/L \cong XL/L \cong X/X \cap L$ is an $\mathfrak{H}$-group and by [Tom92], G/L is a finite primitive or an infinite semiprimitive group. In both cases, $G/L = M/L \ltimes F/L$, where M/L is finite and F/L is the Fitting subgroup of G/L; see [DH92, A, Theorem 15.6] and [Tom95, Proposition 2.3 (i)]. Therefore $NL/L \leq F/L$ and clearly $XL/L \leq M/L$. Since $G = XN$, we have $F = NL$ and $M = XL$.

Suppose first that G/L is finite, then also $G/L \cap N$ is finite, and so there exists a finite subgroup H of G such that $G = H(L \cap N)$. Let Y_0 be an $\mathfrak{H}$-projector of H, then $H = Y_0(H \cap N) = Y_0(H \cap L)$, since $H/H \cap N \cong G/N \in \mathfrak{H}$ and $H/H \cap L \cong G/L \in \mathfrak{H}$. Therefore we have $G = Y_0L = Y_0N$. Let Y be an $\mathfrak{H}$-maximal subgroup of G containing Y_0, then $G = YN$ and so Y is conjugate to X by Proposition 3.2.2. But then $|XL/L| = |YL/L| = |G/L|$ and so $G = XL = M$, a contradiction.

Otherwise G/L is a semiprimitive Černikov group. Since G/L is an $\mathfrak{H}$-group, G/L is the union of an ascending chain of finite $\mathfrak{H}$-groups L_i/L. Moreover, $XL/L \leq M/L$ is finite, and so there exists an integer i such that XL is properly contained in L_i. Therefore L_iN contains $XLN = DM = G$ and so $L_iN = G$. Since G/N is finite, also the group $L_i/L \cap N$ is finite. Thus there exists a finite subgroup H of G such that $L_i = H(L \cap N)$, and it follows that $G = HN$. As above, let Y_0 be an $\mathfrak{H}$-projector of H, then $H/H \cap L \cong HL/L = L_i/L \in \mathfrak{H}$ and so $H = Y_0(H \cap L)$ and $L_i = Y_0L$. Similarly, $H = Y_0(H \cap N)$ and so $G = Y_0N$. Let Y be an $\mathfrak{H}$-maximal subgroup of G containing Y_0, then by Proposition 3.2.2, there exists $g \in G$ such that $X^g = Y$. Therefore $XL < L_i \leq X^gL = G$. But this is impossible since XL/L is finite and so $|XL/L| = |X^gL/L|$. This final contradiction shows that $KX = G$. □

From this, we deduce the crucial covering property of an $\mathfrak{H}$-projector:

3.2.5 Proposition. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups and suppose that G is a periodic soluble group having a nilpotent normal subgroup N of finite index such that $G/N \in \mathfrak{H}$. If X is an $\mathfrak{H}$-maximal supplement of N in G and K is a normal subgroup of G such that $G/K \in \mathfrak{H}$, then $G = XK$.*

Proof. If $K \leq N$, this follows from Proposition 3.2.4. We proceed by induction on the order of the finite group $K/K \cap N$. Let $L = K'(K \cap N)$, then K/L is an abelian normal subgroup of G/L and so the normal subgroup NK/L is nilpotent. Let Y/L be an $\mathfrak{H}$-maximal subgroup of G/L containing the $\mathfrak{H}$-group XL/L, then $G/L = (Y/L)(NKL/L)$, and so by Proposition 3.2.4, it follows that $G = YK$. Now we have $Y = Y \cap XN = X(Y \cap N)$ and $L \cap N = K \cap N$. Moreover, $|L/K \cap N| < |K/K \cap N|$ because K is soluble and $K/K \cap N$ is finite. Since $Y/L \in \mathfrak{F}$, we have $Y = XL$ by induction hypothesis, and so $G = YK = XLK = XK$, as required. □

Now we are ready to prove the existence and conjugacy of $\mathfrak{H}$-projectors of nilpotent-by-finite periodic soluble groups.

3.2.6 Theorem. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups. Then every group $G \in \mathfrak{N}\mathfrak{S}^*$ possesses $\mathfrak{H}$-projectors, and any two are conjugate.*

Proof. Let $G \in \mathfrak{N}\mathfrak{S}^*$ and N a nilpotent normal subgroup of G which has finite index in G. Moreover, let H/N be an $\mathfrak{H}$-projector of the finite soluble group G/N, then by Proposition 3.2.2, there exists an $\mathfrak{H}$-maximal subgroup X of G such that $H = XN$. We show that X is an $\mathfrak{H}$-projector of G.

Let K be a normal subgroup of G and suppose that XK/K is contained in the $\mathfrak{H}$-group Y/K. Then $XNK/NK \leq YN/NK$, and since $H/N = XN/N$ is an $\mathfrak{H}$-projector of G/N, the group XNK/NK is an $\mathfrak{H}$-maximal subgroup of G/NK. Thus $XNK = HK = YN$. Therefore $Y = Y \cap HK = (Y \cap H)K$ and consequently $Y/K = (Y \cap H)K/K \cong (Y \cap H)/(H \cap K)$ is an $\mathfrak{H}$-group. Since $X \leq Y \cap H$, we have $Y \cap H = (Y \cap H) \cap XN = X(Y \cap H \cap N)$ by the modular law. Therefore X is an $\mathfrak{H}$-maximal supplement of the nilpotent normal subgroup $Y \cap H \cap N$ of $Y \cap H$. By Proposition 3.2.5, we have $Y \cap H = X(H \cap K)$ and so $Y = X(H \cap K)K = XK$. This shows that X is an $\mathfrak{H}$-projector of G.

Now let X_1 and X_2 be $\mathfrak{H}$-projectors of G. We show that X_1 and X_2 are conjugate. Since X_1N/N and X_2N/N are $\mathfrak{H}$-projectors of G/N, by the finite case (see e.g. [DH92, III, Theorem 3.21]), there exists an element $g \in G$ such that $X_1N = X_2^gN$. Thus X_1 and X_2^g are $\mathfrak{H}$-maximal supplements of N in the group X_1N, and hence by Proposition 3.2.2, there exists $h \in X_1N$ such that $X_1 = X_2^{gh}$, as required. □

The next corollary shows that $\mathfrak{H}$-projectors of periodic locally nilpotent-by-finite groups are even $\mathfrak{H}$-covering subgroups.

3.2.7 Corollary. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class and suppose that $G \in \mathfrak{N}\mathfrak{S}^*$. If X is an $\mathfrak{H}$-projector of G and $X \leq H \leq G$, then X is also an $\mathfrak{H}$-projector of H.*

Proof. Let N be a normal subgroup of finite index in G. By the finite case (see e.g. [DH92, III, Theorem 3.21]), the $\mathfrak{H}$-projector XN/N of G/N is also an $\mathfrak{H}$-projector of HN/N and so by an isomorphism theorem, $X(H \cap N)/(H \cap N)$ is an $\mathfrak{H}$-projector of $H/H \cap N$. Now let Y be an $\mathfrak{H}$-projector of H, then $Y(H \cap N)/(H \cap N)$ is also an $\mathfrak{H}$-projector of $H/H \cap N$. Therefore we have $X(H \cap N) = Y^h(H \cap N)$ for some $h \in H$ by Theorem 3.2.6. Now X and Y^h are $\mathfrak{H}$-maximal supplements of the nilpotent subgroup $H \cap N$ in the group $X(H \cap N)$. Therefore Proposition 3.2.2 shows that X and Y are conjugate. Hence X is an $\mathfrak{H}$-projector of H. □

It is also possible to extend [DH92, III, Lemma 3.14] to periodic soluble nilpotent-by-finite groups.

3.2.8 Corollary. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class and $G \in \mathfrak{N}\mathfrak{S}^*$. If N is a normal nilpotent subgroup of G such that $G/N \in \mathfrak{H}$, then every $\mathfrak{H}$-maximal supplement of N in G is an $\mathfrak{H}$-projector of G.*

Proof. Let H be an $\mathfrak{H}$-projector of G and L an $\mathfrak{H}$-maximal supplement of N in G. Moreover, by Fitting's theorem, we may assume without loss of generality that N has finite index in G. So H and L are conjugate by Proposition 3.2.2, and so L is an $\mathfrak{H}$-projector of G. □

3.3. Pronormal subgroups of periodic soluble nilpotent-by-finite groups

A subgroup P of a group G is called *pronormal* if, for every $g \in G$, the subgroups P and P^g are conjugate in their join $<P, P^g>$. If $\mathfrak{H}$ is an $\mathfrak{N}\mathfrak{S}^*$-Schunck class, then by Theorem 3.2.6 and Corollary 3.2.7, every $\mathfrak{H}$-projector of an $\mathfrak{N}\mathfrak{S}^*$-group G is a pronormal subgroup of G.

The finite case of following proposition has been proved by Mann [Man69, Corollary]; see also [DH92, I, Theorem 6.6].

3.3.1 Proposition. *Let G be a periodic soluble nilpotent-by-finite group. Then the following statements are equivalent:*

(a) *P is pronormal in G.*

(b) *Every Sylow basis of G reduces into exactly one conjugate of P.*

(c) *If there exists $g \in G$ such that the Sylow bases $\{S_p \mid p \in \mathbb{P}\}$ and $\{S_p^g \mid p \in \mathbb{P}\}$ of G reduce into P, then $P = P^g$.*

Proof. (a) ⇒ (b). Let $g \in G$ and suppose that $\{S_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G which reduces into P and P^g. By transfinite induction, we construct a descending chain

$$G = G_0 > G_1 > \ldots > G_\alpha = <P, P^g>$$

of subgroups G_β of G such that for every $\beta \leq \alpha$, the Sylow basis $\{S_p \mid p \in \mathbb{P}\}$ of G reduces into G_β, and every G_β contains the pronormal subgroups P and P^g. Suppose that we have constructed the subgroup G_β. If $<P, P^g>$ is properly contained in G_β, then by [Tom75, Lemma 2.3], $<P, P^g>$ is contained in a major subgroup M of G_β. Let N be the core of M in G_β, then by [Tom92], the factor group G_β/N is finite or an infinite semiprimitive Černikov group.

Assume first that G_β/N is finite. Since PN/N is a pronormal subgroup of G_β/N and the Sylow basis $\{(S_p \cap G_\beta)N/N \mid p \in \mathbb{P}\}$ of $G_\beta N/N$ reduces into PN/N and $P^g N$, we have $PN = P^g N$ by [DH92, I, Theorem 6.6]. Now $PN \leq M < G_\beta$ and the Sylow basis $\{S_p \cap G_\beta \mid p \in \mathbb{P}\}$ reduces into PN by Lemma 1.2.3 (e). Thus we set $G_{\beta+1} = PN$.

Otherwise, G_β/N is an infinite semiprimitive group. Let D/N be the maximal radicable abelian subgroup of G_β/N, then $G_\beta/N = M/N \ltimes D/N$ and M/N is finite. Therefore also PN/N and $P^g N/N$ are finite. Let K/N be a finite normal subgroup of D/N such that P^g is contained in PK. Then the Sylow basis $\{(S_p \cap G_\beta)N/N \mid p \in \mathbb{P}\}$ of $G_\beta N/N$ reduces into PN/N and $P^g N/N$ and also into the abelian group K/N. Therefore by Lemma 1.2.3 (d), $G_\beta N/N$ also reduces into the finite group $PK/N = P^g K/N$. Thus [DH92, I, Theorem 6.6] yields that $PN = P^g N$. Moreover, the Sylow basis $\{S_p \mid p \in \mathbb{P}\}$ of G reduces into PN by Lemma 1.2.3 (e). Since the subgroup $PN \leq M$ is a proper subgroup of G, we set $G_{\beta+1} = PN$.

If λ is a limit ordinal, then by Lemma 1.2.3 (c), $\{S_p \mid p \in \mathbb{P}\}$ also reduces into $\bigcap_{\beta<\lambda} G_\beta = G_\lambda$. Therefore we have $<P, P^g> = G_\alpha$ for every ordinal α whose cardinality exceeds that of G.

This shows that we may assume without loss of generality that $G = <P, P^g>$. Now suppose that P is properly contained in G. Then P is contained in a major subgroup M of G. Put $N = M_G$, then by the above arguments, PN/N is a finite subgroup of G/N, and hence $G/N = <PN/N, P^g N/N>$ is finite. Hence by [DH92, I, Theorem 6.6], we have $G/N = PN/N = P^g N/N$, contradicting $PN \leq M < G$. Therefore $G = P$, as required.

The implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) can be proved as in [DH92, I, Theorem 6.6]. □

Since $\mathfrak{H}$-projectors of periodic soluble nilpotent-by-finite groups are pronormal by Theorem 3.2.6 and Corollary 3.2.7, we obtain:

3.3.2 Corollary. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups and suppose that G is an $\mathfrak{N}\mathfrak{S}^*$-group. Then every Sylow basis of G reduces into exactly one $\mathfrak{H}$-projector of G.*

Chapter 4

Factorizers of subgroups of products

4.1. Factorizers of $\mathfrak{H}$-subgroups of nilpotent-by-finite groups

In this chapter, we examine under which hypotheses the factorizer of an $\mathfrak{X}$-subgroup of a product of two locally nilpotent subgroups is again an $\mathfrak{X}$-group, where $\mathfrak{X}$ is a Schunck class or a local formation. First, we consider $\mathfrak{H}$-maximal subgroups of nilpotent-by-finite products of two locally nilpotent subgroups. Our theorems generalize the results obtained in [AH94] and [Hoe93].

By a result of Gross [Gro73, Theorem 1], see also [AFG92, Lemma 2.5.2], a finite primitive group G which is the product of two nilpotent subgroups A and B is either a p-group or A and B are a Sylow p-subgroup or a Hall p'-subgroup of G. The following is a weaker version for semiprimitive groups.

4.1.1 Theorem. *Suppose that G is an infinite semiprimitive group with finite residual D and suppose that D is a p-group. If G is the product of two locally nilpotent subgroups A and B, then one of the groups $AO_p(G)/O_p(G)$ and $BO_p(G)/O_p(G)$ is a p-group and the other is a p'-group. In particular, A or B is a p-group.*

Proof. If G is a p-group, the statement is clear. Therefore suppose that G is not a p-group. Since $A_{p'}B_{p'}$ is a Sylow p'-subgroup of G, we must have $A_{p'} \neq 1$ or $B_{p'} \neq 1$. Assume without loss of generality that $B_{p'} \neq 1$. Then $D \cap B_{p'}^G$ is either finite or equals D. Assume first that $D \cap B_{p'}^G$ is finite. Then $D \cap B_{p'}^G \leq D[p^n] = N$ for some integer n and so $D/N \cap B_{p'}^G N/N = 1$, and in particular, $B_{p'}$ is contained in $C_G(D/N)$. As in the proof of [Tom95, Proposition 2.3 (ii)], there exists an isomorphism $G \to G/N$ mapping D to D/N, and so we have $D = C_G(D/N)$. But then $B_{p'}$ is contained in a p-group, contradicting $B_{p'} \neq 1$. This shows that we must have $D \leq B_{p'}^G$.

Now $O_{p'}(G)$ is contained in $C_G(D) = D$ and so $O_{p'}(G) = 1$. Therefore $[A_{p'}, D] \leq [A_{p'}^G, B_{p'}^G] = 1$ by Lemma 2.7 and Lemma 2.1 of [FGS94]. But then $A_{p'}$ is contained in $C_G(D) = D$ and A is a p-group. Now $B_p^G = B_p^A$ is contained in the Sylow p-subgroup AB_p of G. Therefore $B_p^G \leq O_p(G)$ and $BO_p(G)/O_p(G)$ is a p'-group. □

The following lemma further investigates the structure of certain semiprimitive groups.

4.1.2 Lemma. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups of characteristic π and suppose that $G = M \ltimes D$ is an infinite semiprimitive Černikov group, where D is a radicable abelian*

p-group for the prime p and M is finite and soluble. If G/D is an $\mathfrak{H}$-group and $p \in \pi$ but G is not an $\mathfrak{H}$-group, then M does not centralize any M-composition factor of D.

Proof. Since $G \notin \mathfrak{H}$, the $\mathfrak{H}$-subgroup M is an $\mathfrak{H}$-projector of G by Lemma 3.2.1. Let U/V be an M-composition factor of D which is centralized by M. Then $MU/V = MV/V \times U/V$, and since Schunck classes of finite groups are closed with respect to finite direct products by [DH92, III, Corollary 6.2] and U/V is an elementary abelian p-group and $p \in \pi$, we have $MU/V \in \mathfrak{H}$. On the other hand, by Corollary 3.2.7, M is also an $\mathfrak{H}$-projector of MU. This contradiction shows that M does not centralize any M-composition factor of D. □

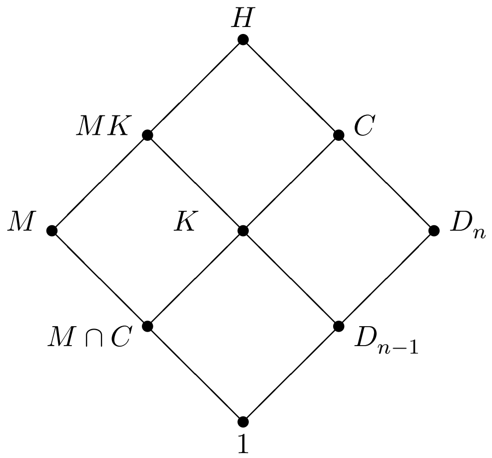
Next, we deduce an important property of groups satisfying the hypotheses of the preceding Lemma 4.1.2.

4.1.3 Lemma. *Suppose that G is an infinite semiprimitive Černikov group which is a semidirect product of a radicable abelian normal p-group D and a finite soluble group M. Further, assume that M does not centralize any M-composition factor of D (of a given M-composition series of D). If M is not a p-group, then $N_D(M_{p'}) = 1$ for every Hall p'-subgroup $M_{p'}$ of M.*

Proof. Let

$$1 = D_0 \lhd D_1 \lhd \ldots \lhd D_\alpha = D$$

be an M-composition series of D for an ordinal α whose factors are not centralized by M. Since D does not contain infinite M-invariant subgroups, we have $\alpha \leq \omega$, the least infinite ordinal number. Therefore it suffices to show that $N_{D_n}(M_{p'}) = 1$ for every integer n. We proceed by induction on n, assuming that $n > 0$ and $N_{D_{n-1}}(M_{p'}) = 1$.



The structure of the group H in the proof of Lemma 4.1.3

Let $H = MD_n$ and $C = C_H(D_n/D_{n-1})$. Put $K = C \cap MD_{n-1} = (C \cap M)D_{n-1}$ and observe that K is a normal subgroup of $H = D_nM$ because K/D_{n-1} is centralized by D_n and normalized by M. Since $D_n \cap K = D_{n-1}(D_n \cap M) = D_{n-1}$ by Dedekind's modular law, the factor group D_n/D_{n-1} is H-isomorphic with $D_nK/K = (C \cap M)D_n/K = C/K$. It follows that C/K is a self-centralized minimal normal subgroup of H/K. Therefore $H/K = (MK/K)(C/K)$ is a primitive group by [DH92, A, Theorem 15.8 (b)]. Let $R/C = O_{p'}(H/C)$ and $Q = M_{p'} \cap R$, then Q is nontrivial because $C/K = O_p(H/K)$.

Since the p'-group $QK/K = O_{p'}(MK/K)$ cannot be normal in H/K and MK/K is a maximal subgroup of H/K, it follows that $MK = N_H(QK/K)$ and since QK/K is a characteristic subgroup of MK/K, we also have $N_H(M_{p'}K/K) \leq N_H(QK/K) = MK$. It follows that $N_H(M_{p'}) \leq N_H(M_{p'}K/K) \leq MK$. Therefore

$$N_{D_n}(M_{p'}) \leq MK \cap D_n = M(C \cap M)D_{n-1} \cap D_n = MD_{n-1} \cap D_n = D_{n-1},$$

and it follows that $N_{D_n}(M_{p'}) = N_{D_{n-1}}(M_{p'}) = 1$, as required. □

Combining Lemma 4.1.2 and Lemma 4.1.3, we obtain a first result about $\mathfrak{H}$-maximal subgroups of infinite semiprimitive Černikov groups which are the product of two locally nilpotent subgroups.

4.1.4 Proposition. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups and suppose that G is an infinite semiprimitive Černikov group. Further, assume that every finite image of G is an $\mathfrak{H}$-group and that G is not an $\mathfrak{H}$-group. If G is the product of two locally nilpotent subgroups A and B, then G possesses an $\mathfrak{H}$-projector which contains A or B, hence is factorized.*

Proof. Let D denote the finite residual of G, which is a radicable abelian p-group for a prime p. Suppose that B is not a p-group, then by Theorem 4.1.1, A is a p-group and $B_{p'}$ is a Sylow p'-subgroup of G. Let M be a complement of D in G which contains $B_{p'}$, then by Lemma 4.1.2 and Lemma 4.1.3, M contains $B \leq N_G(B_{p'})$ because B is locally nilpotent. Since M is an $\mathfrak{H}$-projector of G by Lemma 3.2.1 (e) and M contains B, it follows that M is factorized. □

If G is a $\mathfrak{U}$-group with Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ and H is a subgroup of G, then by [Har71, Lemma 2.1] and [GHT71, Theorem 2.10], there exists a $g \in G$ such that $\{G_p \mid p \in \mathbb{P}\}$ reduces into H^g. Thus, with the notation of the following theorem, every $\mathfrak{H}$-subgroup has a conjugate H into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces. For details, see also the proof of Corollary 4.1.7 below.

4.1.5 Theorem. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class of characteristic π and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G is the product of two locally nilpotent subgroups A and B. Further, let H be an $\mathfrak{H}$-subgroup of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces.*

(a) *If π contains $\pi(A) \cap \pi(B)$, then the factorizer of H is an $\mathfrak{H}$-group.*

(b) *If H is a π-group, then the factorizer of H in $A_\pi B_\pi$ is an $\mathfrak{H}$-group. Hence H is contained in a prefactorized $\mathfrak{H}$-subgroup of G.*

Proof. (a) Let X denote the factorizer of H. Since the Sylow basis

$$\{(X \cap A_p)(X \cap B_p) \mid p \in \mathbb{P}\} = \{(X \cap A_pB_p) \mid p \in \mathbb{P}\}$$

reduces into H, we may assume without loss of generality that $G = X$. Therefore it remains to show that $G \in \mathfrak{H}$. Now $\mathfrak{H}$ is a Schunck class and our hypotheses are inherited by factor groups. Hence it suffices to consider the cases when G is a finite primitive group or an infinite semiprimitive Černikov group.

Suppose first that G is finite and primitive. Then by [Gro73, Theorem 1], either $G = A = B$ is a cyclic p group for some prime p, or A is a Sylow p-subgroup of G and B is a Hall p'-subgroup of G. In the first case, we have $p \in \pi$ and so $G \in \mathfrak{H}$. Otherwise, the Sylow basis

$\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H. Therefore the subgroup $H = (H \cap A_pB_p)(H \cap A_{p'}B_{p'}) = (H \cap A)(H \cap B)$ is factorized and hence $G = H \in \mathfrak{H}$.

If G is an infinite semiprimitive Černikov group, we have $G = M \ltimes D$, where D is a radicable abelian p-group for the prime p and M is finite. Since every primitive image of G/D belongs to $\mathfrak{H}$, we have $M \cong G/D \in \mathfrak{H}$ because $\mathfrak{H}$ is a Schunck class. Now suppose that $G \notin \mathfrak{H}$. Then by Theorem 4.1.1 and Proposition 4.1.4, without loss of generality, A is a p-group containing D and $B \leq M$ is finite. Now every subgroup containing B is factorized, so that the subgroup $MD[p^n] = MD[p^n] \cap AB = (MD[p^n] \cap A)B$ is factorized for every $n \in \mathbb{N}$. Since $G = \bigcup_{n\in\mathbb{N}} MD[p^n]$, this shows that every finite subgroup U of G is contained in a finite factorized subgroup of G. In particular, the factorizer of every finite subgroup of G is finite.

By Proposition 3.1.5, the Černikov group H is the union of an ascending chain $\{H_i \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups. Since $H \cap D$ has finite index in H, we may assume without loss of generality that $H = H_1(H \cap D)$. Since G is an $\mathfrak{U}$-group, there is a $g \in G$ such that the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H_1^g. Replacing H_i by H_i^g for every $i \in \mathbb{N}$, we may assume that the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H_1. Now $H_i = H_1(H_i \cap D)$ by the modular law, and so by Lemma 1.2.3 (d), the Sylow basis $\{A_pB_p\}$ reduces into every H_i. Therefore the factorizers X_i of the H_i are $\mathfrak{H}$-groups by the finite case. Now the union U of the factorizers of the H_i is a factorized subgroup of G which contains H. Thus $G = U$ and $\{X_i \mid i \in \mathbb{N}\}$ is an ascending chain of $\mathfrak{H}$-subgroups of the semiprimitive group G. By the definition of a Schunck class, this contradicts $G \notin \mathfrak{H}$, and so $G \in \mathfrak{H}$.

(b). Since H is a π-group and the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H, we have $H \leq A_\pi B_\pi$. Applying (a) to the group $A_\pi B_\pi$, we obtain that the factorizer of H in $A_\pi B_\pi$ is an $\mathfrak{H}$-group, as required. □

From this theorem, we derive a necessary and sufficient condition for an $\mathfrak{H}$-maximal subgroup of G to be factorized.

4.1.6 Corollary. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class of characteristic π and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G is the product of two locally nilpotent subgroups A and B. If H is an $\mathfrak{H}$-maximal subgroup of G, then:*

(a) *If π contains $\pi(A) \cap \pi(B)$, then H is prefactorized if and only if the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H. Thus an $\mathfrak{H}$-maximal subgroup of G is prefactorized if and only if it is factorized.*

(b) *If H is a π-group, then H is prefactorized if and only if the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

Proof. If H is any prefactorized subgroup of G, then by Theorem 2.3.7, the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H. This shows the necessity of our conditions.

Conversely, if π contains $\pi(A) \cap \pi(B)$ and the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H, then the factorizer X of H is an $\mathfrak{H}$-group by Theorem 4.1.5. Hence $H = X$ by the $\mathfrak{H}$-maximality of H, and so H is factorized.

As in the proof of Theorem 4.1.5, statement (b) now follows by considering the Sylow π-subgroup $A_\pi B_\pi$ instead of G. □

Since every $\mathfrak{H}$-maximal subgroup of an $\mathfrak{N}\mathfrak{S}^*$-group possesses a conjugate into which a given Sylow basis of G reduces, we also have:

4.1.7 Corollary. *Let $\mathfrak{X}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class of characteristic π and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G is the product of two locally nilpotent subgroups A and B.*

(a) *If π contains $\pi(A) \cap \pi(B)$, then every $\mathfrak{H}$-maximal subgroup of G has a factorized conjugate.*

(b) *Every $\mathfrak{H}$-maximal subgroup of G which is a π-group has a prefactorized conjugate.*

Proof. Let H be an $\mathfrak{H}$-maximal subgroup of G, then a Sylow basis of H can be extended to a Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ of G (see [Har71, Lemma 2.1]). Therefore by [GHT71, Theorem 2.10], there exists an element $g \in G$ such that $\{G_p^g \mid p \in \mathbb{P}\} = \{A_pB_p \mid p \in \mathbb{P}\}$. Thus $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into $H^{g^{-1}}$. The result now follows from Corollary 4.1.6. □

Since $\mathfrak{H}$-projectors are in particular $\mathfrak{H}$-maximal subgroups, we also obtain

4.1.8 Corollary. *Let $\mathfrak{X}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class of characteristic π and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G is the product of two locally nilpotent subgroups A and B. If π contains $\pi(A) \cap \pi(B)$ or an $\mathfrak{H}$-projector of G is a π-group, then G possesses a unique $\mathfrak{H}$-projector H which is prefactorized. If π contains $\pi(A) \cap \pi(B)$, then H is even factorized.*

Proof. By Corollary 3.3.2, the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into a unique $\mathfrak{H}$-projector of G. Therefore by Theorem 2.3.7, G has at most one prefactorized $\mathfrak{H}$-projector. Since G possesses a prefactorized $\mathfrak{H}$-projector by Corollary 4.1.7 and this projector is factorized if π contains $\pi(A) \cap \pi(B)$, the proof is complete. □

The above results can also be applied to trifactorized groups.

4.1.9 Corollary. *Let $\mathfrak{H} = QS\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G has a triple factorization $G = AB = AC = BC$ by its subgroups A, B and C, where A and B are locally nilpotent and $C \in \mathfrak{H}$. If $\pi(A) \cap \pi(B)$ is contained in the characteristic of $\mathfrak{H}$, then $G \in \mathfrak{H}$.*

Proof. In view of Lemma 3.1.3 (c), we may assume without loss of generality that C is an $\mathfrak{H}$-maximal subgroup of G. Hence C has a factorized conjugate by Corollary 4.1.7. Therefore $G = C$ by [AH94, Lemma 1]. □

If $\mathfrak{F}$ is a local formation of characteristic π, then by Lemma 1.5.1, every $\mathfrak{F}$-group is a π-group . This shows that the hypothesis of Theorem 4.1.5 (b) is always satisfied if $\mathfrak{H} = \mathfrak{F}$ is a local formation. Thus we obtain:

4.1.10 Theorem. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of $\mathfrak{N}\mathfrak{S}^*$-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Further, suppose that the $\mathfrak{X}$-group G is the product of two locally nilpotent groups A and B. If H is an $\mathfrak{F}$-subgroup of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, then H is contained in a prefactorized $\mathfrak{F}$-subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then H is even contained in a factorized $\mathfrak{F}$-subgroup of G.*

Proof. By Lemma 1.5.1, the $\mathfrak{F}$-group H is a π-group. Hence H is contained in the Sylow π-subgroup $A_\pi B_\pi$ of G. Since by Proposition 3.1.2, $\mathfrak{F}$ is a Schunck class of nilpotent-by-finite groups. Therefore by Theorem 4.1.5, the factorizer X of H in $A_\pi B_\pi$ is an $\mathfrak{F}$-group. Since $A_\pi B_\pi$ is a prefactorized subgroup of G, the subgroup X is the required prefactorized subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then $A \cap B$ is a π-group and so $A \cap B = A_\pi \cap B_\pi$ is contained in X. Hence X is a factorized subgroup of G. □

4.2. Factorizers of $\mathfrak{F}$-subgroups of *FC*- and *CC*-groups

In this section, we will show that for local formations of periodic FC- and CC-groups, results similar to those of Section 4.1 can be obtained. Since the concept of Schunck classes has not yet been extended to the class of all periodic locally soluble CC-groups, we formulate our theorems for local $\mathfrak{X}$-formations of periodic locally soluble CC-groups only. Note also that FC-groups are CC-groups, so that our results hold in particular for local formations of periodic locally soluble FC-groups.

First, we show that, as in the case of $\mathfrak{N}\mathfrak{S}^*$-groups, every $\mathfrak{F}$-subgroup of a CC-group G is contained in an $\mathfrak{F}$-maximal subgroup of G.

4.2.1 Lemma. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of periodic locally soluble CC-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Moreover, let G be an $\mathfrak{X}$-group.*

(a) *The group G is an $\mathfrak{F}$-group if and only if G is a π-group and $G/C_G(x^G) \in \mathfrak{F}$ for every $x \in G$.*

(b) *The class $\mathfrak{F}$ is closed with respect to unions of chains of subgroups.*

Proof. (a) If G is an $\mathfrak{F}$-group, then clearly every factor group of G belongs to $\mathfrak{F}$. Conversely, suppose that $G/C_G(x^G) \in \mathfrak{F}$ for every $x \in G$. Since $Z(G) = \bigcap_{x\in G} C_G(x^G)$, we have $G/Z(G) \in \mathfrak{F}$ by Lemma 1.5.2. Therefore also $G \in \mathfrak{F}$ by Lemma 1.5.3.

(b) Let $\{G_i\}$ be a chain of $\mathfrak{F}$-subgroups of the $\mathfrak{X}$-group G and assume without loss of generality that $G = \bigcup G_i$. If $x \in G$, then $G/C_G(x^G)$ is a Černikov group. Since $\mathfrak{F} \cap \mathfrak{N}\mathfrak{S}^*$ is a local $\mathfrak{N}\mathfrak{S}^*$-formation, hence a $\mathfrak{N}\mathfrak{S}^*$-Schunck class by Proposition 3.1.2, by Lemma 3.1.3 (c) the factor groups $G/C_G(x^G)$ are $\mathfrak{F}$-groups for every $x \in G$. Therefore $G \in \mathfrak{F}$ by Lemma 4.2.1 (a). □

Now we can prove an analogue of Theorem 4.1.5 for periodic CC-groups which are the product of two locally nilpotent subgroups.

4.2.2 Theorem. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of periodic locally soluble CC-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Further, suppose that the $\mathfrak{X}$-group G is the product of two locally nilpotent groups A and B. If H is an $\mathfrak{F}$-subgroup of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, then H is contained in a prefactorized $\mathfrak{F}$-subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then the factorizer of H is an $\mathfrak{F}$-subgroup of G.*

Proof. Suppose first that $\pi(A) \cap \pi(B) \subseteq \pi$ and let X denote the factorizer of H in G. By Theorem 2.3.7, the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G, reduces into X. Therefore we may assume without loss of generality that $G = X$. Hence it remains to show that $G \in \mathfrak{F}$.

Let $x \in G$, then $G/C_G(x^G)$ is a Černikov group. Moreover, the Sylow basis

$$\{A_pB_pC_G(x^G)/C_G(x^G) \mid p \in \mathbb{P}\}$$

of $G/C_G(x^G)$ reduces into the group $HC_G(x^G)/C_G(x^G)$. Since $\mathfrak{F} \cap \mathfrak{N}\mathfrak{S}^*$ is a $\mathfrak{N}\mathfrak{S}^*$-Schunck class by Proposition 3.1.2, the factorizer $Y/C_G(x^G)$ of the $\mathfrak{F}$-group $HC_G(x^G)/C_G(x^G)$ is also an $\mathfrak{F}$-group by Theorem 4.1.5. Now Y is a factorized subgroup of G containing H, and so $G = Y$ and $G/C_G(x^G) \in \mathfrak{F}$. Therefore $G \in \mathfrak{F}$ by Lemma 4.2.1 (a).

In the general case, the π-group H is contained in the Sylow π-subgroup $A_\pi B_\pi$ of G because $\{A_p B_p \mid p \in \mathbb{P}\}$ reduces into H. Therefore by the first part, the factorizer of H in $A_\pi B_\pi$ is a prefactorized $\mathfrak{F}$-subgroup of G which contains H. □

As in the case of Theorem 4.1.5, we deduce a number of useful consequences, whose proofs are similar to the corresponding results about nilpotent-by-finite groups. First, we derive a necessary and sufficient condition for an $\mathfrak{F}$-maximal subgroup of G to be factorized.

4.2.3 Corollary. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of periodic locally soluble CC-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Further, suppose that the $\mathfrak{X}$-group G is the product of two locally nilpotent subgroups A and B and let H be an $\mathfrak{F}$-maximal subgroup of G.*

(a) *H is prefactorized if and only if the Sylow basis $\{A_p B_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

(b) *If π contains $\pi(A) \cap \pi(B)$, then the subgroup H is factorized if and only if the Sylow basis $\{A_p B_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

Since the Sylow bases of a periodic locally soluble CC-groups are locally conjugate by [OP91, Theorem 4.3], the following lemma shows that in Theorem 4.2.2, every $\mathfrak{F}$-subgroup H has a local conjugate into which the Sylow basis $\{A_p B_p \mid p \in \mathbb{P}\}$ of G reduces.

4.2.4 Lemma. *Let G be a periodic locally soluble CC-group and H a subgroup of G. Then every Sylow basis of H can be extended to a Sylow basis of G.*

Proof. Let $\{H_p \mid p \in \mathbb{P}\}$ be a Sylow basis of H. For every prime p, put

$$H_{p'} = <H_q \mid q \in \mathbb{P}, q \neq p>.$$

Moreover, let $G_{p'}$ be a Sylow p'-subgroup of G which contains $H_{p'}$. Define

$$G_p = \bigcap_{q \in \mathbb{P}, q \neq p} G_{q'},$$

then $\{G_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G by [OP91, Lemma 4.2]. Since H_p is contained in G_p for every prime p, the Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ reduces into H. □

For $\mathfrak{F}$-maximal subgroups, this has the following consequence.

4.2.5 Corollary. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Suppose that the CC-group G is the product of two locally nilpotent subgroups A and B. Then:*

(a) *Every $\mathfrak{F}$-maximal subgroup of G is locally conjugate to a prefactorized $\mathfrak{F}$-maximal subgroup of G.*

(b) *If π contains $\pi(A) \cap \pi(B)$, then every $\mathfrak{F}$-maximal subgroup of G is locally conjugate to a factorized $\mathfrak{F}$-maximal subgroup of G.*

To prove that a periodic locally soluble CC-group which is the product of two locally nilpotent subgroups has at most one prefactorized $\mathfrak{F}$-projector, we need the following result.

4.2.6 Proposition. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of periodic locally soluble CC-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. If the $\mathfrak{X}$-group G has an $\mathfrak{F}$-projector, then every Sylow basis of the reduces into a unique $\mathfrak{F}$-projector of G. Thus the $\mathfrak{F}$-projectors of G are locally conjugate.*

Proof. Let H be an $\mathfrak{F}$-projector of G, then by Lemma 4.2.4, there exists a Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ of G which reduces into H. Now assume that L is an $\mathfrak{F}$-projector into which $\{G_p \mid p \in \mathbb{P}\}$ reduces. Let $x \in G$, then the Sylow basis

$$\{G_p C_G(x^G)/C_G(x^G) \mid p \in \mathbb{P}\}$$

of $G/C_G(x^G)$ reduces into both $HC_G(x^G)/C_G(x^G)$ and $LC_G(x^G)/C_G(x^G)$. Therefore

$$HC_G(x^G)/C_G(x^G) = LC_G(x^G)/C_G(x^G)$$

by Corollary 3.3.2. Put $H^* = \bigcap_{x \in G} HC_G(x^G)$, then

$$H^* C_G(x^G)/C_G(x^G) = HC_G(x^G)/C_G(x^G) \in \mathfrak{F}.$$

Thus by Lemma 1.5.2, $H^*/Z(G) \in \mathfrak{F}$, and finally $H^* \in \mathfrak{F}$ by Lemma 1.5.3. Since H^* contains both H and L, it follows that $H = H^* = L$ by the $\mathfrak{F}$-maximality of H and L.

Now let H and H^* be arbitrary $\mathfrak{F}$-projectors of G and suppose that $\{G_p \mid p \in \mathbb{P}\}$ and $\{G_p^* \mid p \in \mathbb{P}\}$ are Sylow bases of G reducing into H and H^*, respectively. Since the Sylow bases of G are locally conjugate by [OP91, Theorem 4.3], there exists a locally inner automorphism ϕ of G such that $G_p^\phi = G_p^*$ for every $p \in \mathbb{P}$. Now the Sylow basis $\{G_p^* \mid p \in \mathbb{P}\}$ reduces into H^ϕ and H^*, and so we have $H^* = H^\phi$ by the first part. □

Although it seems to be an open question whether every periodic locally soluble CC-group possesses $\mathfrak{F}$-projectors, there is also a result analogous to Corollary 4.1.8, provided that the CC-groups in question possess $\mathfrak{F}$-projectors.

4.2.7 Theorem. *Let $\mathfrak{X} = \mathrm{QS}\mathfrak{X}$ be a class of periodic locally soluble CC-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. If G is an $\mathfrak{X}$-group which is the product of two locally nilpotent subgroups A and B, then G has at most one prefactorized $\mathfrak{F}$-projector. If G has $\mathfrak{F}$-projectors, then G possesses a unique $\mathfrak{F}$-projector which is prefactorized. If π contains $\pi(A) \cap \pi(B)$, then this $\mathfrak{F}$-projector is factorized.*

For periodic locally soluble FC-groups, the existence and local conjugacy of $\mathfrak{F}$-projectors has been proved by Tomkinson [Tom69a]; see also [Tom84]. Thus we obtain:

4.2.8 Corollary. *Let $\mathfrak{X} = \mathrm{QS}\mathfrak{X}$ be a class of periodic locally soluble FC-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Then every $\mathfrak{X}$-group G which is the product of two locally nilpotent subgroups A and B possesses a unique $\mathfrak{F}$-projector which is prefactorized. If π contains $\pi(A) \cap \pi(B)$, then this $\mathfrak{F}$-projector is factorized.*

Despite the fact the Sylow bases of a periodic locally soluble CC-group need not be conjugate, also a result similar to Corollary 4.1.9 can be obtained.

4.2.9 Theorem. *Let $\mathfrak{X} = \mathrm{QS}\mathfrak{X}$ be a class of periodic locally soluble CC-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Moreover, suppose that the $\mathfrak{X}$-group G has subgroups A, B and C such that $G = AB = AC = BC$. If A and B are locally nilpotent, $C \in \mathfrak{F}$ and $\pi(A) \cap \pi(B)$ is contained in π, then $G \in \mathfrak{X}$.*

Proof. Let $x \in G$, then $G/C_G(x^G)$ is a Černikov group. Since by Proposition 3.1.2, $\mathfrak{F} \cap \mathfrak{N}\mathfrak{S}^*$ is a $\mathfrak{N}\mathfrak{S}^*$-Schunck class, we have $G/C_G(x^G) \in \mathfrak{F}$ by Corollary 4.1.9. Therefore $G \in \mathfrak{F}$ by Lemma 4.2.1 (a). □

4.3. Factorizers of $\mathfrak{F}$-subgroups of groups with min-p for all primes p

Since periodic locally soluble groups satisfying the minimal condition on p-subgroups for every prime p are residually Černikov groups by [KW73, Theorem 3.17], the methods applied to periodic CC-groups which are the product of two locally nilpotent subgroups yield essentially the same results for periodic locally soluble groups satisfying min-p for every prime p. The main difficulties are due to the fact that Sylow bases of the latter class of groups are not so well-behaved as in the case of CC-groups.

4.3.1 Theorem. *Let $\mathfrak{X} = \text{QS}\mathfrak{X}$ be a class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Further, suppose that the $\mathfrak{X}$-group G is the product of two locally nilpotent groups A and B. If H is an $\mathfrak{F}$-subgroup of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, then H is contained in a prefactorized $\mathfrak{F}$-subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then H is even contained in a factorized $\mathfrak{F}$-subgroup of G.*

Proof. Let X denote the factorizer of H in $A_\pi B_\pi$, then we may assume without loss of generality that $G = X$. Since the factor group $G/O_{\pi'}(G)$ is a Černikov group for every finite set π of primes by [KW73, Theorem 3.17], an argument similar to that in the proof of Theorem 4.2.2 shows that $G/O_{\pi'}(G) \in \mathfrak{F}$. Now the intersection of all subgroups $O_{\pi'}(G)$, where π is a finite set of primes, is trivial, we have $G \in \mathfrak{F}$ by Lemma 1.5.2. □

For $\mathfrak{F}$-maximal subgroups, this has the following consequence.

4.3.2 Corollary. *Let $\mathfrak{X} = \text{QS}\mathfrak{X}$ be a class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Further, suppose that the $\mathfrak{X}$-group G is the product of two locally nilpotent subgroups A and B and let H be an $\mathfrak{F}$-maximal subgroup of G.*

(a) *H is prefactorized if and only if the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

(b) *If π contains $\pi(A) \cap \pi(B)$, then the subgroup H is factorized if and only if the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

The proof of the next theorem does not use the above results about periodic locally soluble products satisfying min-p. Instead, it relies on the nilpotent-by-finite case.

4.3.3 Theorem. *Let $\mathfrak{X} = \text{QS}\mathfrak{X}$ be a class of periodic locally soluble groups satisfying min-p for all primes p and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Further, suppose that the $\mathfrak{X}$-group G has a triple factorization $G = AB = AC = BC$ by its subgroups A, B and C, where A and B are locally nilpotent and $C \in \mathfrak{F}$. If $\pi(A) \cap \pi(B) \subseteq \pi$, then $G \in \mathfrak{F}$.*

Proof. Let π be a finite set of primes. By [KW73, Theorem 3.17], the factor group $G/O_{\pi'}(G)$ is a Cernikov group. So by Corollary 4.1.9, we have $G/O_{\pi'}(G) \in \mathfrak{F}$ for every finite set π of primes. Since the intersection of the subgroups $O_{\pi'}(G)$, where π is a finite set of primes, is trivial, we have $G \in \mathfrak{F}$ by Lemma 1.5.2. □

The next proposition will be used to show that a periodic locally soluble group satisfying min p for every prime p which is the product of two locally nilpotent subgroups has at most one prefactorized $\mathfrak{F}$-projector.

4.3.4 Proposition. *Let $\mathfrak{X} = \mathit{QS}\mathfrak{X}$ be a class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a local $\mathfrak{X}$-formation. If the $\mathfrak{X}$-group G has an $\mathfrak{F}$-projector, then every Sylow basis of the reduces into at most one $\mathfrak{F}$-projector of G.*

Proof. Let H and L be $\mathfrak{F}$-projectors of G into which the Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ of G reduces. Let $p \in \mathbb{P}$, then the Sylow basis $\{G_pO_{p'}(G)/O_{p'}(G) \mid p \in \mathbb{P}\}$ of $G/O_{p'}(G)$ reduces into $HO_{p'}(G)/O_{p'}(G)$ and $LO_{p'}(G)/O_{p'}(G)$. Thus by Corollary 3.3.2, we have $HO_{p'}(G) = LO_{p'}(G)$. Since $\bigcap_{p\in\mathbb{P}} O_{p'}(G) = 1$, it follows from Lemma 1.6.1 that $H = L$. □

Although the Sylow bases of a periodic locally soluble group G satisfying min-p for every prime p are locally conjugate by [DT80], G may have $\mathit{L}\mathfrak{N}$-projectors into which no Sylow basis reduces [Dix82, Section 5], even if G is countable. Therefore our next result might also be of independent interest. Recall that a group G is *co-Hopfian* if it does not contain a proper subgroup isomorphic with G. In particular, every periodic radical group satisfying min-p is co-hopfian (cf. [Bae70]).

4.3.5 Proposition. *Let $\mathfrak{X} = \mathit{QS}\mathfrak{X}$ be a class of countable locally finite-soluble group satisfying min-p for all primes p. If $G \in \mathfrak{X}$ and $\mathfrak{F}$ is a class of co-Hopfian groups, then every Sylow basis of G reduces into a unique $\mathfrak{F}$-projector of G.*

Proof. Let $\{G_p \mid p \in \mathbb{P}\}$ be a Sylow basis of G and let $\{p_1, p_2, \ldots\}$ denote the set of all primes in their natural order and set $N_i = O_{\{p_{i+1},p_{i+2},\ldots\}}$ for every $i \in \mathbb{N}$. Then G/N_i is a Černikov group by [KW73, Theorem 3.17]. Hence it has an $\mathfrak{F}$-projector H_i/N_i into which the Sylow basis $\{G_pN_i/N_i \mid p \in \mathbb{P}\}$ of G/N_i reduces. Let $H = \bigcap_{n\in\mathbb{N}} H_i$, then by Lemma 1.2.3 (c), the Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ also reduces into H. Continuing as in the proof of [Dix82, Theorem 3.4], H is an $\mathfrak{F}$-projector of G. The uniqueness statement now follows from Proposition 4.3.4. □

Since every countable periodic locally soluble group satisfying min-p for every prime p possesses $\mathfrak{F}$-projectors by [Dix82, Theorem 3.4], we thus obtain:

4.3.6 Corollary. *Let $\mathfrak{X} = \mathit{QS}\mathfrak{X}$ be a class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. Then every $\mathfrak{X}$-group G which is the product of two locally nilpotent subgroups A and B has at most one prefactorized $\mathfrak{F}$-projector. If $\mathfrak{F}$ is a class of co-Hopfian groups, then G possesses a unique $\mathfrak{F}$-projector which is prefactorized. If, in addition, π contains $\pi(A) \cap \pi(B)$, then this $\mathfrak{F}$-projector is factorized.*

4.4. Triply factorized groups

Throughout this chapter, we have frequently encountered *trifactorized groups*, i.e. groups G possessing subgroups A, B and C such that $G = AB = AC = BC$. In Corollary 4.1.9, Theorem 4.2.9 and Theorem 4.3.3, we have shown that under certain additional assumptions, G is an $\mathfrak{F}$-group if A and B are locally nilpotent and $C \in \mathfrak{F}$, where $\mathfrak{F}$ is a local formation.

Since by [AFG92, Lemma 1.1.4], the factorizer X of a normal subgroup N of a product $G = AB$ has a triple factorization

$$X = (A \cap BN)(AN \cap B) = (A \cap BN)N = (AN \cap B)N,$$

it is also of interest to study triply factorized groups, i.e. groups G which possess subgroups A and B and a normal subgroup N such that $G = AB = AN = BN$. We will show below that in this case, in order to prove that G is an $\mathfrak{F}$-group, it suffices that A and B are $\mathfrak{F}$-groups and N is locally nilpotent. The following example shows that even in the finite case, the assumption that N is normal in G cannot be replaced by the assumption that A and B are normal subgroups of G. It also shows that in Corollary 4.1.9, Theorem 4.2.9 and Theorem 4.3.3 it is not enough to assume that A is locally nilpotent and B and C belong to the local formation $\mathfrak{F}$.

4.4.1 Example. Let p be a prime, P an extraspecial p-group of order p^3 and let A_0 and B_0 be distinct maximal subgroups of P. Let $q \neq p$ be a prime and $F = GF(q)$ the field with q elements, then by [DH92, B, Corollary 10.7], the p-group P has a faithful irreducible FP-module N. Put $G = P \ltimes N$ and let $A = A_0N$ and $B = B_0N$, then $G = AB = AP = BP$, and the normal subgroups A and B of G belong to the local $\mathfrak{S}^*$-formation $\mathfrak{F} = \mathfrak{N}^*\mathfrak{A}^*$of all finite nilpotent-by-abelian groups. But since $N = F(G)$ and P is nonabelian, we have $G \notin \mathfrak{N}^*\mathfrak{A}^*$. Moreover, if we choose $q > p$, then A and B even belong to the local $\mathfrak{S}^*$-formation of all finite supersoluble groups, but since G is not nilpotent-by-abelian, it cannot be supersoluble.

Now we come to our first theorem about triply factorized groups.

4.4.2 Theorem. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups. Suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G has $\mathfrak{H}$-subgroups A and B and a normal locally nilpotent subgroup R such that $G = AB = AR = BR$. Then G is an $\mathfrak{H}$-group.*

Proof. We may assume without loss of generality that A and B are $\mathfrak{H}$-maximal subgroups of G. Then A and B are conjugate by Proposition 3.2.2. Then by direct calculation or by [Wie58, *Hilfssatz* 10], we have $G = A = B$, and so G is an $\mathfrak{H}$-group. □

Theorem 4.4.2 may be used to obtain a similar theorem for local formations of FC- and CC-groups.

4.4.3 Theorem. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of periodic locally soluble CC-groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation. Suppose that the $\mathfrak{X}$-group G has $\mathfrak{F}$-subgroups A and B and a normal locally nilpotent subgroup R such that $G = AB = AR = BR$. Then G is an $\mathfrak{F}$-group.*

Proof. Let $x \in G$ and put $N = C_G(x^G)$. Then the factor group $G/C_G(x^G)$ is a Černikov group and $G/N = (AN/N)(BN/N) = (AN/N)(RN/N) = (BN/N)(RN/N)$ and so $G/N \in \mathfrak{F}$ by Theorem 4.4.2. Since this holds for every $x \in G$, we have $G \in \mathfrak{F}$ by Lemma 4.2.1 (a). □

The same argument can also be used to prove Theorem 4.4.2 for groups satisfying the minimal condition on p-subgroups.

4.4.4 Theorem. *Let $\mathfrak{X} = QS\mathfrak{X}$ be a class of periodic locally soluble groups satisfying min-p for every prime p. Moreover, let $\mathfrak{F}$ be a local $\mathfrak{X}$-formation. Suppose that the $\mathfrak{X}$-group G has $\mathfrak{F}$-subgroups A and B and a normal locally nilpotent subgroup R such that $G = AB = AR = BR$. Then G is an $\mathfrak{F}$-group.*

Proof. Let π be a finite set of primes and put $N = O_{\pi'}(G)$. Then G/N is a Černikov group by [KW73, Theorem 3.17]. Moreover, $G/N = (AN/N)(BN/N) = (AN/N)(RN/N) = (BN/N)(RN/N)$ and so $G/N \in \mathfrak{F}$ by Theorem 4.4.2. Since the intersection of all $O_{\pi'}(G)$, where π is a finite set of primes, is trivial, we have $G \in \mathfrak{F}$ by Lemma 1.5.2. □

In order to obtain a theorem similar to Theorem 4.4.3 and Theorem 4.4.4 for $\mathfrak{U}$-groups, we need a result like Proposition 3.2.2 for local formations of $\mathfrak{U}$-groups.

4.4.5 Theorem. *Let $\mathfrak{X}$ be a* QS*-closed subclass of $\mathfrak{U}$ and $\mathfrak{F}$ a local $\mathfrak{X}$-formation. Further, suppose that G is $\mathfrak{X}$-group such that G/R is in $\mathfrak{F}$ for some locally nilpotent subgroup R of G. Then the $\mathfrak{F}$-maximal supplements of R in G coincide with the $\mathfrak{F}$-projectors of G and the $\mathfrak{F}$-normalizers of G. Hence the $\mathfrak{F}$-maximal supplements of R are conjugate in G.*

Proof. Let H be an $\mathfrak{F}$-maximal supplement of R in G. Since $H \in \mathfrak{F}$, by [GHT71, Theorem 4.6 (iii)], the group H coincides with its $\mathfrak{F}$-normalizer. Therefore by [GHT71, Theorem 4.9], there exists an $\mathfrak{F}$-normalizer D of G containing H. Since $D \in \mathfrak{F}$ by [GHT71, Theorem 4.6 (vi)] and H is $\mathfrak{F}$-maximal, it follows that $D = H$. Now [GHT71, Theorem 5.1] shows that the $\mathfrak{F}$-projectors of G coincide with the $\mathfrak{F}$-normalizers of G. The conjugacy of the $\mathfrak{F}$-maximal supplements of R now follows directly from the fact that the $\mathfrak{F}$-projectors of G are conjugate by [GHT71, Theorem 5.4]. □

Our theorem about triply factorized $\mathfrak{U}$-groups generalizes a result of B. Amberg and A. Fransman [AF94, Corollary 2], replacing the nilpotency hypothesis on the normal subgroup by local nilpotency, at the same time shortening the proof considerably.

4.4.6 Theorem. *Let $\mathfrak{X}$ be a* QS*-closed subclass of $\mathfrak{U}$ and $\mathfrak{F}$ a local $\mathfrak{X}$-formation. Suppose that the $\mathfrak{X}$-group G has $\mathfrak{F}$-subgroups A and B and a normal locally nilpotent subgroup R, such that $G = AB = AR = BR$. Then G is an $\mathfrak{F}$-group.*

Proof. Without loss of generality, we may assume that A and B are $\mathfrak{F}$-maximal subgroups of G. Therefore A and B are conjugate by Theorem 4.4.5. As in Theorem 4.4.2, this yields $G = A = B$ and so G is an $\mathfrak{F}$-group. □

Chapter 5

Projectors and injectors of products

5.1. Projectors in soluble and hypoabelian $\mathfrak{U}$-groups

Let $\mathfrak{F}$ be a local $\mathfrak{U}$-formation. Although we have not been able to prove the existence of prefactorized $\mathfrak{F}$-maximal subgroups of a $\mathfrak{U}$-group G which is the product of two locally nilpotent subgroups, we have nevertheless obtained positive results for the most important class of $\mathfrak{F}$-maximal subgroups of G, namely for $\mathfrak{F}$-projectors of G. As a first step, we consider periodic locally soluble groups which are the extension of a p-group by an $\mathfrak{F}$-group.

Let G be a group and suppose that $\mathfrak{F}$ is any class of groups. Then $G^{\mathfrak{F}}$ denotes the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{F}$. Observe that if $\mathfrak{F}$ is an $\mathfrak{X}$-formation for some Q-closed class of groups, then $G/G^{\mathfrak{F}} \in \mathfrak{F}$.

5.1.1 Proposition. *Suppose that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation of characteristic π for some QS-closed class $\mathfrak{X}$ of locally finite groups. Let G be an $\mathfrak{X}$-group such that $G^{\mathfrak{F}}$ is a p-group for some $p \in \pi$ and suppose that H is an $\mathfrak{F}$-maximal subgroup of G which satisfies $G = HG^{\mathfrak{F}}$. Then:*

(a) *$H = N_G(O_{p'}(H))$.*

(b) *If the Sylow p'-subgroups of every subgroup S of G are conjugate in S, then every Sylow p'-subgroup of G reduces into at most one conjugate of H.*

(c) *If $G^{\mathfrak{F}}$ is abelian, then H complements $G^{\mathfrak{F}}$.*

(d) *If $G^{\mathfrak{F}}$ is abelian, then then every Sylow p'-subgroup of G reduces into at most one complement of $G^{\mathfrak{F}}$.*

Proof. (a) Let $Q = O_{p'}(H)$ and set $L = N_G(Q)$, then clearly, $H \leq L$. We will show that $L \in \mathfrak{F}$. Then the desired result will follow from the $\mathfrak{F}$-maximality of H. Since $\mathfrak{F}$ is a local $\mathfrak{X}$-formation, we have

$$\mathfrak{F} = \mathfrak{X}_\pi \cap \bigcap_{q\in\pi} \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$$

by Lemma 1.5.1 (d). Now if $q \neq p$ is a prime, then $G/N \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$ by hypothesis, where $N = G^{\mathfrak{F}}$, and so also $L/L \cap N$ belongs to that class. Since N is a q'-group, this shows that $L \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$ for every prime $q \neq p$.

Now $L = L \cap HN = H(L \cap N)$ and $(H \cap N) \cap Q(L \cap N) = Q(H \cap N)$ by the modular law, and so

$$L/Q(L \cap N) = H(L \cap N)/Q(L \cap N) \cong H/Q(H \cap N) \in \mathfrak{S}_p f(p)$$

because $H/Q \in \mathfrak{S}_p f(p)$. Therefore also $L/Q \in \mathfrak{S}_p f(p)$ and consequently $L \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$. Since G is a π-group contained in $\mathfrak{X}$, the same is true for L, and we have $L \in \mathfrak{F}$ by Lemma 1.5.1. Therefore $H = N_G(O_{p'}(H))$.

(b) Suppose that the Sylow p'-subgroup $G_{p'}$ reduces into H and H^g. Then $G_{p'}^{g^{-1}}$ reduces into H. Let H_p be a Sylow p-subgroup of H, then $H = (H \cap G_{p'})H_p$ by [GHT71, Lemma 2.1]. Therefore $G_{p'} = G_{p'} \cap HN = G_{p'} \cap (H \cap G_{p'})H_pN = (H \cap G_{p'})(G_{p'} \cap H_pN) = (H \cap G_{p'})$ is a Sylow p'-subgroup of H, and by the same argument, also $G_{p'}^{g^{-1}}$ is a Sylow p'-subgroups of H. Since H is a $\mathfrak{U}$-group, it follows that $G_{p'}^{g^{-1}} = G_{p'}^h$ for some $h \in H$. Therefore $gh \in N_G(G_{p'})$. Since $G_{p'}$ is contained in H, we clearly have $N_G(G_{p'}) \le N_G(O_{p'}(H))$ and so $gh \in H$ by (a). This shows that $g \in H$, proving that $H = H^g$.

(c) Put $N = G^{\mathfrak{F}}$ and $Q = O_{p'}(H)$ and observe that NQ is a normal subgroup of G. Therefore also $K = [N,Q] = [N,NQ]$ is normal in G.

First, we show that $G/K \in \mathfrak{F}$. Since N/K is a p-group, we have $G/K \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$ for every prime $q \ne p$. Now $G/NQ \in \mathfrak{S}_p f(p)$ as in the proof of (a). Since $Q^N = Q[Q,N] = QK$ and $Q \trianglelefteq H$, the subgroup QK is normalized by $NH = G$ and so QK is a normal subgroup of G. Now QN/QK is a p-group, and so also $G/QK \in \mathfrak{S}_p f(p)$. But then $G/K \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$, and so $G/K \in \mathfrak{F}$. Therefore we have $N = G^{\mathfrak{F}} \le K$ and so $N = [N,Q]$.

Next, we show that $C_N(Q) = 1$. Let $x \in C_N(Q)$. Since $x \in N$, we have $x = \prod_{i=1}^n [y_i, q_i]$, where $y_i \in N$ and $q_i \in Q$. Let $Q_0 = <q_1, \ldots, q_n> \le Q$ which is a finitely generated subgroup of Q, hence is finite, and so also $Y = <x, y_1, \ldots y_n>^{Q_0} \le N$ is finite. Applying [Hup67, III.13.4] to the finite group Q_0Y, we obtain that $Y = [Y,Q_0] \times C_Y(Q_0)$. In particular, we have $x \in [Y,Q_0] \cap C_Y(Q_0) = 1$ and so $C_N(Q) = 1$.

Now the normal p-subgroup $H \cap N$ of H centralizes $Q = O_{p'}(H)$ and so $H \cap N = 1$, as required.

(d) Suppose that the Sylow p'-subgroup $G_{p'}$ of G reduces into H and H^*. Since both H and H^* complement $N = G^{\mathfrak{F}}$ by (c), we have $O_{p'}(H)N/N = O_{p'}(G/N) = O_{p'}(H^*)N/N$. So $O_{p'}(H^*) = G_{p'} \cap NO_{p'}(H) = O_{p'}(H)$ and thus $H = H^*$ by (a). □

Our next lemma is the key for finding a prefactorized $\mathfrak{F}$-projectors.

5.1.2 Lemma. *Let π be a set of primes and suppose that the group G is the product of two subgroups A and B. Further, assume that A and B have Sylow subgroups A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ respectively such that $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. If $A_\pi B_\pi$ is a Sylow π-subgroup of G and N is a normal π'-subgroup of G such that $L/N = O_\pi(G/N)$ is a prefactorized subgroup of G/N, then $L \cap A_\pi B_\pi$ is a prefactorized Sylow π-subgroup of L.*

Proof. By hypothesis, we have $L/N = (L/N \cap AN/N)(L/N \cap BN/N)$ and so

$$L = (L \cap AN)(L \cap BN) = (L \cap A)N(L \cap B)$$

by the modular law. Since L/N is a π-group, it follows that $A_{\pi'} \cap L \le N$ and $B_{\pi'} \cap L \le N$. Since $A = A_\pi \times A_{\pi'}$, we have $L \cap A = (L \cap A_\pi) \times (L \cap A_{\pi'})$, and hence we obtain $L = (L \cap A_\pi)(L \cap B_\pi)N$. Now the set $(L \cap A_\pi)(L \cap B_\pi)$ is clearly contained in $L \cap A_\pi B_\pi$ which is a π-group. Put $A^* = (L \cap A_\pi)N$ and $B^* = (L \cap B_\pi)N$, then Proposition 2.1.4 (a), applied to $L = A^*B^*$, shows that $(L \cap A_\pi)(L \cap B_\pi)$ is a Sylow π-subgroup of L, and so $L \cap A_\pi B_\pi = (L \cap A_\pi)(L \cap B_\pi)$, as required. □

Recall that a group is hypoabelian if it has a descending series with abelian factors. Hence every soluble group is hypoabelian. Note also that the following theorem does not claim that

$\mathfrak{F}$-projectors do exist in the group G or, in case they exist, that any Sylow basis of G reduces into an $\mathfrak{F}$-projector of G.

5.1.3 Theorem. *Let $\mathfrak{X}$ be a* QS*-closed class of periodic locally soluble groups and suppose that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation. Assume that $G \in \mathfrak{X}$ and that H is an $\mathfrak{F}$-projector of G. If G is hypoabelian or an $\mathfrak{U}$-group, then every Sylow basis of G reduces into at most one $\mathfrak{F}$-projector of G.*

Proof. Suppose that $\{G_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G and that H and L are $\mathfrak{F}$-projectors of G into which $\{G_p \mid p \in \mathbb{P}\}$ reduces.

Since G is hypoabelian or an $\mathfrak{U}$-group, there exists an ordinal α such that G possesses a descending series

$$G = N_0 \rhd N_1 \rhd \cdots \rhd N_\alpha = 1$$

whose factors $N_\beta/N_{\beta+1}$ are p-groups for some prime p depending on $\beta < \alpha$. In case G is hypoabelian, we may also assume that every factor $N_\beta/N_{\beta+1}$ is abelian. Let $\beta < \alpha$, then $\{G_pN_\beta/N_\beta \mid p \in \mathbb{P}\}$ reduces into the $\mathfrak{F}$-projectors HN_β/N_β and LN_β/N_β of G/N_β, and so by transfinite induction, we have $HN_\beta = LN_\beta$ for all $\beta < \alpha$. Thus if α is a limit ordinal, then we have

$$H = \bigcap_{\beta<\alpha} HN_\beta = \bigcap_{\beta<\alpha} LN_\beta = L$$

by Lemma 1.6.1.

Otherwise, α has a predecessor $\alpha - 1$. Then $N_{\alpha-1}$ is a p-group for a prime p, and $HN_{\alpha-1} = LN_{\alpha-1}$. Now H and L are $\mathfrak{F}$-maximal subgroups of $HN_{\alpha-1}$ and $\{G_p \mid p \in \mathbb{P}\}$ reduces into $HN_{\alpha-1}$ by Lemma 1.2.3 (d). In particular, if $G_{p'} = <G_q \mid q \in \mathbb{P}, q \neq p>$, then $G_{p'}$ reduces into $HN_{\alpha-1}$, H and L. The result now follows from Proposition 5.1.1 (b) if $G \in \mathfrak{U}$ and from Proposition 5.1.1 (d) if G is hypoabelian. □

Since every $\mathfrak{U}$-group G possesses $\mathfrak{F}$-projectors by [GHT71] and by [Har71, Lemma 2.1], there exists a Sylow basis of G reducing into a given subgroup of G, we have:

5.1.4 Corollary. *Let $\mathfrak{X}$ be a* QS*-closed class of $\mathfrak{U}$-groups and suppose that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation. If $G \in \mathfrak{X}$, then every Sylow basis of G reduces into exactly one $\mathfrak{F}$-projector of G.*

Now we are ready to prove the main theorem of this section.

5.1.5 Theorem. *Let $\mathfrak{X}$ be a* QS*-closed class of $\mathfrak{U}$-groups and suppose that $\mathfrak{F}$ is a local $\mathfrak{X}$-formation of characteristic π. Moreover, let the $\mathfrak{X}$-group G be the product of two locally nilpotent subgroups A and B. If G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and N has a hypoabelian Sylow π-subgroup, then G has a unique prefactorized $\mathfrak{F}$-projector H, and this $\mathfrak{F}$-projector contains $A_\pi \cap B_\pi$. Thus if the characteristic π of $\mathfrak{F}$ contains $\pi(A) \cap \pi(B)$, then H is factorized.*

Proof. By Corollary 5.1.4, there exists a unique $\mathfrak{F}$-projector H of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, and by Theorem 2.3.7, this is the only $\mathfrak{F}$-projector of G which may be prefactorized.

In view of Lemma 1.5.1, every $\mathfrak{F}$-group is a π-group. Thus H is contained in the Sylow π-subgroup $A_\pi B_\pi$ of G. Since H is also an $\mathfrak{F}$-projector of G by [GHT71, Theorem 5.4], it will suffice to show that H is a factorized subgroup of $A_\pi B_\pi$. Since $N \cap A_\pi B_\pi$ is hypoabelian, we may assume without loss of generality that $G = A_\pi B_\pi$ and that N is hypoabelian.

Now let

$$N = N_1 \rhd N_2 \rhd \ldots \rhd N_\alpha = 1$$

be a descending normal series of N with abelian factors which are p-groups for suitable primes p. Clearly, we may assume that $\alpha > 1$. Let $\beta < \alpha$, then the Sylow basis

$$\{A_p B_p N_\beta / N_\beta \mid p \in \mathbb{P}\}$$

of G/N_β reduces into the $\mathfrak{F}$-projector HN_β/N_β of G/N_β and hence by induction on α, the subgroup HN_β is factorized for all $\beta < \alpha$. If α is a limit ordinal, then by Lemma 1.6.1,

$$H = \bigcap_{\beta<\alpha} HN_\beta$$

and so H is factorized. Therefore assume that α has a predecessor. Now the Sylow basis $\{A_p B_p \mid p \in \mathbb{P}\}$ of G reduces into the factorized subgroup $HN_{\alpha-1}$, and consequently it suffices to consider the case $G = HN_{\alpha-1}$. Since $G/N_{\alpha-1} \in \mathfrak{F}$ and N is an abelian p-group, also the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is an abelian p-group. Thus we may assume without loss of generality that $N = G^{\mathfrak{F}}$. Then H complements N by Proposition 5.1.1 (c), and so $O_{p'}(G/N) = O_{p'}(H)N/N$. Since $O_{p'}(G/N)$ is a prefactorized subgroup of G/N by Theorem 2.4.1, it follows from Lemma 5.1.2 that $O_{p'}(H) = A_{p'}B_{p'} \cap O_{p'}(H)N$ is prefactorized. Moreover, $A_{p'} \cap O_{p'}(H)N = A_{p'} \cap O_{p'}(H)$ is a normal subgroup of $A_{p'}$, hence of A, and similarly, $B_{p'} \cap O_{p'}(H)$ is a normal subgroup of B. Therefore by [Wie58, *Hilfssatz* 7] (see also [AFG92, Lemma 1.2.2]), the normalizer $N_G(O_{p'}(H))$ of $O_{p'}(H) = (A_{p'} \cap O_{p'}(H))(B_{p'} \cap O_{p'}(H))$ is factorized. Since we have $H = N_G(O_{p'}(H))$ by Proposition 5.1.1 (a), it follows that H is factorized. □

Since by [Weh68, Theorem A1], every periodic locally soluble linear group is a soluble $\mathfrak{U}$-group, we also have:

5.1.6 Corollary. *Let $\mathfrak{X}$ be a class of periodic locally soluble linear groups and suppose that $\mathfrak{F}$ is a local $\textsc{qs}\mathfrak{X}$-formation of characteristic π. Moreover, let the $\textsc{qs}\mathfrak{X}$-group G be the product of two locally nilpotent subgroups A and B. Then G has a unique prefactorized $\mathfrak{F}$-projector, and this $\mathfrak{F}$-projector contains $A_\pi \cap B_\pi$. Thus if the characteristic π of $\mathfrak{F}$ contains $\pi(A) \cap \pi(B)$, then this $\mathfrak{F}$-projector is factorized.*

5.2. System normalizers and Carter subgroups of $\mathfrak{U}$-groups

Let G be an $\mathfrak{U}$-group which is the product of two locally nilpotent subgroups. If G is not hypoabelian, the techniques used in the last section to prove the existence of a prefactorized $\mathfrak{F}$-projector of G cannot be applied any more. This is mainly due to the fact that then Proposition 5.1.1 (c) does not hold if $G^{\mathfrak{F}}$ is a nonabelian p-group. However, we have a positive result about $\textsc{l}\mathfrak{N}$-projectors of G which will be proved using the following proposition. If G is a group with Sylow basis $\{G_p \mid p \in \mathbb{P}\}$, then the subgroup $H = \bigcap_{p\in\mathbb{P}} N_G(G_p)$ is the *system normalizer of G associated with the Sylow basis* $\{G_p \mid p \in \mathbb{P}\}$.

5.2.1 Proposition. *Suppose that the $\mathfrak{U}$-group G is the product of two locally nilpotent subgroups. Then G has a factorized system normalizer.*

Proof. Let $\{A_pB_p \mid p \in \mathbb{P}\}$ be the Sylow basis of G consisting of prefactorized Sylow subgroups of G. Then for each $p \in \mathbb{P}$, A_p and B_p are normal subgroups of A and B, respectively, and so by [Wie58, *Hilfssatz* 7], $N_G(A_pB_p)$ is factorized. Therefore also the system normalizer $D = \bigcap_{p\in\mathbb{P}} N_G(A_pB_p)$ is factorized. □

We define a *Carter subgroup* of an $\mathfrak{U}$-group to be an $L\mathfrak{N}$-projector. For equivalent definitions of a Carter subgroup, see also [GHT71, Lemma 5.6]. The preceding result can now be used to prove the existence of a unique factorized Carter subgroup.

5.2.2 Theorem. *Suppose that the $\mathfrak{U}$-group G is the product of two locally nilpotent subgroups. Then G has a unique prefactorized Carter subgroup, and this Carter subgroup is factorized.*

Proof. By Corollary 5.1.4, there exists a unique Carter subgroup C of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces. Therefore by Theorem 2.3.7, this is the only Carter subgroup of G which may be prefactorized.

Let n denote the length of the Hirsch-Plotkin series of G. If $n \leq 2$, the Carter subgroups of G coincide with its system normalizers (see [GHT71, Theorem 5.1]). So in this case, the result follows from Proposition 5.2.1. Therefore assume that $n \geq 3$ and let R denote the Hirsch-Plotkin radical of G. Then CR/R is a Carter subgroup of G/R into which the Sylow basis $\{A_pB_pR/R \mid p \in \mathbb{P}\}$ of G/R reduces. Thus by induction on n, the subgroup CR of G is factorized. Since C is also a Carter subgroup of CR and $n(CR) = 2 < n$, the subgroup C is factorized in CR, hence in G. □

5.3. Injectors and radicals of *FC*-groups

Let G be a locally soluble FC-group and $\mathcal{F}$ a Fitting set of G. Then by a result of Beidleman and Karbe [BK87], G possesses $\mathcal{F}$-injectors, and the $\mathcal{F}$-injectors of G are locally conjugate. A similar result about $\mathfrak{F}$-injectors, where $\mathfrak{F}$ is a Fitting class of FC-groups, has been proved by Tomkinson [Tom69b]; see also [Tom84].

The following theorem is the key to apply the results on finite products of two nilpotent subgroups obtained in [AH94] to FC-groups being the product of two locally nilpotent subgroups.

5.3.1 Theorem. *Suppose that the periodic FC-group is the product of two locally nilpotent subgroups A and B. Then every finite subset of G is contained in a finite subnormal prefactorized subgroup of G.*

Proof. Let X be a finite subset of G, then by [Tom84, Theorem 1.8], X is contained in a finite normal subgroup N of G. By [Cer80, Lemma 3], cf. also [Keg65], N is contained in a finite prefactorized subgroup H of G, and we may assume without loss of generality that no proper prefactorized subgroup of H contains N.

Similarly, $K = H^G$ is contained in a finite prefactorized subgroup L of G. Now let $Y/N = F(L/N)$, then by [Amb73] or [Pen73], Y is a factorized subgroup of L, and so $H \cap Y$ is a prefactorized subgroup of G containing N. Therefore $H \leq Y$ by the choice of H and $H/N \leq F(G/N)$ is a subnormal subgroup of L/N. Since H^G is contained in L, the subgroup H is also subnormal in $H^G \trianglelefteq G$ and so H is a prefactorized subnormal subgroup of G which contains X. □

For our purposes, it will be necessary to consider finite subsets of a factorizer of a normal subgroup.

5.3.2 Proposition. *Suppose that the periodic FC-group is the product of two locally nilpotent subgroups A and B. If N is a normal subgroup of G and $X = AN \cap BN$, then every finite subset of X is contained in a finite prefactorized subgroup of X which is subnormal in G.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a finite subset of X. By Theorem 5.3.1, $\{x_1, \ldots, x_n\}$ is contained in a finite prefactorized subset S of G. We show that the prefactorized subgroup $X \cap S$ is also subnormal in G. Let $R/N = R(G/N)$, then R is factorized by Theorem 2.4.5, we have $X/N \leq R/N$. Thus $S \cap X$ is contained in $S \cap R$. Since S is finite, $(S \cap R)/(S \cap N) \cong (S \cap R)N/N$ is nilpotent, and so $S \cap X$ is a subnormal subgroup of $S \cap R$. Since $R \cap S \trianglelefteq S$ is subnormal in G, this proves that $X \cap S$ is also a subnormal subgroup of G. □

The next proposition follows directly from Proposition 5.3.2 and Lemma 1.6.3. It can be used to reduce questions about injectors and radicals in FC-groups to factorizers of normal subgroups.

5.3.3 Proposition. *Suppose that the periodic FC-group is the product of two locally nilpotent subgroups A and B. Let N be a normal subgroup of G and $X = AN \cap BN$. If $\mathcal{F}$ is a Fitting set of G and I is an $\mathcal{F}$-injector of G, then $X \cap I$ is an $\mathcal{F}$-injector of X and $X_{\mathcal{F}} = X \cap G_{\mathcal{F}}$.*

For finite groups, the following theorem has been proved in [AH94, Proposition 3]. A similar result also holds for CC-groups, see Theorem 5.4.4 below.

5.3.4 Theorem. *Suppose that the periodic FC-group is the product of two locally nilpotent subgroups A and B. If $\mathcal{F}$ is a Fitting set of G and I is a prefactorized $\mathcal{F}$-injector of G, then $G_{\mathcal{F}}$ is prefactorized.*

Proof. Let $X = AG_{\mathcal{F}} \cap BG_{\mathcal{F}}$, then $X \cap I$ is a prefactorized $\mathcal{F}$-injector of X and $X_{\mathcal{F}} = G_{\mathcal{F}}$ by Proposition 5.3.3. Therefore we may assume that $G = X$. We show that in this case $I = G_{\mathcal{F}}$. Let $g \in I$ and let N be a finite normal subgroup of G that contains g. Then $N/N \cap G_{\mathcal{F}} \cong NG_{\mathcal{F}}/G_{\mathcal{F}}$ is finite. Since $G/G_{\mathcal{F}}$ is locally nilpotent, this shows that $I \cap N$ is subnormal in N and hence $I \cap N$ is contained in $N_{\mathcal{F}} = G_{\mathcal{F}} \cap N$. Thus $g \in G_{\mathcal{F}}$, as required. □

The following example shows that even a finite product of two nilpotent subgroups having a factorized $\mathcal{F}$-radical need not have a prefactorized $\mathcal{F}$-injector.

5.3.5 Example. Let p and q be distinct primes and $F = GF(q)$ Moreover, let H be the semidirect product of a cyclic group H_p of order p with a faithful irreducible FH_p-module H_q (such a FH_p-module exists by [DH92, B, Corollary 10.7]). Now let $G = H \times K$, where $K \cong H$, and put $A = H_p \times K_q$ and $B = H_q \times K_p$. If h and k are generators of H_p and K_p, respectively, set $I = \langle hk \rangle$ and $\mathcal{F} = \{1, I^g \mid g \in G\}$. Now I is a Sylow p-subgroup of the normal subgroup H_qK_qI of G, and so it follows from [DH92, VIII, Theorem 3.8] or by direct calculation that $\mathcal{F}$ is a Fitting set of G. Clearly, I is an $\mathcal{F}$-injector of G and $G_{\mathcal{F}} = 1$. Since $A \cap B = 1$, this shows that the $\mathcal{F}$-radical of G is factorized.

Moreover, $N_G(I) \cap H_qK_q = 1$ and so $N_G(I) = H_pK_p$ is factorized. Thus [AH94, Proposition 1] shows that I is the only candidate for a prefactorized $\mathcal{F}$-injector. But evidently I is not prefactorized.

Let G be a periodic FC-group which is the product of two locally nilpotent subgroups A and B and suppose that $\mathcal{F}$ is a Fitting set of G. The next lemma shows that in order to prove that $G_{\mathcal{F}}$ is prefactorized, it suffices to investigate the $\mathcal{F}$-radicals of the finite prefactorized subnormal subgroups of G.

5.3.6 Lemma. *Let the periodic FC-group G be the product of two locally nilpotent subgroups A and B and $\mathcal{F}$ a Fitting set of G. Furthermore, suppose that for every finite subnormal prefactorized subgroup S of G, the $\mathcal{F}$-radical $S_{\mathcal{F}}$ is prefactorized (factorized) in S. Then $G_{\mathcal{F}}$ is prefactorized (factorized). Conversely, if $G_{\mathcal{F}}$ is factorized, then $S_{\mathcal{F}}$ is a factorized subgroup of S for every finite prefactorized subgroup S of G.*

Proof. Suppose first that $S_{\mathcal{F}}$ is prefactorized for every finite prefactorized subnormal subgroup S of G and let $g \in G_{\mathcal{F}}$. Then by Theorem 5.3.1, g is contained in a finite subnormal prefactorized subgroup S of G. Therefore $g \in S \cap G_{\mathcal{F}} = S_{\mathcal{F}}$. Since the latter subgroup is prefactorized, it follows that $g \in (A \cap S_{\mathcal{F}})(B \cap S_{\mathcal{F}}) \subseteq (A \cap G_{\mathcal{F}})(B \cap G_{\mathcal{F}})$. Therefore $G_{\mathcal{F}} = (A \cap G_{\mathcal{F}})(B \cap G_{\mathcal{F}})$, as required. Now suppose that, in addition, the $\mathcal{F}$-radical of every prefactorized finite subnormal subgroup S of G contains $A \cap B \cap S$. Then every $g \in A \cap B$ is contained in a finite subnormal prefactorized subgroup V and $g \in V \cap A \cap B \subseteq V_{\mathcal{F}} \subseteq G_{\mathcal{F}}$, as required.

Conversely, if $G_{\mathcal{F}}$ is a factorized subgroup of G and S is a subnormal prefactorized subgroup of G, then $S_{\mathcal{F}} = S \cap G_{\mathcal{F}}$, hence is a factorized subgroup of S. □

The following proposition shows that an FC-group which is the product of two locally nilpotent subgroups cannot have more than one prefactorized $\mathcal{F}$-injector.

5.3.7 Proposition. *Let $\mathcal{F}$ be a Fitting set of the FC-group G. If G is the product of two locally nilpotent subgroups A and B, then G has at most one prefactorized $\mathcal{F}$-injector.*

Proof. Suppose that I and J are prefactorized $\mathcal{F}$-injectors of G. By Theorem 2.4.5, the set $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G, and by Theorem 2.3.7, $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into both I and J.

Now by Theorem 5.3.1, every element $g \in I$ is contained in a finite prefactorized subnormal subgroup S of G. By Theorem 2.3.7, the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ also reduces into S. Therefore by Lemma 1.2.3 (c), the Sylow basis $\{A_pB_p \cap S \mid p \in \mathbb{P}\}$ reduces into $S \cap I$ and $S \cap J$. Since $S \cap I$ and $S \cap J$ are $\mathcal{F}$-injectors of S and the $\mathcal{F}$-injectors of S are pronormal in S by [DH92, VIII, Proposition 2.14], it follows from [DH92, I, Theorem 6.6] that $I \cap S = J \cap S$. Therefore $g \in J$ and consequently $I = J$, as required. □

Now the main theorem of this section can be proved. It shows that, in order to determine whether an $\mathcal{F}$-injector or the $\mathcal{F}$-radical of an FC-group which is the product of two locally nilpotent subgroups is factorized, it suffices to consider $\mathcal{F}$-injectors or the $\mathcal{F}$-radical of its finite prefactorized subgroups.

5.3.8 Theorem. *Suppose that the periodic FC-group is the product of two locally nilpotent subgroups A and B and let $\mathcal{F}$ be a Fitting set of G. Then the following statements are equivalent:*

(a) *For every prefactorized subgroup S of G, there exists a unique $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(b) *For every prefactorized subgroup S of G, there exists an $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(c) *For every prefactorized subgroup S of G, the $\mathcal{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

(d) *For every finite prefactorized subgroup S of G, there exists a unique $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(e) *For every finite prefactorized subgroup S of G, there exists an $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(f) *For every finite prefactorized subgroup S of G, the $\mathcal{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

Proof. The implications (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (d), (b) $\Rightarrow$ (e), (c) $\Rightarrow$ (f) and (d) $\Rightarrow$ (e) are trivial. Moreover, (b) $\Rightarrow$ (c) and (e) $\Rightarrow$ (f) are a direct consequence of Theorem 5.3.4.

(f) $\Rightarrow$ (d). Let S be a finite prefactorized subgroup of G. Since the $\mathcal{F}$-radical of every factorized subgroup of S is prefactorized (factorized), S has a unique prefactorized (factorized) $\mathcal{F}$-injector by [AH94, Theorem C*].

Thus it remains to show that (d) implies (a). Since G has at most one prefactorized $\mathcal{F}$-injector by Proposition 5.3.7, we only have to prove the existence of a prefactorized (factorized) $\mathcal{F}$-injector.

Let I be an $\mathcal{F}$-injector of G into which the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces and let $g \in I$. By Theorem 5.3.1, there exists a finite prefactorized subnormal subgroup S of G which contains g. Since S is prefactorized, the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into S. Therefore by Lemma 1.2.3 (c), the Sylow basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G also reduces into the intersection $S \cap I$. Since S satisfies the hypothesis of [AH94, Theorem C*], $S \cap I$ is a prefactorized $\mathcal{F}$-injector of S. Therefore $g \in (S \cap I \cap A)(S \cap I \cap B) \subseteq (I \cap A)(I \cap B)$. This shows that I is the unique prefactorized $\mathcal{F}$-injector of G.

Now suppose that $S_{\mathcal{F}}$ is factorized for every finite prefactorized subgroup of G. Then $G_{\mathcal{F}}$ is factorized by Lemma 5.3.6. Therefore we have $A \cap B \leq G_{\mathcal{F}} \leq I$ and so I is factorized. □

For Fitting classes of periodic FC-groups, we thus obtain:

5.3.9 Corollary. *Let $\mathfrak{X}$ be a subgroup-closed class of periodic FC-groups and $\mathfrak{F}$ an $\mathfrak{X}$-Fitting class. If the $\mathfrak{X}$-group G is the product of two locally nilpotent subgroups A and B, then the following statements are equivalent:*

(a) *For every prefactorized subgroup S of G, there exists a unique $\mathfrak{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(b) *For every prefactorized subgroup S of G, the $\mathfrak{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

(c) *For every finite prefactorized subgroup S of G, there exists an $\mathfrak{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(d) *For every finite prefactorized subgroup S of G, the $\mathfrak{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

The following is a simple criterion for a prefactorized $\mathfrak{F}$-radical and an $\mathfrak{F}$-injector of a periodic FC-group to be factorized.

5.3.10 Proposition. *Let $\mathfrak{F}$ be a Fitting class of periodic locally soluble FC-groups of characteristic π and suppose that the periodic FC-group G is the product of two locally nilpotent subgroups A and B. Then $A \cap B \cap G_{\mathfrak{F}} = A_\pi \cap B_\pi$. Thus if $G_{\mathfrak{F}}$ is prefactorized, then $G_{\mathfrak{F}}$ is factorized if and only if $A \cap B$ is a π-group. Moreover, a prefactorized $\mathfrak{F}$-injector is factorized if and only if $A \cap B$ is a π-group*

Proof. It follows from Proposition 5.3.3 that $(A \cap B)_{\mathfrak{F}} = A \cap B \cap G_{\mathfrak{F}}$. Since $G_{\mathfrak{F}}$ is a π-group by [Dix88, Lemma 2.2] and by Theorem 2.4.5 (e), the intersection $A \cap B$ is contained in the Hirsch-Plotkin radical of G, we have $A \cap B \cap G_{\mathfrak{F}} = A_\pi \cap B_\pi$. The statement about $\mathfrak{F}$-injectors follow from the fact that $G_{\mathfrak{F}}$ is contained in every $\mathfrak{F}$-injector of G by Lemma 1.6.2. □

Although we are mainly concerned with locally finite groups, it is also possible to deduce from the above theorems some results about Fitting sets $\mathcal{F}$ of FC-groups which are not necessarily periodic. For example, it is clear that Theorem 5.3.8 holds for non-periodic FC-groups if we assume that $\mathcal{F}$ consists of periodic subgroups of G only and that the torsion subgroup of G is factorized. For the prefactorized case, it obviously suffices to assume that the torsion subgroup of G is prefactorized. Trivial examples show that the torsion subgroup of an FC-group need not be prefactorized.

For Fitting classes of FC-groups containing infinite cyclic groups, a different result is possible. To prove this, we need the following lemma, which might also be of independent interest.

5.3.11 Lemma. *Let G be a group such that $G/Z(G)$ is periodic (locally finite) and let G be the product of two subgroups A and B. Then there exists a prefactorized torsion-free subgroup M of $Z(G)$ such that G/M is periodic (locally finite).*

Proof. By Zorn's lemma, there exists a maximal torsion-free subgroup N of $Z(G)$. Then G/N is periodic. Let $M = (A \cap N)(B \cap N)$, then also $M \leq Z(G)$ is a normal subgroup of G. Let $g \in Z(G)$, then there exist $a \in A$ and $b \in B$ such that $g = ab$. Since $A/A \cap N$ and $B/B \cap N$ are periodic, there exists an integer n such that $a^n \in A \cap N$ and $b^n \in B \cap N$. Since $g \in Z(G)$, we have $g^n = a^n b^n \in M$ and so $Z(G)/M$ is a periodic abelian group, hence is locally finite. Therefore also G/M is periodic, and if $G/Z(G)$ is locally finite, then also G/M is locally finite. □

We state the result about Fitting classes of FC-groups containing an infinite cyclic subgroup in a slightly more general form.

5.3.12 Theorem. *Let the FC-group G be the product of two locally nilpotent subgroups A and B. If $\mathcal{F}$ is a Fitting set of G which contains every torsion-free subgroup of $Z(G)$, then the following statements are equivalent:*

(a) *For every prefactorized subgroup S of G, there exists a unique $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(b) *For every prefactorized subgroup S of G, the $\mathcal{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

(c) *For every central-by-finite prefactorized subgroup S of G, there exists an $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(d) *For every central-by-finite prefactorized subgroup S of G, the $\mathcal{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

Proof. By Lemma 5.3.11, G possesses a torsion-free central prefactorized subgroup M such that G/M is periodic, and since $\mathcal{F}$ contains every torsion-free subgroup of G, we have $M \in \mathcal{F}$. Now let

$$\mathcal{F}_{G/M} = \{U/M \mid M \leq U \leq G, U \in \mathfrak{F}\},$$

then it is easy to verify that $\mathcal{F}_{G/M}$ is a Fitting set of G/M. If S is a prefactorized subgroup S of G containing M, then $S_{\mathcal{F}}/M$ is the $\mathcal{F}_{G/M}$-radical of G/M, and the subgroup I/M is an $\mathcal{F}_{G/M}$-injector of G/M if and only if I is an an $\mathcal{F}$-injector of G; see e.g. [Ens90, Proposition 8.1]. Therefore Theorem 5.3.8 may be applied to the factor group G/M with the Fitting set $\mathcal{F}_{G/M}$, and so the desired result follows from Proposition 1.1.3 (h). □

For Fitting classes, this may be formulated as follows.

5.3.13 Corollary. *Let the FC-group G be the product of two locally nilpotent subgroups A and B. If $\mathfrak{F}$ is a Fitting class of locally soluble FC-groups which contains an infinite cyclic group, then the following statements are equivalent:*

(a) *For every prefactorized subgroup S of G, there exists a unique $\mathfrak{F}$-injector which is a factorized subgroup of S.*

(b) *For every prefactorized subgroup S of G, the $\mathfrak{F}$-radical of S is a factorized subgroup of S.*

(c) *For every central-by-finite prefactorized subgroup S of G, there exists an $\mathfrak{F}$-injector which is a prefactorized subgroup of S.*

(d) *For every central-by-finite prefactorized subgroup S of G, the $\mathfrak{F}$-radical of S is a prefactorized subgroup of S.*

Proof. Let $\mathcal{F}$ be the Fitting set consisting of all $\mathfrak{F}$-subgroups of G. Since $\mathfrak{F}$ contains an infinite cyclic group, $\mathcal{F}$ clearly contains all torsion-free subgroups of $Z(G)$. Assume now that G satisfies one of the above statements, then by the prefactorized case of Theorem 5.3.12, every prefactorized subgroup S of G has a unique prefactorized $\mathfrak{F}$-injector, and $S_{\mathfrak{F}}$ is prefactorized. Thus it remains to prove that the latter subgroups are factorized. Since $\mathfrak{F}$ contains an infinite cyclic group, by [Dix88, Lemma 2.2], it contains every locally nilpotent FC-group. Moreover, by Theorem 2.4.8, the Hirsch-Plotkin radical of the prefactorized subgroup S of G is factorized in S, and so $A \cap B \cap S \leq R(S) \leq S_{\mathfrak{F}}$. Since $S_{\mathfrak{F}}$ is contained in every $\mathfrak{F}$-injector of S, also every prefactorized $\mathfrak{F}$-injector of G is factorized. □

5.4. Injectors and radicals of CC-groups

We will now generalize some of the results about injectors and radicals of FC-groups which are the product of two locally nilpotent subgroups to CC-groups. Note that if $\mathfrak{F}$ is a Fitting class of CC-groups, then by [Dix88], every locally soluble CC-group has $\mathfrak{F}$-injectors.

The following elementary observation is the key to transfer our results about FC-groups to CC-groups.

5.4.1 Lemma. *Suppose that the periodic CC-group is the product of two locally nilpotent subgroups A and B. Then every finite subset of G is contained in a subnormal factorized subgroup of G which is locally nilpotent-by-finite.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a finite subset of G. If R denotes the Hirsch-Plotkin radical of G, then G/R is an FC-group by Lemma 2.4.7. Therefore by Theorem 5.3.1, HR/R is contained in a finite prefactorized subnormal subgroup K/R of G/R. Since R is factorized by Theorem 2.4.5 (e), also H is a factorized subgroup of G by Proposition 1.1.3. □

For our results about $\mathcal{F}$-radicals of CC-groups, we need a result similar to Proposition 5.3.2.

5.4.2 Proposition. *Let G be a periodic CC-group which is the product of two locally nilpotent subgroups A and B. If N is a normal subgroup of G and $X = AN \cap BN$, then every finite subset of X is contained in a (locally nilpotent)-by-finite subgroup of X which is serial in G.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a finite subset of X and S a locally nilpotent-by-finite subnormal factorized subgroup of G containing $\{x_1, \ldots, x_n\}$. Then also $X \cap S$ is factorized. Now let $R/N = R(G/N)$, then $(X \cap S)N/N \leq R/N$ because R is factorized by Theorem 2.4.5. Therefore $X \cap S$ is a subgroup of $R \cap S$, hence is locally nilpotent. This shows that $X \cap S$ is a serial subgroup of $R \cap S$. Now $R \cap S$ is a normal subgroup of the subgroup S, which is in turn subnormal in G. Hence $X \cap S$ is the required serial subgroup of G. □

The next Proposition 5.4.3 is now a direct consequence of Lemma 1.6.3.

5.4.3 Proposition. *Suppose that the periodic CC-group is the product of two locally nilpotent subgroups A and B. Let N be a normal subgroup of G and $X = AN \cap BN$. If $\mathcal{F}$ is a Fitting set of G and I is an $\mathcal{F}$-injector of G, then $X \cap I$ is an $\mathcal{F}$-injector of X and $X_{\mathcal{F}} = X \cap G_{\mathcal{F}}$.*

We are now able to prove the equivalent of Proposition 5.3.10.

5.4.4 Theorem. *Suppose that the periodic CC-group G is the product of two locally nilpotent subgroups A and B. Let $\mathcal{F}$ be a Fitting set of G and suppose that G has a prefactorized (factorized) $\mathcal{F}$-injector I. Then $G_{\mathcal{F}}$ is prefactorized (factorized).*

Proof. Let $X = AG_{\mathcal{F}} \cap BG_{\mathcal{F}}$, then $X_{\mathcal{F}} = X \cap G_{\mathcal{F}} = G_{\mathcal{F}}$ and $I \cap X$ is a prefactorized (factorized) injector of X. Therefore it suffices to consider the case when $G = AG_{\mathcal{F}} = BG_{\mathcal{F}}$. But then $G/G_{\mathcal{F}}$ is locally nilpotent, and hence I is serial in G. It follows that $I \leq G_{\mathcal{F}}$ and $I = G_{\mathcal{F}}$ which is prefactorized (factorized), as required. □

In view of Lemma 5.4.1, it is natural to study (locally nilpotent)-by-finite products of two locally nilpotent subgroups. Note that by [Dix88, Lemma 2.2], the condition that $G_{\mathcal{F}}$ must contain the Hirsch-Plotkin radical of $A_\pi B_\pi$ in the next proposition is automatically satisfied if $\mathcal{F}$ is the set of all $\mathfrak{F}$-subgroups of G, where $\mathfrak{F}$ is a Fitting class of CC-groups. We have not been able to provide a version which avoids this condition.

5.4.5 Proposition. *Suppose that the periodic (locally nilpotent)-by-finite group G is the product of two locally nilpotent subgroups and let $\mathcal{F}$ be a Fitting set of G and let π be a set of primes such that every $\mathcal{F}$-subgroup of G is a π-group. If $G_{\mathcal{F}}$ contains the Hirsch-Plotkin radical of $A_\pi B_\pi$ and $S_{\mathcal{F}}$ is a prefactorized (factorized) subgroup of S for every prefactorized subgroup S of G, then G has a prefactorized (factorized) $\mathcal{F}$-injector I. Moreover, I is the unique prefactorized $\mathcal{F}$-injector of G.*

Proof. By Lemma 1.6.3 and Lemma 5.4.1, $N = G_{\mathcal{F}}$ is the union of the subgroups $S_{\mathcal{F}}$, where S is a prefactorized (locally nilpotent)-by-finite subnormal subgroups of G. Therefore by Proposition 1.1.3 (d), N is prefactorized if the $S_{\mathcal{F}}$ are prefactorized. Since $A \cap B$ is the union of the subgroups $A \cap B \cap S$, it follows that $G_{\mathcal{F}}$ is factorized if the $S_{\mathcal{F}}$ are factorized. In this case, by Lemma 1.6.2, also every $\mathcal{F}$-injector of G contains $A \cap B$. Hence it suffices to show that I prefactorized.

Let I be an $\mathcal{F}$-injector of G into which the Sylow basis $\{A_pB_p \mid p \in \pi\}$ of $A_\pi B_\pi$ reduces. Then the π-subgroup I is contained in $A_\pi B_\pi$ and so by [Ens90, *Satz* 7.2], I is also an $\mathcal{F}$-injector of $A_\pi B_\pi$. Now assume that J is a prefactorized $\mathcal{F}$-injector of G, then $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into J, and since J is a π-group, it follows that $J \leq A_\pi B_\pi$. Therefore we may assume without loss of generality that $G = A_\pi B_\pi$. Let R denote the Hirsch-Plotkin radical of G, then by hypothesis, R is contained in N. Now by[Ens90, Lemma 8.1], the set

$$\mathcal{F}_{G/N} = \{H/G_{\mathcal{F}} \mid G_{\mathcal{F}} \leq H \leq G, H \in \mathcal{F}\}$$

is a Fitting set of G/N, and I/N and J/N are $\mathcal{F}_{G/N}$-injectors of G/N. Let S be a prefactorized subgroup of G containing N, then $S_{\mathcal{F}}/N$ is the $\mathcal{F}_{G/N}$-radical of S/N and $S \cap I$ and $S \cap J$ are $\mathcal{F}$-injectors of S. Therefore G/N, together with its Fitting set $\mathcal{F}_{G/N}$, satisfies the hypothesis of Theorem 5.3.8 (e), whence $I/N = J/N$ is the unique prefactorized $\mathcal{F}_{G/N}$-injector of G/N. Since N is prefactorized, it follows that I itself is prefactorized and $I = J$, as required. □

As in the case of FC-groups, we state first the result about Fitting classes of CC-groups containing an infinite cyclic subgroup in a slightly more general form.

5.4.6 Theorem. *Let the CC-group G be the product of two locally nilpotent subgroups A and B and suppose that $\mathcal{F}$ is a Fitting set of G. If $G_{\mathcal{F}}$ contains the Hirsch-Plotkin radical of G, then the following statements are equivalent:*

(a) *For every prefactorized subgroup S of G, there exists a unique $\mathcal{F}$-injector which is a factorized subgroup of S.*

(b) *For every prefactorized subgroup S of G, the $\mathcal{F}$-radical of S is a factorized subgroup of S.*

(c) *For every (locally nilpotent)-by-finite prefactorized subgroup S of G, there exists an $\mathcal{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(d) *For every (locally nilpotent)-by-finite prefactorized subgroup S of G, the $\mathcal{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

Proof. Let $R = R(G)$ and put

$$\mathcal{F}_{G/R} = \{U/R \mid R \leq U \leq G, U \in \mathcal{F}\},$$

then $\mathcal{F}_{G/R}$ is a Fitting set of G/R. Let S be a prefactorized subgroup S of G containing R, then $S_{\mathcal{F}}/R$ is the $\mathcal{F}_{G/R}$-radical of G/R, and the subgroup I/R is an $\mathcal{F}_{G/R}$-injector of G/R if and only if I is an an $\mathcal{F}$-injector of G; see e.g. [Ens90, Proposition 8.1]. In view of Proposition 1.1.3 (h), the result now follows by applying Theorem 5.3.8 to the Fitting set $\mathcal{F}_{G/R}$ of G/R. □

Note that, as in Corollary 5.3.13, the radicals and injectors in the following theorem are factorized if the Fitting class contains an infinite cyclic group.

5.4.7 Theorem. *Let the CC-group G be the product of two locally nilpotent subgroups A and B and suppose that $\mathfrak{F}$ is a Fitting class of CC-groups. If G is periodic or $\mathfrak{F}$ contains an infinite cyclic group, then the following statements are equivalent:*

(a) *For every prefactorized subgroup S of G, there exists a unique $\mathfrak{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(b) *For every prefactorized subgroup S of G, there exists an $\mathfrak{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(c) *For every prefactorized subgroup S of G, the $\mathfrak{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

(d) *For every central-by-finite prefactorized subgroup S of G, there exists an $\mathfrak{F}$-injector which is a prefactorized (factorized) subgroup of S.*

(e) *For every central-by-finite prefactorized subgroup S of G, the $\mathfrak{F}$-radical of S is a prefactorized (factorized) subgroup of S.*

Proof. Suppose first that $\mathfrak{F}$ contains an infinite cyclic group. Then by [Dix88, Lemma 2.2], $\mathfrak{F}$ contains the Hirsch-Plotkin radical of G. Therefore the Fitting set $\mathcal{F}$ consisting of all $\mathfrak{F}$-subgroups of G satisfies the hypotheses of Theorem 5.4.6, and by that theorem, the above statements are equivalent.

Otherwise G is periodic. Let π denote the characteristic of $\mathfrak{F}$, then by [Dix88, Lemma 2.2], every $\mathfrak{F}$-group is a π-group, where π is the characteristic of $\mathfrak{F}$, and every CC-group which is a locally nilpotent π-group is contained in $\mathfrak{F}$. Now consider the Sylow π-subgroup $A_\pi B_\pi$ of G. Then the Hirsch-Plotkin radical of $A_\pi B_\pi$ is contained in the $\mathfrak{F}$-radical of $A_\pi B_\pi$ and so it satisfies the hypotheses of Theorem 5.4.6. Thus if G satisfies (d) or (e), then Proposition 2.1.9 shows that also $A_\pi B_\pi$ satisfies (d) or (e). Therefore every prefactorized subgroup of $A_\pi B_\pi$ has a factorized $\mathfrak{F}$-injector, and in view of [Ens90, *Satz* 7.2], the fact that every $\mathfrak{F}$-subgroup is a π-group and Theorem 2.3.7, these $\mathfrak{F}$-injectors are the $\mathfrak{F}$-injectors of the prefactorized subgroups of G. So by Theorem 5.4.4, also the $\mathfrak{F}$-radical of every prefactorized subgroup is prefactorized. □

Remark. The proof periodic case of Theorem 5.4.7 can also be formulated for Fitting sets $\mathcal{F}$ of G. In this case, the following additional assumptions have to be made: (1) G is periodic; (2) π is a set of primes such that every $\mathcal{F}$-subgroup of G is a π-group; (3) the Hirsch-Plotkin radical of the Sylow π-subgroup $A_\pi B_\pi$ of G is contained in $\mathcal{F}$.

Chapter 6

Miscellaneous results

6.1. The class of all subgroups of products of two finite nilpotent groups

In this section, we will study groups which can occur as subgroups of a product of two nilpotent subgroups. First, we collect some properties of a product of two nilpotent groups.

6.1.1 Lemma. *Suppose that the periodic radical group G is the product of two nilpotent subgroups A and B. Then $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G and $A_{p'}B_{p'}/O_{p'}(G)$ is a nilpotent p'-group of class $c+d$, where c and d are the nilpotency classes of $A_{p'}$ and $B_{p'}$.*

Proof. By [FGS94, Proposition 2.6] and Lemma 1.2.2, $\{A_pB_p \mid p \in \mathbb{P}\}$ is a Sylow basis of G. Moreover, [FGS94, Lemmas 2.1 and 2.7], we have $[A_{p'}, B_{p'}] \leq O_{p'}(G)$. Therefore the group $A_{p'}B_{p'}/O_{p'}(G)$ is the product of its normal nilpotent subgroups $A_{p'}O_{p'}(G)/O_{p'}(G)$ and $B_{p'}O_{p'}(G)/O_{p'}(G)$ of classes c and d, respectively. Thus by Fitting's theorem (see e.g. [Rob82, Theorem 5.2.8]), $A_{p'}B_{p'}/O_{p'}(G)$ is nilpotent of class at most $c+d$. □

Conversely, groups satisfying the properties obtained in Lemma 6.1.1 can often be embedded in a product of two nilpotent subgroups.

6.1.2 Proposition. *Suppose that the group G possesses a Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ and for every set of primes π, let $G_\pi = <G_p \mid p \in \pi>$. Then:*

(a) *If $\pi(G)$ is finite and $G_{p'}/O_{p'}(G)$ is nilpotent (locally nilpotent) for every prime p, then G can be embedded into a periodic product H of two nilpotent (locally nilpotent) groups which satisfies $|\pi(H)| < \infty$.*

(b) *If there exist integers c and d such that for every prime p, the group G_p is nilpotent of class at most c and the factor group $G_{p'}/O_{p'}G)$ is nilpotent of class at most d, then G can be embedded into a product H of two nilpotent groups of classes at most c and d, respectively. If G has finite exponent n, then H can be chosen to have exponent n.*

Proof. Form the cartesian product

$$H = \prod_{p\in\mathbb{P}} G/O_{p'}(G),$$

then clearly the map $\alpha : g \to (gO_{p'}(G))_{p\in\mathbb{P}}$ is a monomorphism from G to H. Now put

$$A = \prod_{p\in\mathbb{P}} G_pO_{p'}(G)/O_{p'}(G) \qquad \text{and} \qquad B = \prod_{p\in\mathbb{P}} G_{p'}/O_{p'}(G),$$

then we have $H = AB$ since it follows from Lemma 1.2.2 that

$$G/O_{p'}(G) = (G_pO_{p'}(G)/O_{p'}(G))(G_{p'}/O_{p'}(G))$$

for every prime p.

If $\pi(G)$ is finite, then the above cartesian products are in fact finite direct products, and so A and B are nilpotent (locally nilpotent) and $\pi(H)$ is finite. If G is finite, then also H, A and B are finite. This proves (a).

Now suppose that for every prime p, the groups G_p and $G_{p'}/O_{p'}(G)$ have nilpotency classes at most c and d, respectively. Then also their cartesian products A and B have nilpotency classes c and d, respectively. Moreover, if G has finite exponent n, then also the cartesian product H has exponent dividing n. Since the homomorphism α is a monomorphism, H has exponent n. □

Thus from Lemma 6.1.1 and Proposition 6.1.2 (a), we obtain:

6.1.3 Corollary. *Let G be a periodic radical group such that $\pi(G)$ is finite Then G can be embedded into a group which is the product H of two nilpotent subgroups if and only if G possesses a Sylow basis $\{G_p \mid p \in \mathbb{P}\}$ such that $G_{p'}/O_{p'}(G)$ is nilpotent for every prime p, where $G_{p'} = \langle G_q \mid q \in \mathbb{P}, q \neq p\rangle$. Moreover, in this case, H may be chosen such that H is periodic and $\pi(H)$ is finite.*

A similar argument also allows to characterize the class of all finite groups which can be embedded into a product of two finite nilpotent subgroups. In the next lemma, we investigate this class of groups further. Note also that every finite group G satisfying one of the statements of the following lemma possesses Hall π-subgroups for every set π of primes, hence is soluble by a well-known theorem of Hall.

6.1.4 Lemma. *Let G be a finite group. Then the following statements about G are equivalent.*

(a) *$G/O_\pi(G)$ has a nilpotent Hall π-subgroup for all sets π of primes.*

(b) *$G/O_{\pi,\pi'}(G)$ has a nilpotent Hall π-subgroup for all sets π of primes.*

(c) *$G/O_{p'}(G)$ has a nilpotent Hall p'-subgroup for all primes p.*

(d) *$G/O_{p',p}(G)$ has a nilpotent Hall p'-subgroup for all primes p.*

(e) *$G/O_{\{p,q\}}(G)$ has a nilpotent Hall $\{p,q\}$-subgroup for all primes p, q.*

Proof. Clearly, (a) implies (c) and (e); moreover, (d) follows from (b). If $G/O_\pi(G)$ has a nilpotent Hall π-subgroup, the same is clearly true for $G/O_{\pi,\pi'}(G)$. Therefore (a) implies (b) and (d) is a consequence of (c). Thus it remains to show that (a) follows from both (d) and (e).

Suppose first that G satisfies (d) and let π be a set of primes. Moreover, for every prime p, let $H_{p'}/O_{p',p}(G)$ be a nilpotent Hall p'-subgroup of G. Then by the Schur-Zassenhaus theorem, $O_{p',p}(G)/O_{p'}(G)$ has a complement $L_{p'}/O_{p'}(G)$ in $H_{p'}/O_{p'}(G)$. Clearly, $L_{p'}$ is a Hall p'-subgroup of G and $L_{p'}/O_{p'}(G)$ is nilpotent. Now $L = \bigcap_{p\in\pi'} L_{p'}$ is a Hall π-subgroup of G and $L/O_\pi(G) \cong L/L \cap \bigcap_{p\in\pi'} O_{p'}(G)$ which is nilpotent. Therefore G satisfies (a).

Finally, assume that the finite group G satisfies (e). As a first step, we prove that G is a soluble group. Since condition (e) is inherited by factor groups of G, by induction on the group order of G, we may assume that G possesses a unique minimal normal subgroup K such that G/K is soluble. Since $O_{\{p,q\}}(G)$ is soluble by Burnside's p-q-theorem, we may assume that $O_{\{p,q\}}(G) = 1$ for all primes p, q. Therefore G has a nilpotent Hall $\{p, q\}$-subgroup $G_{\{p,q\}}$ for all primes p and q. Fix a prime p dividing $|G|$ and let P be a Sylow p-subgroup of G. If q is a prime distinct from p and G_p and G_q are the unique Sylow p- and Sylow q-subgroup of $G_{\{p,q\}}$, then $P = G_p^g$ for a suitable element $g \in G$. So the Sylow q-subgroup $G_q^* = G_q^g$ of G centralizes P. Therefore also the subgroup $C = \langle G_q^* \mid q \in \mathbb{P}, q \neq p \rangle$ centralizes G_p and so P is a normal subgroup of $\langle P, C \rangle = G$. But then $K \leq P$ is soluble, hence G is soluble. This proves that every finite group G satisfying (e) is soluble.

Thus if π is an arbitrary set of primes, and G is a group satisfying (e), then G possesses a Hall π-subgroup H, and since for every $p \in \pi$, a Sylow p-subgroup H_p of H is a Sylow p-subgroup of G , we have $[H_p, H_q] \leq O_{\{p,q\}}(G) \leq O_\pi(G)$ for all primes p, $q \in \pi$. This shows that $H/O_\pi(G)$ is nilpotent. Therefore G satisfies (a). □

Let $\mathfrak{X}$ be a class of periodic groups and $\mathfrak{F}$ a local $\mathfrak{X}$-formation of characteristic π. A preformation function f for $\mathfrak{F}$ is called *full* if $\mathfrak{S}_p f(p) = f(p)$ for every prime $p \in \pi$. It is called *integrated* if $f(p) \subseteq \mathfrak{F}$ for every $p \in \pi$. Note that by [DH92, IV, Theorem 3.7], every local formation of finite groups has a unique formation function which is both full and integrated.

6.1.5 Theorem. *Let $\mathfrak{F}$ denote the class of all finite groups such that $G/O_\pi(G)$ has a nilpotent Hall π-subgroup for every set π of primes. Moreover, let $\mathfrak{Y}$ be the class of all groups which are the product of two finite nilpotent subgroups. Then the class $\mathfrak{F}$ has the following properties:*

(a) *$\mathfrak{F}$ is a class of finite soluble groups; hence if $G \in \mathfrak{F}$, then every Hall π-subgroup of $G/O_\pi(G)$ is nilpotent.*

(b) *For every prime p, let $f(p)$ be the class of all finite soluble groups having a nilpotent Hall p'-subgroup. Then $f(p)$ is closed with respect to subgroups, factor groups and products of finitely many normal subgroups. In particular, $f(p)$ is a formation of finite soluble groups.*

(c) *$\mathfrak{F}$ is the local formation of finite soluble groups defined by the formation function f. Moreover, f is a (the unique) full and integrated local function for $\mathfrak{F}$.*

(d) *$\mathfrak{F}$ is a subgroup-closed Fitting class of finite soluble groups.*

(e) *$\mathfrak{F} = S\mathfrak{Y}$, i.e. $\mathfrak{F}$ is the class of all subgroups of products of two finite nilpotent subgroups.*

(f) *$\mathfrak{F}$ is the smallest Schunck class of finite soluble groups which contains all products of two finite nilpotent subgroups.*

(g) *$\mathfrak{F}$ is the smallest subgroup-closed Fitting class of finite soluble groups which contains all products of two finite nilpotent groups.*

(h) *$\mathfrak{F}$ is the smallest formation of finite soluble groups which contains $\mathfrak{F}$.*

Proof. (a) If $H/O_\pi(G)$ is a Hall π-subgroup of $G/O_\pi(G)$, then H is a Hall π-subgroup of G. Therefore every $G \in \mathfrak{F}$ possesses Hall subgroups for every set π of primes, and so every $\mathfrak{F}$-group is soluble. In particular, the Hall π-subgroups of $G/O_\pi(G)$ are conjugate, hence isomorphic.

(b) Let p be a prime. Then it is straightforward to check that $f(p)$ is closed with respect to factor groups and subgroups. Now suppose that the finite soluble group G has normal subgroups N_1 and $N_2 \in f(p)$ such that $G = N_1 N_2$ and let G_p and $G_{p'}$ be a Sylow p-subgroup and a Hall p'-subgroup of G. Since G_p and $G_{p'}$ reduce into every normal subgroup of G, we have $N_1 = (G_p \cap N_1)(G_{p'} \cap N_1)$ and $N_2 = (G_p \cap N_2)(G_{p'} \cap N_2)$; in particular $G_{p'} \cap N_1$ and $G_{p'} \cap N_2$ are Hall p'-subgroups of N_1 and N_2, respectively. Therefore by order reasons $G_{p'}$ is the product of its normal nilpotent subgroups $G_{p'} \cap N_1$ and $G_{p'} \cap N_2$. It follows from Fitting's theorem that $G_{p'}$ is nilpotent. Consequently, $f(p)$ is closed with respect to products of finitely many normal subgroups. Since $f(p)$ is in particular closed with respect to subgroups of finite direct products, it is residually closed with respect to the class of all finite soluble groups.

(c) If $G/O_p(G) \in f(p)$, then $G \in f(p)$ by the Schur-Zassenhaus theorem. so that f is a full formation function. Now let $\mathfrak{G}$ be the saturated formation defined by f, then

$$\mathfrak{G} = \bigcap_{p \in \mathbb{P}} \mathfrak{S}_{p'} \mathfrak{S}_p f(p) = \bigcap_{p \in \mathbb{P}} \mathfrak{S}_{p'} f(p)$$

and so $\mathfrak{G}$ is the class of all groups G such that $G/O_{p'}(G)$ has a nilpotent Hall p'-subgroup for every prime p. Therefore $\mathfrak{G} = \mathfrak{F}$ by Lemma 6.1.4. Since every group in $f(p)$ is the product two nilpotent subgroups, namely of a Sylow p-subgroup and a Hall p'-subgroup, we have $f(p) \subseteq \mathfrak{F}$ by Lemma 6.1.1 and so f is integrated.

(d) By [DH92, IV, Proposition 3.14], $\mathfrak{F}$ is closed with respect to subgroups and products of finitely many normal subgroups. Therefore by [DH92, II, Proposition 2.11], $\mathfrak{F}$ is a Fitting class of finite groups.

(e) Suppose that the group G is the product of two nilpotent subgroups A and B. Then by Lemma 6.1.1 and Lemma 6.1.4, for every set of primes π, the group $G/O_\pi(G)$ has a nilpotent Hall π-subgroup. Therefore $\mathfrak{Y}$ is contained in $\mathfrak{F}$, and since $\mathfrak{F}$ is closed with respect to subgroups, we also have $s\mathfrak{Y} \subseteq \mathfrak{F}$.

Conversely, if $G \in \mathfrak{F}$, then by Proposition 6.1.2, G can be embedded in a product of two finite nilpotent groups, and so $\mathfrak{F} \subseteq s\mathfrak{Y}$.

(f) Let $\mathfrak{H}$ be the intersection of all Schunck classes of finite soluble groups containing $\mathfrak{Y}$. Then $\mathfrak{H} \subseteq \mathfrak{F}$ because every local formation of finite soluble groups is a Schunck class by [DH92, IV, Theorem 3.3] and [DH92, III, Proposition 4.1]. Conversely, in order to show that $\mathfrak{F}$ is contained in $\mathfrak{H}$, it suffices to show that every primitive $\mathfrak{F}$-group G belongs to $\mathfrak{H}$. Since G is soluble, G has a unique minimal normal subgroup N which is an elementary abelian p-group for some prime p and $O_{p'}(G) = 1$; see [DH92, A, Theorem 15.6]. But then the $\mathfrak{F}$-group G has a nilpotent Hall p'-group and so G is the product of a Sylow p-subgroup and a nilpotent Hall p'-subgroup. Thus $G \in \mathfrak{Y} \subseteq \mathfrak{H}$.

(g) Since $\mathfrak{F}$ is evidently the smallest subgroup-closed class containing all products of two finite nilpotent groups, this follows at once from (d).

(h) Since $\mathfrak{Y}$ is closed with respect to finite direct products, we have $R\mathfrak{Y} \cap \mathfrak{S}^* \subseteq SD\mathfrak{Y} \cap \mathfrak{S}^* \subseteq S(D\mathfrak{F} \cap \mathfrak{S}^*) = \mathfrak{F}$; see [DH92, II, Lemma 1.18]. On the other hand, if $G \in \mathfrak{F}$, then $G/O_{p'}(G)$ is the product of a Sylow p- and a nilpotent Hall p'-subgroup. Since $\bigcap_{p \in \mathbb{P}} O_{p'}(G) = 1$, it follows that $G \in R\mathfrak{Y} \cap \mathfrak{S}^*$ and so $\mathfrak{F} = R\mathfrak{Y} \cap \mathfrak{S}^*$. Now let $\mathfrak{G}$ be an $\mathfrak{S}^*$-formation containing $\mathfrak{Y}$, then $\mathfrak{F} = R\mathfrak{Y} \cap \mathfrak{S}^*$ is contained in $R\mathfrak{G} \cap \mathfrak{S}^* = \mathfrak{G}$. Since $\mathfrak{F}$ is in particular a formation, this shows that $\mathfrak{F}$ is the unique smallest formation which contains $\mathfrak{Y}$. □

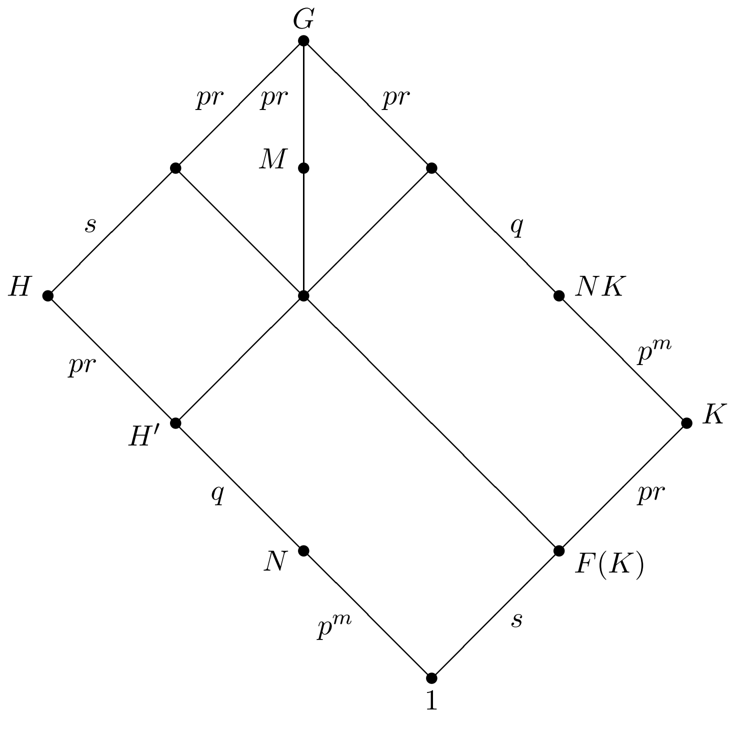
Recall that a class $\mathfrak{F}$ is *saturated* if $G \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$ for some normal subgroup N of G which is contained in the Frattini subgroup of G. In particular, every Schunck class, and thus every local formation is saturated; see [DH92, III, Lemma 2.10].

It seems to be an open question whether $\mathfrak{F}$ is the smallest Fitting class (the smallest S_n-closed class, the smallest N_0-closed class, the smallest saturated class) containing every product of two finite nilpotent subgroups.

The following example shows that the class of all products of two finite nilpotent subgroups is not closed with respect to subnormal subgroups, and in particular that a subgroup of a product of two finite nilpotent groups need not be a product of two nilpotent groups.

6.1.6 Example. Let p and r be distinct primes. Then by Dirichlet's remainder theorem, for every choice of p and r, there are infinitely many primes q and s such that p divides $q-1$ and pr divides $s-1$. In particular, q and s may be chosen such that p, q, r and s are distinct. (For instance, take $p=2$, $r=3$, $q=5$ and $r=7$.)

Now let $C_{qr} = <x_q> \times <x_r>$ be a direct product of a cyclic group $<x_q>$ of order q and a cyclic group $<x_r>$ of order r. Let n be a primitive p-th root modulo q and let x_p be the automorphism of C_{qr} defined by $x_q^{x_p} = x_q^n$ and $x_r^{x_p} = x_r$. Let $L = C_{qr} \rtimes <x_p>$, then $L' = <x_q>$ is cyclic of order q and $L = L' \rtimes <x_p x_r>$. Moreover, $F(L) = C_{qr}$ and so by [DH92, B, Corollary 10.7], L has a faithful irreducible $GF(p)$-module N of order p^m, say. (If $p=2$, $r=3$ and $q=5$, then N has order 2^4). Form the semidirect product $H = L \ltimes N$. Furthermore, let y_{pr} be an automorphism of order pr of a cyclic group $<y_s>$ of order s and put $K = <y_{pr}> \ltimes <y_s>$. Form the direct product $G = H \times K$, then clearly, G is the product of its nilpotent subgroups $H_p \times K_{s'}$ and $H_{p'} \times K_s$.



The structure of the group G in Example 6.1.6

Now let $M = <N, x_q, x_p \cdot x_r \cdot y_{pr}, y_s>$ which is a normal subgroup of G with $|G : M| = pr$. We show that M is not the product of two nilpotent subgroups.

Assume that $M = AB$ with A, B nilpotent. Since $G = MH = MK$, the factor groups $MH/H \cong K$ and $MK/K \cong H$ are primitive. Then by [Gro73, Theorem 1], without loss of generality AK/K is a Sylow p-subgroup of G/K. Since the order of a Sylow p-subgroup of G/K equals that of M, the p-component A_p of A is a Sylow p-subgroup of M. Since A is nilpotent, the Sylow r-subgroup A_r of A centralizes $N \leq M_p \leq A$, hence A_rK/K is contained in $C_{G/K}(NK/K)$. Since $C_{G/K}(NK/K) = NK/K$ by [DH92, A, Theorem 15.6] and NK/K is a p-group, it follows that $A_r \leq K \cap M$, which is an s-group. Consequently $A_r = 1$. If q divides the order of A, then A_q is a Sylow q-subgroup of M, hence $G/NK \cong L$ has a nilpotent Hall $\{p, q\}$-subgroup, a contradiction. This shows that $A_q = 1$. Similarly, if we had $A_s \neq 1$, then G/H would have a nilpotent Hall $\{p, s\}$-subgroup. This contradiction shows that $A_s = 1$ and A is a Sylow p-subgroup of M. Thus B must contain a Hall p'-subgroup of M. But then G/K and G/H would have to have nilpotent Hall p'-subgroups. Since this is not the case, M is not the product of two nilpotent subgroups.

6.2. Products of more than two finite nilpotent groups

While the main results in Section 1.1 hold for arbitrary products of groups, most theorems about prefactorized subgroups products of two nilpotent groups and products of locally nilpotent groups do not hold for products of more than two subgroups. The only exception known to the author is the existence of prefactorized Sylow bases in such products; see [Wie51, *Satz* 1]. The next proposition shows that even in the finite case, the main results of Chapter 2 cannot be extended to products of more than two locally nilpotent subgroups.

6.2.1 Proposition. *There is a finite group which is the product of three pairwise permutable nilpotent subgroups A, B and C such that $F(G) \neq (A \cap F(G))(B \cap F(G))(C \cap F(G))$. In addition, G may be chosen such that*

(a) $|G| = p^\alpha q^\beta$ *for given distinct primes p and q and suitable integers α and β.*

(b) $A = G' = O_q(G)$ *and BC is a Sylow p-subgroup of G.*

(c) $A < F(G)$ *and* $B \cap F(G) = C \cap F(G) = 1$.

(d) *The subgroups $O_p(G)$, $F(G)$, $O_{q',q}(G)$ of G are not prefactorized.*

(e) *If $B^g \neq B$, then B^g does not permute with C. In particular, there exists a $g \in G$ such that C does not permute with B^g.*

Proof. Let p and q be distinct primes and let H be the product of a normal q-group A with a cyclic group $B = <b>$ of order p such that B does not centralize A (for example, let H be the regular wreath product of a group of order q with a group of order p). Now let $G = H \times <x>$, where $<x>$ is a cyclic group of order p. Let $C = <bx>$, then $A = O_q(G)$ and BC is a Sylow p-subgroup of G. This shows that AB, AC and BC are subgroups of G and $G = ABC$. Now $B \cap O_q(G) = C \cap O_q(G) = 1$ and so $O_q(G) = <x>$ is not prefactorized. Also, $F(G) = A \times <x>$ and $B \cap F(G) = C \cap F(G) = 1$ which shows that $F(G) = O_{q',q}(G)$ is not prefactorized.

We show that the conjugates of B do not permute with C: Let $g \in G$ with $B^gC = CB^g$. Then B^gC is a Sylow p-subgroup of G and so $<x> = O_p(G)$ is contained in B^gC. Since $bx \in C$, the subgroup B^gC contains b. Since $H \trianglelefteq G$, we have $B^g \leq H$ and so $B^gC \cap H = B^g$

is a Sylow p-subgroups of H containing b and b^g. Since B^g is a cyclic p-group, we have $B^g = <b^g> = <b> = B$ and so $g \in N_G(B)$. This shows that B^gC is a group if and only if $B = B^g$. □

Let G be the product of three pairwise permutable finite nilpotent subgroups. The following proposition shows that in general, no term of the upper Fitting series of G except 1 and G is prefactorized.

6.2.2 Proposition. *For every integer $k \geq 2$, there is a finite group of Fitting length k which is the product of three pairwise permutable nilpotent subgroups A, B and C, such that for every n with $1 \leq n \leq k-1$, $F_n(G) \neq (A \cap F_n(G))(B \cap F_n(G))(C \cap F_n(G))$. The example may be chosen such that*

(a) *For any two prescribed primes p and q, G is a $\{p,q\}$-group.*

(b) *The subgroups $A \cap F_n(G)$, $B \cap F_n(G)$ and $C \cap F_n(G)$ are mutually permutable.*

Proof. Let p and q be distinct primes and suppose that Z_p and Z_q are groups of order p and q respectively. Further, set $H_0 = 1$ and $H_1 = Z_p$ and for every integer $n \geq 2$, let $H_n = (H_{n-2} \wr Z_q) \wr Z_p$. Then each H_n has Fitting length n and $H_n = Z_p \ltimes O^p(H_n)$. Now let $G_n = Z_p \times H_n$, then G_n can be identified with the group

$$\{(x,y,z) \mid x,\, y \in Z_p,\, z \in O^p(H_n)\}$$

with $(x_1,y_1,z_1)(x_2,y_2,z_2) = (x_1x_2, y_1y_2, z_1^{y_2}z_2)$ as the rule of multiplication. Now let A_n and B_n be a Sylow p- and a Sylow q-subgroup of $H_n = \{(1,y,z) \mid y \in Z_p,\, z \in O^p(H_n)\}$ respectively and $C_n = \{(x,x,1) \mid x \in Z_p\}$. Then $A_nB_n = H_n$, A_nC_n is a Sylow p-subgroup of G_n and it is straightforward to check that also $B_nC_n = \{(x,x,z) \mid x \in Z_p,\, z \in B_n\}$ is a subgroup of G_n. Therefore $G_n = A_nB_nC_n$ is the product of the pairwise permutable subgroups A_n, B_n and C_n. Now let $K_n = F_{n-1}(G_n) = Z_p \times O^p(H_n)$, then $A_n \cap K_n$ and $B_n = B_n \cap K_n$ are a Sylow p- and a Sylow q-subgroup of $O^p(H_n)$ respectively and $C_n \cap K_n = 1$. Therefore the (pairwise permutable) subgroups $A_n \cap K_n$, $B_n \cap K_n$ and $C_n \cap K_n$ do not generate K_n. Therefore K_n is not prefactorized.

Now let $G = \bigtimes_{i=2}^k G_i$, then G is the product of its pairwise permutable subgroups $A = \bigtimes_{i=2}^k A_i$, $B = \bigtimes_{i=2}^k B_i$ and $C = \bigtimes_{i=2}^k C_i$. If $1 \leq n \leq k-1$, then $F_n(G) = \bigtimes_{i=2}^k F_n(G_i)$. Let ϕ be the canonical projection of G onto G_{n+1}. Then $A^\phi = A_{n+1}$, $B^\phi = B_{n+1}$ and $C^\phi = C_{n+1}$. Moreover, $F_n(G)^\phi = F_n(G_{n+1}) = K_{n+1}$. So if $F_n(G)$ were prefactorized, then K_{n+1} would be a prefactorized subgroup of $G_{n+1} = A_{n+1}B_{n+1}C_{n+1}$, a contradiction. □

Let the finite group G be the product of two subgroups A and B. If A_0 and B_0 are normal subgroups of A and B, then by a result of Wielandt [Wie58, Hilfssatz 7], see also [AFG92, Lemma 1.2.5], the normalizer $N_G(<A_0, B_0>)$ is factorized. However, also this important result does not hold in finite products of more than two nilpotent subgroups.

6.2.3 Example. Let p and q be two primes with $p \mid q-1$. Let H and K be nonabelian groups of order pq and put $G = H \times K$. Let $A = H_p \times K_q$ and $B = H_q \times K_p$. Then $G = AB$. Let x and y be generators of H_q and K_q respectively, then $C = <xy>$ permutes with A and B. Now $L = N_G(C)$ has index p in G but $A \cap L = K_q$ and $B \cap L = H_q$ which shows that the group $(L \cap A)(L \cap B)(L \cap C) = H_q \times H_q$ has index p^2 in G. Therefore L is not prefactorized in $G = ABC$.

Appendix A

List of symbols

Generally, uppercase Latin letters denote groups (A, B, G, H, ...) or sets, lowercase (Latin) letters symbolize elements of sets or groups. Uppercase Fraktur letters ($\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{X}$, ...) represent classes of groups, while Script ($\mathcal{F}$, $\mathcal{G}$, ...) is used for sets of (sub)groups. Lowercase and uppercase Greek letters usually denote homomorphisms of groups (α, β, ...) and sets of automorphisms (Γ) or sets of primes (π, σ, τ ...).

In the following, G and H will be groups, A and B are subgroups of G and g, $h \in G$. Γ will be a set acting on G via endomorphisms and $\alpha \in \Gamma$. X and Y represent sets, while the letters k, m, n and p denote integers and p is a prime. Moreover π is a set of primes. $\mathcal{S}$ is a set of subgroups of G and $\mathfrak{X}$ is a class of groups.

$X \subseteq Y$	the set X is contained in the set Y
$X \setminus Y$	the difference of the set X and the set Y
$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0$	the set of nonnegative integers
$GF(p^n)$	the finite field of order p^n
(m, n)	the greatest common divisor of the integers m and n
$\mathbb{P}$	the set of primes
π'	the set $\mathbb{P} \setminus \pi$
p'	the set $\mathbb{P} \setminus \{p\}$

$G \leq H$	G is a subgroup of the group H
$G \cong H$	G is isomorphic with H
$G \times H$	the direct product of G and H
$G \wr H$	the regular wreath product of G and H
$<X>$	the subgroup of G generated by the elements of $X \subseteq G$
$<x_1, x_2, \ldots>$	the subgroup generated by the set $\{x_1, x_2, \ldots\}$
g^h	$= h^{-1}gh$
X^α	the set $\{x^\alpha \mid x \in X\}$
X^Γ	the subgroup of G generated by the set $\{x^\alpha \mid x \in X,\ \alpha \in \Gamma\}$
X_Γ	$= \bigcap_{\alpha \in \Gamma} X^\alpha$
$[g, \alpha]$	the commutator of g and α; $[g, \alpha] = g^{-1}g^\alpha$
$[A, B]$	the subgroup of G generated by all $[a, b]$ where $a \in A$ and $b \in B$
$N_\Gamma(X)$	normalizer of the set X: $N_\Gamma(X) = \{\alpha \in \Gamma \mid [x, \alpha] \in X \text{ for all } x \in X\}$
$C_\Gamma(X)$	centralizer of the set X: $C_\Gamma(X) = \{\alpha \in \Gamma \mid [x, \alpha] = 1 \text{ for all } x \in X\}$
$Z(G)$	centre of the group G; $Z(G) = C_G(G)$
$G^{(n)}$	n-th derived subgroup of G defined recursively by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geq 0$

G', G''	$= G^{(1)}, G^{(2)}$
$R(G)$	Hirsch-Plotkin radical of G, the subgroup generated by the normal locally nilpotent subgroups of G
$F(G)$	Fitting subgroup of G, the subgroup generated by the normal nilpotent subgroups of G
$J(G)$	the intersection of all normal subgroups of G which have finite index in G
$\Phi(G)$	Frattini-subgroup of G, the intersection of all maximal subgroups of G, or $\Phi(G) = G$ if no maximal subgroups exists.
$O^{\pi}(G)$	the intersection of all normal subgroups N of G such that G/N is a π-group
$O^{\pi',\pi}(G)$	$= O^{\pi}(O^{\pi'}(G))$
$O_{\pi}(G)$	the maximal normal π-subgroup of G
$O^{\pi',\pi}(G)$	defined by $O^{\pi',\pi}(G)/O^{\pi'}(G) = O_{\pi}(G/O^{\pi'}(G))$
G_{π}	a Sylow π-subgroup of the group G
$G[n]$	the subgroup of G generated by all $g \in G$ with $g^n = 1$
$G^{\mathfrak{X}}$	the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{X}$
$G_{\mathcal{S}}$	the subgroup of G generated by the subgroups $S \in \mathcal{S}$ which are serial in G
$\|G\|$	the cardinality of the set G
$\pi(G)$	the set of primes dividing the order of some element of G
$\mathfrak{A}$	the class of all periodic abelian groups
$\mathfrak{N}$	the class of all periodic nilpotent groups
$\mathfrak{S}$	the class of all periodic locally soluble groups
$\mathfrak{X}_{\pi}$	the class of all $\mathfrak{X}$-groups which are π-groups
$\mathfrak{X}^{*}$	the class of all finite $\mathfrak{X}$-groups
$Q\mathfrak{X}$	the class of all factor groups of $\mathfrak{X}$-groups
$S\mathfrak{X}$	the class of all subgroups of $\mathfrak{X}$-groups
$L\mathfrak{X}$	the class of all group G such that every finite subset of G is contained in an $\mathfrak{X}$-subgroups of G
$S_n\mathfrak{X}$	the class of all subnormal subgroups of $\mathfrak{X}$-groups
$N\mathfrak{X}$	the class of all groups which are the product of their normal $\mathfrak{X}$-subgroups
$R\mathfrak{X}$	the class of all groups which possess a set $\mathcal{N}$ of normal subgroups such that $\bigcap_{N \in \mathcal{N}} N = 1$ and $G/N \in \mathfrak{X}$ for every $N \in \mathcal{N}$
$D\mathfrak{X}$	the class of all groups which are the direct product of $\mathfrak{X}$-groups

Appendix B

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Appendix C

Curriculum vitae

Dec 4, 1966	Born as the first child of Edgar Höfling, electrical engineer, and Erika Höfling, teacher, in Koblenz-Horchheim, Federal Republic of Germany.
1973–1977	Elementary school "Koblenz-Karthause" in Koblenz.
1977–1986	"Staatliches Gymnasium auf der Karthause", Koblenz. Final degree: *Abitur* with grade "*sehr gut*" (1.0).
1986–1987	Study of electrical engineering at the RWTH (Technical University), Aachen.
since 1987	Study of physics at the Johannes-Gutenberg-University, Mainz.
since 1988	Study of mathematics at the University of Mainz.
1990	*Vordiplom* in Mathematics.
1990/1991	One-year stay at the Department of Mathematics of the *Università degli studi Federico II* in Naples, Italy, with an *Erasmus* grant.
1992/1993	Scholarship of the Johannes-Gutenberg-Universität.
June 15, 1993	*Diplom* (Master degree) in Mathematics with final grade: "*sehr gut*".
1993–1995	PhD scholarship from the German federal state Rheinland-Pfalz.
1995–1996	*Wissenschaftlicher Angestellter* (Scientific Assistant) at the Department of Mathematics, Mainz.
June 21, 1996	*Promotion zum Dr. rer. nat* (PhD) in Mathematics with final grade "mit Auszeichnung bestanden" (summa cum laude).

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[1] On Direct Summands of Butler $\mathcal{B}^{(1)}$-Groups, *Comm. Alg.* 21(6) (1993), 1849–1856.

[2] *Produkte endlicher nilpotenter Gruppen*, Diploma Thesis, Mainz 1993.

[3] On finite products of nilpotent groups (together with B. Amberg), *Arch. Math* 63 (1994), 1–8.

[4] Finite groups with multiple factorizations (together with B. Amberg and L. S. Kazarin), to appear in *Fundamental and Applied Algebra*, Moscow.