FinLie

Computation with finite Lie algebras

Version 1.1

01 March 2018

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Chapter 1

Introduction

This package contains methods to compute with finite Lie algebras; that is, with Lie algebras over finite fields. The functions in this package are often particularly effective if the considered Lie algebras are solvable, since there are various special methods for finite solvable Lie algebras implemented. The GAP system contains various methods to compute with Lie algebras, see [DeG00] for background. This package is based on these methods and extends them in the case of a finite Lie algebra, see [Eic04] for a description of some of the methods. The package also contains a database of small-dimensional simple Lie algebras over the field with two elements, see [Eic10] for additional information.

1.1 Tables

Computations with Lie algebras in FinLie are usually carried out using tables. These tables wrt. to a basis *B* of *L* contain the structure constants for the Lie algebra *L*. The following functions allow the computation of such a table and the construction of a Lie algebra from a given table.

1.1.1 LieTableByBasis

▷ `LieTableByBasis(L, B)` (function)

Determines a table for the given Lie algebra *L* wrt. to the given basis *B*.

1.1.2 LieAlgebraByTable

▷ `LieAlgebraByTable(T)` (function)

Constructs a Lie algebra described by the given table *T*.

1.2 Constructions

There are several constructions of Lie algebras implemented in FinLie; in particular the following constructions are available.

1.2.1 AbelianLieAlgebra

▷ `AbelianLieAlgebra(n, p)` (function)

Constructs the n-dimensional abelian Lie algebra L over the field with p elements.

1.2.2 UnipotentLieAlgebra

▷ `UnipotentLieAlgebra(n, p)` (function)

Constructs the unipotent Lie algebra L of $n \times n$ matrices over the field with p elements.

1.2.3 TriangularLieAlgebra

▷ `TriangularLieAlgebra(n, p)` (function)

Constructs the triangular Lie algebra L of $n \times n$ matrices over the field with p elements.

1.2.4 LieAlgebraTensor

▷ `LieAlgebraTensor(L, K)` (function)

Constructs the tensor product of a given Lie algebra L with a finite field K.

1.2.5 DerivationsLieAlgebra

▷ `DerivationsLieAlgebra(L)` (function)

Constructs the Lie algebra of derivations a given Lie algebra L.

Chapter 2

Cohomology

2.1 First cohomology

2.1.1 LieOneCohomology

▷ `LieOneCohomology(L, B, M)` (function)
▷ `LieOneCobounds(L, B, M)` (function)
▷ `LieOneCocycles(L, B, M)` (function)

These functions take as input a Lie algebra `L`, a basis `B` of `L` and a list of matrices `M` such that the map mapping the elements of `B` to the elements of `M` induces a representation of `L`.

Let $B = \{b_1, \ldots, b_n\}$ and `V` be the natural `L`-module defined by the given representation. Then the map $Z^1(L,V) \to V^n : \varphi \mapsto (b_1^{\varphi}, \ldots, b_n^{\phi})$ is a faithful representation of $Z^1(L,V)$.

The functions LieOneCobounds and LieOneCocycles return lists of vectors which form a basis for the images of the representations of $Z^1(L,V)$ and $B^1(L,V)$, respectively.

The functions LieOneCohomology returns a linear map onto the factor of the results of LieOneCocycles modulo LieOneCobounds. Thus the image of this map is a faithful representation of $H^1(L,V)$.

2.2 Second cohomology

2.2.1 LieTwoCohomology

▷ `LieTwoCohomology(L, B, M)` (function)
▷ `LieTwoCobounds(L, B, M)` (function)
▷ `LieTwoCocycles(L, B, M)` (function)

These functions take as input a Lie algebra `L`, a basis `B` of `L` and a list of matrices `M` such that the map mapping the elements of `B` to the elements of `M` induces a representation of `L`.

The functions LieTwoCobounds and LieTwoCocycles return lists of vectors which form a basis for the images of the representations of $Z^2(L,V)$ and $B^2(L,V)$, respectively.

The functions LieTwoCohomology returns a linear map onto the factor of the results of LieTwoCocycles modulo LieTwoCobounds. Thus the image of this map is a faithful representation of $H^2(L,V)$.

2.3 Extensions by cocycles

2.3.1 LieExtensionByCocycle

▷ `LieExtensionByCocycle(L, B, M, c)` (function)

This function takes as input a Lie algebra `L`, a basis `B` of `L` and a list of matrices `M` such that the map mapping the elements of `B` to the elements of `M` induces a representation of `L` and a 2-cocycle `c` in the representation of $Z^2(L,V)$.

The function returns a Lie algebra which is an extension of `L` by the natural `L`-module `V` defined by the representation via `v`.

Chapter 3

Automorphism groups and isomorphism testing

3.1 Calculation of automorphism groups

3.1.1 AutomorphismGroupOfLieAlgebra

▷ `AutomorphismGroupOfLieAlgebra(L)` (function)

Computes the automorphism group of the finite Lie algebra L.

3.2 Isomorphism testing

3.2.1 CanonicalFormOfSolvableLieAlgebra

▷ `CanonicalFormOfSolvableLieAlgebra(L)` (function)

Takes as input a finite solvable Lie algebra L and it returns another finite solvable Lie algebra K isomorphic to L whose structure constants are in a canonical form. Two Lie algebras L and K are isomorphic if and only if the structure constants of their canonical forms are equal.

3.2.2 AreIsomorphicLieAlgebras

▷ `AreIsomorphicLieAlgebras(L, K)` (function)

This function checks whether the finite Lie algebras L and K are isomorphic. The function can be really slow if L or H are not solvable.

Chapter 4

Simple Lie algebras

4.1 Simple Lie algebras and ideals

4.1.1 IsSimpleLieAlgebra

▷ IsSimpleLieAlgebra(*L*) (function)

This function determines whether a given Lie algebra *L* is simple.

4.1.2 IdealsOfLieAlgebra

▷ IdealsOfLieAlgebra(*L*) (function)

This function determines the ideals of a given Lie algebra *L*.

4.1.3 SimpleFactorsLieAlgebra

▷ SimpleFactorsLieAlgebra(*L*) (function)

This function determines the simple factors of a given Lie algebra *L*.

4.2 Small-dimensional simple Lie algebras over *GF(2)*

FinLie contains a database of small-dimensional simple Lie algebras over the field with two elements. This database provides a complete list of isomorphism type representatives up to dimension *9*, see [VL06] for the classification. In dimensions *10* to *20* the database contains non-isomorphic simple Lie algebras, but it might not be complete. These are either obtained by the method described in [Eic10] or are contained in one of the known infinite families of simple Lie algebras over *GF(2)*. These infinite series are also available in FinLie (see next section).

4.2.1 LieAlgebraByLibrary

▷ LieAlgebraByLibrary(*p*, *d*, *n*) (function)

This function returns the `n`-th Lie algebra in the database of `d`-dimensional simple Lie algebras over `GF(p)`. Note that currently on `p=2` is supported.

4.3 Infinite families of simple Lie algebras over `GF(2)`

Kaplansky [Kap82] introduced several infinite series of simple Lie algebras in characeristic two. These are available in the following functions.

4.3.1 SimpleLieAlgebraByGramMatrix1

▷ `SimpleLieAlgebraByGramMatrix1(n)` (function)

For given $n \geq 4$ this returns a simple Lie algebra of dimension $2^n - 2$ (Kaplansky Type I).

4.3.2 SimpleLieAlgebraByGramMatrix2

▷ `SimpleLieAlgebraByGramMatrix2(n)` (function)

For a given even n this returns a simple Lie algebra of dimension $2^n - 1$ (Kaplansky Type II).

4.3.3 SimpleLieAlgebraAlternateMats

▷ `SimpleLieAlgebraAlternateMats(n)` (function)

For given n this returns a simple Lie algebra of dimension $n(n-1)/2$ (Kaplansky Type III).

4.3.4 SimpleLieAlgebraByQuadraticForm

▷ `SimpleLieAlgebraByQuadraticForm(Q)` (function)

Let $m \geq 3$. Given Q a nonsingular quadratic form on a $2m$-dimensional vector space V, this function returns a simple Lie algebra of dimension $2^{m-1}(2^m - 1)$ if the Arf invariant of Q is 0 or $2^{m-1}(2^m + 1)$ if the Arf invariant of Q is 1 (Kaplansky Type IV).

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