Computing the Schur multiplicator and the nonabelian tensor square of a polycyclic group

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Abstract

We describe effective algorithms for computing a polycyclic presentation of the Schur multiplicator, a Schur cover, the nonabelian exterior square and the nonabelian tensor square of a polycyclic group given by a polycyclic presentation. Additionally, we introduce a method to check whether a polycyclic group is capable.

1 Introduction

The nonabelian tensor product $G \otimes H$ for two arbitrary groups $G$ and $H$ has been introduced by Brown and Loday [5, 6] following ideas of Dennis [7]. It is an interesting group theoretic construction which arises from applications in homotopy theory of a generalised Van Kampen theorem.

An important special case is the nonabelian tensor square $G \otimes G$ of a group $G$. For $g, h \in G$ let $[g, h] = ghg^{-1}h^{-1}$ and $^h g = hgh^{-1}$. Then $G \otimes G$ is defined as the group generated by the symbols $g \otimes h$ for $g, h \in G$ subject to the defining relations

$$gh \otimes k = (^g h \otimes ^g k)(g \otimes k) \quad \text{and} \quad g \otimes hk = (g \otimes h)(^h g \otimes ^h k) \quad \text{for } g, h, k \in G.$$ 

Thus $G \otimes G$ is defined by a presentation on $|G|^2$ generators and $2 \cdot |G|^3$ relations. This presentation exhibits the universal properties of the nonabelian tensor square, but it does not reflect its group structure very well and it is unsuitable for computational purposes.

The structure of $G \otimes G$ in terms of central extensions has been investigated in [5, 6]. Define $\nabla(G) := \langle g \otimes g \mid g \in G \rangle$ and the nonabelian exterior square $G \wedge G := (G \otimes G)/\nabla(G)$. Further, let $J(G)$ denote the kernel of $G \otimes G \to G : g \otimes h \mapsto [g, h]$ and let $\Gamma(G/G')$ be Whitehead’s quadratic functor [22]. Then by [5, 6] the kernel of $G \wedge G \to G : g \wedge h \mapsto [g, h]$ is isomorphic to the Schur multiplicator $M(G)$ and there exists a commutative diagram whose rows and columns all represent central extensions:

$$
\begin{array}{c}
\Gamma(G/G') \\
\downarrow \\
\nabla(G) \\
\downarrow \\
1
\end{array} \quad \rightarrow \quad
\begin{array}{c}
J(G) \\
\downarrow \\
G \otimes G \\
\downarrow \\
G \wedge G
\end{array} \rightarrow
\begin{array}{c}
M(G) \\
\downarrow \\
G \wedge G \\
\downarrow \\
1
\end{array} \rightarrow
1
\begin{array}{c}
(*)
\end{array}
$$
If $G/G'$ is finitely generated, then $\Gamma(G/G')$ is finitely generated abelian and it can be determined easily, see [22] or [4]. All other groups in the diagram (*) are more difficult to investigate (theoretically and computationally). Brown, Johnson and Robertson [4] computed $G \otimes G$ for all groups $G$ of order at most 30 using its defining presentation and the Todd-Coxeter algorithm. Ellis and Leonard [11] describe a more effective method to determine $G \otimes H$ for finite $G$ and $H$ and they compute the nonabelian tensor square of the Burnside group $B(2,4)$ of order $2^{12}$ as an example application. Holt [12] describes a method to compute the Schur multiplicator of a finite group.

Using the diagram (*), Blyth and Morse [3] prove that if $G$ is polycyclic, then $G \otimes G$ is polycyclic. Hence $G \otimes G$ has a consistent polycyclic presentation in this case. Such a consistent polycyclic presentation would allow one to exploit the structure of $G \otimes G$ using computational methods. Hence it is of interest to determine such a presentation.

It is the aim of this paper to introduce effective algorithms for computing a consistent polycyclic presentation for $G \otimes G$ and all other groups in the diagram (*) for a group $G$ given by a consistent polycyclic presentation. For $\Gamma(G/G')$ this has been achieved in [22], see also [4], and hence we consider the remaining groups of the diagram (*) only. Further, we introduce a method to check whether a group $G$ given by a consistent polycyclic presentation is capable; that is, whether there exists a group $H$ with $H/Z(H) \cong G$. Our main tool for all of these algorithms is a method to compute certain central extensions of a group given by a consistent polycyclic presentation.

An implementation of our algorithm is available as part of the Polycyclic package [9] of GAP [21]. It is significantly more effective than the previously known methods for many finite polycyclic groups and it also extends the range of possible applications to infinite polycyclic groups. We report on some sample applications in Section 6 below.

2 Central extensions

In this section we describe our main algorithmic tool: a method to compute consistent polycyclic presentations for certain central extensions of a polycyclic group. Our method is a variation on an algorithm described in the thesis [16]. In the following we give a self-contained account of our method.

Let $G$ be a polycyclic group defined by a consistent polycyclic presentation $F_n/R$, where $F_n$ is the free group on generators $g_1, \ldots, g_n$. Let $H$ be a finitely presented group defined by a finite presentation $F_m/S$, where $F_m$ is the free group on generators $f_1, \ldots, f_m$ and $m \leq n$. Suppose that $\varphi : H \to G$ with $\varphi(f_i) = g_i$ defines an epimorphism and denote the kernel of $\varphi$ by $K/S$. We define

$$G^* := F_n/[R,F_n] \quad \text{and} \quad \hat{G} := F_m/[K,F_m]S.$$ 

By construction, both groups $G^*$ and $\hat{G}$ are central extensions of $G = F_n/R \cong F_m/K$.

The group $\hat{G}$ can also be described as the largest central extension of $G$ so that $\varphi$ extends to an epimorphism of $F_m/S$ to $\hat{G}$.

Here we describe effective algorithms to determine consistent polycyclic presentations for $G^*$ and $\hat{G}$. Our methods make use of the well-known theory of polycyclic presentations and consistency. We refer to Chapter 8 in [13], Chapter 9 in [20], or to [8] for an introduction and further information on this topic.
2.1 A consistent polycyclic presentation for \( G^* \)

The relations of a consistent polycyclic presentation \( F_n/R \) have the form:

\[
\begin{align*}
g_i^{e_i} &= g_{i+1}^{x_i} \cdots g_n^{x_n} & \text{for } & i \in I, \\
g_j^{-1} g_i g_j &= g_{j+1}^{y_{i,j}} \cdots g_n^{y_{i,j,n}} & \text{for } & j < i, \\
g_j g_i g_j^{-1} &= g_{j+1}^{z_{i,j}} \cdots g_n^{z_{i,j,n}} & \text{for } & j < i \text{ and } j \notin I
\end{align*}
\]

for some \( I \subseteq \{1, \ldots, n\} \), certain exponents \( e_i \in \mathbb{N} \) for \( i \in I \) and \( x_{i,j}, y_{i,j,k}, z_{i,j,k} \in \mathbb{Z} \) for all \( i,j \) and \( k \). To simplify notation, we rewrite the relations of \( G \) as relators and label them as \( r_1, \ldots, r_l \). Thus every relator \( r_j \) is a word in the generators \( g_1, \ldots, g_n \); that is, \( r_j = r_j(g_1, \ldots, g_n) \).

We introduce \( l \) new abstract generators \( t_1, \ldots, t_l \) and we define \( G^* \) as the group generated by \( g_1, \ldots, g_n, t_1, \ldots, t_l \) subject to the following defining relators:

\[
\begin{align*}
(1) & \quad r_i(g_1, \ldots, g_n)t_i \text{ for } 1 \leq i \leq l, \\
(2) & \quad [t_i, g_j] \text{ for } 1 \leq j \leq n, 1 \leq i \leq l, \\
(3) & \quad [t_i, t_j] \text{ for } 1 \leq j < i \leq l.
\end{align*}
\]

It follows directly from these relators that \( G^* \) is a central extension of \( T := \langle t_1, \ldots, t_l \rangle \) by \( G \). The next lemma states a few more elementary facts about \( G^* \).

1 Lemma: Let \( G \) be defined by the consistent polycyclic presentation \( F_n/R \) as above. Using the above notation we obtain the following.

(a) \( G^* \cong F_n/[R, F_n] \), \( T \cong R/[R, F_n] \) and \( G^*/T \cong F_n/R = G \).

(b) \( G^* \) is defined by a (possibly inconsistent) polycyclic presentation.

Proof: (a) The relators in (1) yield that \( G^*/T \cong G \) and the relators in (2) – (3) ensure that \( T \) is a central subgroup of \( G^* \). It remains to be shown that \( G^* \cong F_n/[R, F_n] \). It follows then that \( T = R/[R, F_n] \). Define \( \sigma : F_n \rightarrow G^* \) via \( \sigma(g_i) = g_i \) for \( 1 \leq i \leq n \). Then \( \sigma \) is surjective by the relations in (1). As \( G^* \) is a central extension of \( G \), it follows that \( [R, F_n] \leq Ker(\sigma) \leq R \). On the other hand, the group \( F_n/[R, F_n] \) is polycyclic and a central extension of \( G \). Hence it has a polycyclic presentation of the same form as \( G^* \) and thus it is a quotient of \( G^* \). This yields that \( [R, F_n] = Ker(\sigma) \) as desired.

(b) This follows directly from the definition of \( G^* \).

We now determine a consistent polycyclic presentation for \( G^* \) from the possibly inconsistent one with relations as in (1)-(3) above. We adapt the method described by Sims [20], page 424, to our setting: See also [8] for a discussion. For this purpose we evaluate the following consistency relations in \( G^* \):

\[
\begin{align*}
g_k(g_j g_i) &= (g_k g_j) g_i & \text{for } & k > j > i, \\
(g_j g_i) g_i &= g_j g_i & \text{for } & j > i, j \in I, \\
g_j(g_i^{e_i}) &= (g_j g_i) g_i^{e_i-1} & \text{for } & j > i, i \in I, \\
(g_i^{e_i}) g_i &= g_i (g_i^{e_i}) & \text{for } & i \in I, \\
g_j &= (g_j g_i^{-1}) g_i & \text{for } & j > i, i \notin I.
\end{align*}
\]
These consistency relations can be evaluated in the polycyclic presentation of $G^*$ using collection from the left. Since $G$ is given by a consistent polycyclic presentation and $T$ is central in $G^*$, it follows that every consistency relation yields an equation of the form

$$t_1^{a_{i1}} \cdots t_l^{a_{il}} = 1$$

for certain $a_{ij} \in \mathbb{Z}$.

Let

$$A = \begin{pmatrix}
a_{11} & \cdots & a_{1l} \\
\vdots & \ddots & \vdots \\
a_{w1} & \cdots & a_{wl}
\end{pmatrix} \in \mathbb{Z}^{w \times l}.$$ 

If $w < l$, then add $l - w$ zero-rows to $A$. Using a Smith-normal-form algorithm, we can compute invertible matrices $P$ and $Q$ such that $A = PDQ$, where $D$ is a diagonal matrix with diagonal entries $d_1, \ldots, d_l \in \mathbb{N}_0$. As in [20], Section 8.3, this yields that

$$T \cong C_1 \times \cdots \times C_l,$$

where $C_{d_i}$ is the cyclic group with $d_i$ elements for $d_i > 0$ and $C_0 = \mathbb{Z}$. A consistent polycyclic presentation for $G^*$ can now be read off from $D$ using the isomorphism $Q^{-1} = (q_{ij})_{1 \leq i, j \leq l}$. It has generators $g_1, \ldots, g_n, t_1, \ldots, t_l$ and defining relations of the form

1. $r_i(g_1, \ldots, g_n)t_1^{a_{i1}} \cdots t_l^{a_{il}}$ for $1 \leq i \leq l$,
2. $[t_i, g_j]$ for $1 \leq j \leq n, 1 \leq i \leq l$,
3. $[t_i, t_j]$ for $1 \leq j < i \leq l$,
4. $t_i^{d_i}$ for $1 \leq i \leq l$ with $d_i > 0$.

Note that $d_i = 1$ for certain $i \in \{1, \ldots, l\}$ is possible. In this case the corresponding generator $t_i$ is redundant in the consistent polycyclic presentation and can be eliminated to shorten the presentation.

2 Example: We consider the infinite dihedral group as an example. This group has the consistent polycyclic presentation

$$D_\infty = \langle g_1, g_2 \mid g_1^2 = 1, g_1^{-1} g_2 g_1 = g_2^{-1} \rangle.$$

The defining presentation for its covering group $D_{\infty}^*$ can then be described by

$$D_{\infty}^* = \langle g_1, g_2, t_1, t_2 \mid g_1^2 = t_1, g_1^{-1} g_2 g_1 = g_2^{-1} t_2, (t_1, t_2 \text{ central}) \rangle.$$

It now remains to perform the consistency check on this presentation. The only non-trivial consistency relations to consider for this purpose are

$$g_2(g_1^2) = (g_2 g_1) g_1 \quad \text{and} \quad (g_1^2) g_1 = g_1 (g_1^2).$$

It is easily checked that these two consistency relations are satisfied in the presentation for $D_{\infty}^*$. Hence that presentation is consistent.
2.2 A consistent polycyclic presentation for $\hat{G}$

Our next aim is to compute a consistent polycyclic presentation for the group $\hat{G}$ as introduced in the beginning of Section 2. Recall that we assume that $G$ is defined by a consistent polycyclic presentation $F_n/R$ on the generators $g_1, \ldots, g_n$. Let $H$ be defined by a finite presentation $F_m/S$, where $F_m$ is free on the generators $f_1, \ldots, f_m$ with $m \leq n$. Suppose that $\varphi : H \to G$ with $\varphi(f_i) = g_i$ defines an epimorphism with kernel $K/S$. Then $G = F_n/R \cong F_m/K$ follows and $\hat{G} := F_m/[K,F_m]S$.

As a first step towards our aim, we determine a consistent polycyclic presentation for $G^*$ on the generators $g_1, \ldots, g_n, t_1, \ldots, t_l$ as in Section 2.1. That presentation can be used to construct a consistent polycyclic presentation of $\hat{G}$ using the following lemma.

3 Lemma: Define $\tau : F_m \to G^*$ by $\tau(f_i) = g_i$ for $1 \leq i \leq m$.

(a) $\text{Ker}(\tau) = [K,F_m]$.
(b) $\hat{G} \cong \text{Im}(\tau)/\tau(S)$.

Proof: (a) This is proved similar to Lemma 1 as follows. First, note that $F_m/\text{Ker}(\tau) \cong \text{Im}(\tau) \leq G^*$ and $\text{Im}(\tau)$ covers $G \cong G^*/T$. Hence $F_m/\text{Ker}(\tau)$ is a central extension of $G$. Using that $G \cong F_m/K$, this yields that $[K,F_m] \leq \text{Ker}(\tau) \leq K$. On the other hand, the group $F_m/[K,F_m]$ is polycyclic and, as a central extension of $G$, has a consistent polycyclic presentation. By the construction of $G^*$, the consistent polycyclic presentation of $G^*$ is the largest possible presentation with this property and thus $G^*$ contains $F_m/[K,F_m]$ as a subquotient via $\tau$. Hence $\text{Ker}(\tau) = [F_m,K]$ as desired.

(b) This follows directly from (a).

Thus it remains to determine generators of the subgroups $\text{Im}(\tau)$ and $\tau(S)$ of $G^*$, because general methods for polycyclic groups can then be used to construct a consistent polycyclic presentation for the quotient $\text{Im}(\tau)/\tau(S)$, see [13], Chapter 8, or [8]. Clearly, a generating set for $\text{Im}(\tau)$ is given by $g_1, \ldots, g_m$.

Let $s_1, \ldots, s_h$ be a set of defining relators for the finitely presented group $H = F_m/S$. Then $\tau(S)$ is generated by $\tau(s_1), \ldots, \tau(s_h)$ as a subgroup, since $\tau(S) \leq T$ is central in $G^*$. Thus generators for $\tau(S)$ can be determined readily by evaluating the relators $s_1, \ldots, s_h$ in $G^*$.

4 Example: (Continuation of Example 2)

Let $H = F_2/S = \langle f_1, f_2 \mid f_1^{-1}f_2f_1 = f_2^{-1} \rangle$. Then $\varphi : H \to D_\infty$ defined by $\varphi(f_i) = g_i$ for $i = 1, 2$ is an epimorphism. We determine $\hat{D}_\infty$ as follows.

Take the induced polycyclic presentation of $D_\infty^* = \langle g_1, g_2, t_1, t_2 \rangle$ as in Example 2 and define $\tau : F_2 \to D_\infty^*$ by $\tau(f_i) = g_i$ for $i = 1, 2$ as in Lemma 3. Then $\text{Im}(\tau) = D_\infty^*$, since $g_1, g_2 \in \text{Im}(\tau)$ by definition, and $t_1 = g_1^{-1} \in \text{Im}(\tau)$ and $t_2 = g_2g_1^{-1} \in \text{Im}(\tau)$ follows. Further, $\tau(S) = \langle t_2 \rangle$, since $\tau(f_2^{-1}f_1) = g_2g_1^{-1}g_2g_1 = t_2$.

Thus we obtain that $\hat{D}_\infty = D_\infty^*/\langle t_2 \rangle$. A consistent polycyclic presentation for this group can be described by $\langle g_1, g_2, t_1 \mid g_1^{-1} = t_1, g_1^{-1}g_2g_1 = g_2^{-1}, (t_1 \text{central}) \rangle$. 

5
3 Schur multiplicator, Schur cover and nonabelian exterior square

Suppose that the polycyclic group \( G \) is given by a consistent polycyclic presentation \( F_n/R \). In this section we show that Section 2 applies readily to the determination of consistent polycyclic presentations for the Schur multiplicator of \( G \), a Schur cover of \( G \) (as defined in [4], page 182) and the nonabelian exterior square of \( G \).

We continue to use the notation of Section 2; in particular, let \( G = F_n = \langle R; F_n \rangle \) and \( T = R/[R,F_n] \). Note that a consistent polycyclic presentation of \( G \) that exhibits \( T \) can be computed readily using the methods of Section 2.

We also apply various standard methods for groups defined by consistent polycyclic presentations; we refer to [8] for background. In particular, we exploit the fact that we can compute with subgroups and quotient groups defined by generators and we can determine generators for the derived subgroup and intersections of subgroups.

3.1 The Schur multiplicator and a Schur cover

Hopf’s formula \( M(G) \cong (F_0 \cap R)/[R,F_n] \) translates directly to the following characterisation of the Schur multiplicator of \( G \), which, in turn, yields that a consistent polycyclic presentation of \( M(G) \) can be computed from \( G \).

5 Remark: \( M(G) \cong (G') \cap T \).

Next we consider the Schur covers of \( G \) in the following straightforward and well-known observation.

6 Remark: Every complement \( C \) to \( (G') \cap T \) in \( T \) yields a Schur cover \( G^* / C \) of \( G \).

Note that \( G^*/(G^*)' \cong F_n/F'_n \) and \( T/((G^*)' \cap T) \cong T(G^*)'/(G^*)' \) are free abelian. Hence \( T \) splits over \( (G^*)' \cap T \). If \( \epsilon : T \to T/((G^*)' \cap T) \) is the natural homomorphism and \( \epsilon(a_1), \ldots, \epsilon(a_r) \) is a minimal generating set of \( T/((G^*)' \cap T) \), then \( a_1, \ldots, a_r \) generates a complement \( C \) to \( (G^*)' \cap T \) in \( T \). Hence we can determine generators for a suitable subgroup \( C \) of \( G^* \) and a consistent polycyclic presentation for its quotient \( G^* / C \). This yields a method for determining a consistent polycyclic presentation of a Schur cover of the group \( G \).

7 Example: (Continuation of Example 2)

We obtain that \( (D_{\infty})' = \langle g_2^3 t_2^{-1} \rangle \), which intersects trivially with \( T = \langle t_1, t_2 \rangle \). Hence it follows that \( M(D_{\infty}) = 1 \) and \( D_{\infty} \) is the unique Schur cover of itself.

3.2 The nonabelian exterior square

Let \( \epsilon : G^* \to G \) be the natural epimorphism and for every \( g \in G \) let \( \overline{g} \) be an arbitrary preimage of \( g \) under this epimorphism. Note that every polycyclic group \( G \) has a finitely generated Schur multiplicator, so that the following theorem applies.

8 Theorem: ([4], Corollary 2) Let \( G \) be a group with finitely generated Schur multiplicator. Then \( \beta : \langle G \land G \rangle \to (G^*)' : (g \land h) \mapsto [\overline{g}, \overline{h}] \) is an isomorphism.
A consistent polycyclic presentation for \((G^*)'\) can be determined from such a presentation for \(G^*\) using standard methods for polycyclic presentations. By Theorem 8, this also yields a consistent polycyclic presentation for \(G \wedge G\). For later application we add the following remark.

9 Remark:

a) \(G\) acts naturally on \(G \wedge G\) via \(^k(g \wedge h) = {}^k g \wedge {}^kh\) for \(g, h, k \in G\). This action is compatible with \(\beta\) and \(^k(g \wedge h)\) corresponds to \(^k[\overline{g}, \overline{h}] = [k\overline{g}, k\overline{h}]\). The image \(\overline{w}\) of \(w \in G \wedge G\) is obtained by writing \(w\) as a product of commutators and then computing the action of \(g\) on each factor.

b) By construction, the map \(\lambda : G \times G \to (G^*)' : (g, h) \mapsto [\overline{g}, \overline{h}]\) is a crossed pairing (see [4]); that is, it satisfies for all \(g, g', h, h' \in G\) the two equations

\[
\lambda(gg', h) = \lambda(g', g)\lambda(g, h) \\
\lambda(g, hh') = \lambda(g, h)\lambda(gh, h')
\]

Applying \(\beta\), it corresponds to the crossed pairing \(\lambda : G \times G \to G \wedge G : (g, h) \mapsto g \wedge h\).

Note that it is straightforward to determine images of the action of \(G\) and images of the crossed pairing \(\lambda\) in the polycyclic presentation of \(G \wedge G\) using Remark 9.

10 Example: (Continuation of Example 2)

We determine \(D_\infty \wedge D_\infty\) by identifying it with \((D_\infty^*)' = \langle w \rangle \cong \mathbb{Z}\) for \(w = g_2^2t_2^{-1}\).

a) We show how to evaluate the image of the natural action \(g_1 \wedge g_2\) in the consistent polycyclic presentation of \(D_\infty \wedge D_\infty\). By Remark 9, this element corresponds to \(g_1[g_1, g_2]\) in \((D_\infty^*)'\). In turn, using the relations of \(D_\infty^*\), this element evaluates to \(g_1^2g_2g_1^{-1}g_2^{-1}g_1^{-1} = g_2^2t_2^{-1} = w\), which is the desired image.

b) We show how to evaluate the image \(\lambda(g_1, g_2)\) in the consistent polycyclic presentation of \(D_\infty \wedge D_\infty\). By Remark 9, this element corresponds to \([g_1, g_2]\) in \((D_\infty^*)'\) and thus evaluates to \(g_2^{-2}t_2 = w^{-1}\), which is the desired image.

4 The epicenter and capability

A group \(G\) is called capable if there exists a group \(H\) with \(H/Z(H) \cong G\). We define the epicenter \(Z^*(G)\) of \(G\) as the smallest subgroup of \(G\) so that \(G/Z^*(G)\) is capable, see Corollary 2.2 of [1]. Hence a group \(G\) is capable if and only if \(Z^*(G)\) is trivial.

The following theorem shows how the epicenter of a polycyclic group \(G\) can be determined from \(G^*\). This theorem can also be obtained as a special case of [2], Corollary 3.7 (b), page 208. We include a proof here for completeness.

11 Theorem: Let \(G\) be a polycyclic group and let \(\epsilon : G^* \to G\) be the natural epimorphism. Then \(Z^*(G) = \mathrm{c}(\epsilon(Z(G^*))).\)

Proof: For every \(g \in G\) let \(\overline{g}\) be an arbitrary preimage in \(G^*\) under \(\epsilon\). Then \([\overline{x}, \overline{y}] = 1\) for all \(a \in G\) if and only if \([x, y] = 1\) for all \(x \in G^*\), as \(G^*\) is a central extension of \(G\). By
Theorem 8, the map $\beta : G \times G \to (G^*)' : (g \times h) \mapsto \langle \overline{g}, \overline{h} \rangle$ is an isomorphism. This yields

\[
Z^*(G) = \{ g \in G \mid (a \times g) = 1 \text{ for all } a \in G \} \quad \text{(see [10])}
\]
\[
= \{ g \in G \mid \overline{a}, \overline{g} = 1 \text{ for all } a \in G \} \quad \text{(use } \beta) \\
= \{ g \in G \mid [x, \overline{g}] = 1 \text{ for all } x \in G^* \}
\]
\[
= \{ g \in G \mid \overline{g} \in Z(G^*) \}
\]
\[
= \epsilon(Z(G^*)),
\]

which completes the proof.

**Theorem 11** yields that the epicenter of a group $G$ given by a consistent polycyclic presentation can be computed readily: we first determine a consistent polycyclic presentation for $G^*$ and its corresponding natural epimorphism $\epsilon$ using the method of Section 2, then we compute the center $Z(G^*)$ using standard methods for polycyclically presented groups, and finally we apply $\epsilon$ to obtain $Z^*(G) = \epsilon(Z(G^*))$.

**12 Example:** (Continuation of Example 2)

It is straightforward to see from the consistent polycyclic presentation of $D_1$ that $Z^*(D_1) = \langle t_1, t_2 \rangle$. Hence $Z^*(D_\infty) = 1$ and thus $D_\infty$ is capable.

## 5 The nonabelian tensor square

Suppose that the polycyclic group $G$ is given by a consistent polycyclic presentation $F_n/R$. In this section we describe an effective method to compute a consistent polycyclic presentation for its nonabelian tensor square $G \otimes G$.

We use that $G \otimes G$ embeds into a finitely presented group $\nu(G)$; this group has already been considered in various other publications, see [19, 11, 14, 15, 3]. In this section we first recall the definition of $\nu(G)$ and some basic results for this group. The major part of this section is devoted to the description of an algorithm for computing a consistent polycyclic presentation for $\nu(G)$. We can then determine a consistent polycyclic presentation of $G \otimes G$ as subgroup of $\nu(G)$.

### 5.1 The group $\nu(G)$

Suppose that $G$ is given by a consistent polycyclic presentation on the generators $g_1, \ldots, g_n$ with relators $r_1, \ldots, r_l$. Let $\overline{g}_1, \ldots, \overline{g}_n$ be a new set of abstract generators and define $\nu(G)$ as the group generated by $g_1, \ldots, g_n, \overline{g}_1, \ldots, \overline{g}_n$ subject to the defining relations

\[
(1) \quad r_i(g_1, \ldots, g_n) = 1 \text{ for } 1 \leq i \leq l, \\
(2) \quad r_i(\overline{g}_1, \ldots, \overline{g}_n) = 1 \text{ for } 1 \leq i \leq l, \\
(3) \quad g_k [g_i, \overline{g}_j] = [g_k, g_i, \overline{g}_j] \text{ for } 1 \leq i, j, k \leq n, \\
(4) \quad \overline{g}_k [g_i, \overline{g}_j] = [g_k, g_i, \overline{g}_j] \text{ for } 1 \leq i, j, k \leq n.
\]

This definition of $\nu(G)$ follows [15], Theorem 5 (which, in turn, follows [14]). Note that $\nu(G)$ as defined here coincides with $\nu(G)$ as defined in [15, 19] and with $G \ast G/J$ of [11].
We recall some of the known features of $\nu(G)$ that are important for our applications. Let $\overline{G}$ be the isomorphic copy of $G$ on the generators $\overline{g}_1, \ldots, \overline{g}_n$ and for every $h \in G$ let $\overline{h}$ be the corresponding element in $\overline{G}$. Define

$$\gamma : G \otimes G \to \nu(G) : (g \otimes h) \mapsto [g, \overline{h}].$$

**13 Theorem:** ([11]) The map $\gamma$ is a monomorphism. Its image $[G, \overline{G}]$ is normal in $\nu(G)$ with quotient $\nu(G)/[G, \overline{G}] \cong G \times G$.

The monomorphism $\gamma$ maps $\nabla(G) = \langle (g \otimes g) \mid g \in G \rangle$ to the group

$$\kappa(G) = \langle [g, \overline{g}] \mid g \in G \rangle \leq [G, \overline{G}] \leq \nu(G).$$

This group $\kappa(G)$ is denoted as $\Delta(G)$ in [19], where it is proved that $\kappa(G)$ is a central subgroup of $\nu(G)$. Using the definition of $\nabla(G)$ and $\gamma$, it is easy to see that the group $\kappa(G)$ is the kernel of the natural epimorphism

$$\delta : [G, \overline{G}] \to G \wedge G : [g, \overline{h}] \mapsto g \wedge h.$$

### 5.2 A consistent polycyclic presentation for $\nu(G)/\kappa(G)$

Let $F_n/R$ be the given consistent polycyclic presentation for $G$ on the generators $g_1, \ldots, g_n$ and the relators $r_1, \ldots, r_l$ with index set $I$. Using the ideas of Section 3, we can determine a consistent polycyclic presentation $F_n/U$ for $G \wedge G$ on generators $w_1, \ldots, w_r$ and relators $u_1, \ldots, u_s$, say. Our next aim is to derive a consistent polycyclic presentation for $\nu(G)/\kappa(G)$ from this presentation of $G \wedge G$.

Recall that we can determine images of $\lambda : G \times G \to G \wedge G : (g, h) \mapsto g \wedge h$ in the constructed consistent polycyclic presentation of $G \wedge G$ as outlined in Remark 9. Similarly, we can construct the natural action of $G$ on the determined consistent polycyclic presentation of $G \wedge G$ as defined by $k(g \wedge h) = k(g) \wedge k(h)$ using Remark 9.

We define $\tau(G)$ as the group generated by $g_1, \ldots, g_n, \overline{g}_1, \ldots, \overline{g}_n, w_1, \ldots, w_r$ subject to the defining relations

1. $r_i(g_1, \ldots, g_n) = 1$ for $1 \leq i \leq l$,
2. $r_i(\overline{g}_1, \ldots, \overline{g}_n) = 1$ for $1 \leq i \leq l$,
3. $u_i(w_1, \ldots, w_r) = 1$ for $1 \leq i \leq s$,
4. $g_i^{-1}\overline{g}_j g_i = \overline{g}_j \lambda(g_i^{-1}, g_j^{-1})$ for $1 \leq i, j \leq n, g_i \overline{g}_j g_i^{-1} = \overline{g}_j \lambda(g_j^{-1}, g_i)$ for $1 \leq i, j \leq n, i \notin I$,
5. $g_j^{-1} w_i g_j = g_j^{-1} w_i$ for $1 \leq i \leq r, 1 \leq j \leq n, g_j w_i g_j^{-1} = g_j w_i$ for $1 \leq i \leq r, 1 \leq j \leq n, j \notin I,
\overline{g}_j^{-1} \overline{w}_i \overline{g}_j = g_j^{-1} \overline{w}_i$ for $1 \leq i \leq r, 1 \leq j \leq n,
\overline{g}_j w_i \overline{g}_j^{-1} = g_j w_i$ for $1 \leq i \leq r, 1 \leq j \leq n, j \notin I$.

Note that we can compute the right hand sides of the relations (4) and (5) as words in $w_1, \ldots, w_r$ by Remark 9.
Let $\overline{G}$ denote the isomorphic copy of $G$ on the generators $\overline{g}_1, \ldots, \overline{g}_n$ in $\tau(G)$ and for every $h \in G$ let $\overline{h}$ be the corresponding element in $\overline{G}$. Corresponding to the subgroup $[G,\overline{G}]$ of $\nu(G)$, we write $[G,\overline{G}] = \langle [g,\overline{h}] \mid g, h \in G \rangle$ as a subgroup of $\tau(G)$.

14 Theorem: Let $W = \langle w_1, \ldots, w_r \rangle \leq \tau(G)$.

(a) $W$ is normal in $\tau(G)$ with $\tau(G)/W \cong G \times G$.

(b) The presentation of $\tau(G)$ is a consistent polycyclic presentation.

(c) $W = [G,\overline{G}]$ and $W \cong G \cap G$.

(d) $\varphi : \nu(G) \to \tau(G)$ defined by $\varphi(g_i) = g_i$ and $\varphi(\overline{g}_i) = \overline{g}_i$ for $1 \leq i \leq n$ extends to a well-defined epimorphism with kernel $\kappa(G)$.

Proof: As a preliminary step of the proof, we discuss the relation of $\tau(G)$ in more detail. The relations in (5) ensure that $\tau(G)/W \cong G \times G$. The relations in (1),(2) and (4) yield that $\tau(G)/W \cong G \times \overline{G}$ with $\overline{G} \cong G$. Further, the relations in (3) show that $W$ is a factor of $G \cap G$. Hence $\tau(G)$ satisfies the exact sequence

$$G \cap G \to \tau(G) \to G \times \overline{G} \to 1.$$ 

Next, the relations in (5) imply that $G \times \overline{G}$ acts by conjugation on $W$ as $G \times G$ acts naturally on $G \cap G$. In particular, we obtain that $[w, g] = w(g(w^{-1}))$ and similarly $[w, \overline{h}] = w(h(w^{-1}))$ for every word $w$ in $w_1, \ldots, w_r$, every word $g$ in $g_1, \ldots, g_n$ and every word $\overline{h}$ in $\overline{g}_1, \ldots, \overline{g}_n$. Moreover, the definition of a crossed pairing and the relations in (4) ensure that $[g, \overline{h}] = \lambda(g, h)$ for every word $g$ in $g_1, \ldots, g_n$ and every word $\overline{h}$ in $\overline{g}_1, \ldots, \overline{g}_n$.

Proof of (a). This follows directly from the preliminary step of the proof.

Proof of (b). The relations in (1)-(5) have the form of a polycyclic presentation. Thus it remains to check the consistency of this presentation. Every consistency relation in the generators $g_1, \ldots, g_n$ is satisfied, since the relations in (1) arise from a consistent polycyclic presentation of $G$. Similarly, the relations in (2) and (3) yield that every consistency relation in the generators $\overline{g}_1, \ldots, \overline{g}_n$ and in the generators $w_1, \ldots, w_r$ is satisfied. Further, if a consistency relation involves a generator from $w_1, \ldots, w_r$, then it is satisfied, since $G \times \overline{G}$ acts on $W$ as $G \times G$ acts naturally on $G \cap G$. It remains to consider the consistency relations in $g_1, \ldots, g_n, \overline{g}_1, \ldots, \overline{g}_n$ involving generators $g_i$ and $\overline{g}_j$. These are:

$$\overline{g}_k(g_j g_i) = (\overline{g}_k g_j) g_i \quad \text{for} \quad j > i,$$

$$\overline{g}_k(\overline{g}_j g_i) = (\overline{g}_k \overline{g}_j) g_i \quad \text{for} \quad k > j,$$

$$(\overline{g}_j^{-1}) g_i = \overline{g}_j^{-1} \overline{g}_j g_i \quad \text{for} \quad j \in I,$$

$$\overline{g}_j (g_i^{-1}) = (\overline{g}_j g_i) g_i^{-1} \quad \text{for} \quad i \in I,$$

$$\overline{g}_j = (\overline{g}_j g_i^{-1}) g_i \quad \text{for} \quad i \not\in I.$$ 

In the following we consider the first of these consistency relations as a sample. We suppose that $g_i^{-1} g_j g_i = r_{ij}(g_1, \ldots, g_n) = r_{ij}$ in the relations of $G$ and we use that $\lambda$ is a crossed pairing.

$$\overline{g}_k(g_j g_i) = (g_j g_i) \overline{g}_k^{g_i} = (g_j r_{ij})(\overline{g}_k \lambda(g_k^{-1}, (g_j g_i)^{-1}))$$

$$= (g_j r_{ij})(\overline{g}_k \lambda(g_k^{-1}, (g_j g_i)^{-1}))$$

10
\[
(g_k g_j) g_i = (g_j g_k \lambda (g_k^{-1}, g_j^{-1})) g_i \\
= g_j g_k g_i \lambda (g_k^{-1}, g_j^{-1}) g_i \\
= g_j g_k g_i \lambda (g_k^{-1}, g_i^{-1}) \lambda (g_k^{-1}, g_j^{-1}) g_i \\
= g_i r_i g_k \lambda (g_k^{-1}, g_i^{-1}) \lambda (g_k^{-1}, g_j^{-1}) g_i \\
= g_i r_i g_k \lambda (g_k^{-1}, g_i^{-1} g_j^{-1})
\]

It can be checked by similar computations that all other consistency relations are also satisfied. Hence, in summary, we obtain that \( \tau(G) \) is given by a consistent polycyclic presentation.

**Proof of (c).** The relations (4) and the preliminary step of this proof imply that \( W = [G, G] \). Further, it follows from (b) and the theory of polycyclic presentations (see [13], Section 8.3) that the subgroup \( W \) has a consistent polycyclic presentation on the generators \( w_1, \ldots, w_r \) and the relations \( u_1, \ldots, u_s \). By our setup, this yields that \( W \cong G \cap G \).

**Proof of (d).** First, we observe that the relations of \( \nu(G) \) hold in \( \tau(G) \). The relations (1) and (2) of \( \nu(G) \) coincide with the relations (1) and (2) of \( \tau(G) \) and hence are satisfied. The relations (3) of \( \nu(G) \) can be rewritten in \( \tau(G) \) as

\[
g_k [g_i, g_j] = k \lambda (g_i, g_j) = \lambda (g_k g_i, g_k g_j) = [g_k, g_i, g_j]
\]

and hence they are satisfied in \( \tau(G) \). Similarly, the relations (4) of \( \nu(G) \) are satisfied in \( \tau(G) \). Hence \( \varphi \) is a well-defined homomorphism. Further, the homomorphism \( \varphi \) is surjective, as \( \tau(G) \) is generated by \( g_1, \ldots, g_n, \overline{f}_1, \ldots, \overline{f}_n \) by relations (4) of \( \tau(G) \) and the definition of \( \lambda \). Moreover, it follows that \( \varphi([g, h]) = [g, h] = \lambda(g, h) \) for every word \( g \) in \( g_1, \ldots, g_n \) and every word \( \overline{f} \) in \( \overline{f}_1, \ldots, \overline{f}_n \). Thus the map induced by \( \varphi \) on the subgroup \( [G, G] \) coincides with \( \delta : [G, G] \to G \cap G : [g, h] \mapsto g \cap h \) by construction. The situation is summarized by the following commutative diagram.

\[
\begin{array}{ccc}
1 & \rightarrow & [G, G] \\
\downarrow \delta & & \downarrow \varphi \\
1 & \rightarrow & \tau(G) \\
\end{array}
\]

It follows that \( \ker(\varphi) \leq [G, G] \) and thus \( \ker(\varphi) = \ker(\delta) = \kappa(G) \). In summary, we obtain that \( \varphi \) is an epimorphism with kernel \( \kappa(G) \) as desired.

**15 Example:** (Continuation of Example 2 and 10)

The group \( \tau(D_\infty) \) has the generators \( g_1, g_2, \overline{f}_1, \overline{f}_2, w \) and the following relations:

1. \( g_1^2 = 1, g_1^{-1} g_2 g_1 = g_2^{-1} \),
2. \( \overline{f}_1^2 = 1, \overline{f}_1^{-1} \overline{f}_2 \overline{f}_1 = \overline{f}_2^{-1} \),
3. no relation,
4. \( g_1^{-1} \overline{f}_1 g_1 = \overline{f}_1, \overline{f}_2 \overline{f}_1 g_2 = \overline{f}_1 w \),
\[ g_1^{-1}g_2g_1 = g_2w^{-1}, \]
\[ g_2^{-1}g_2g_2 = g_2, \]
\[ g_2g_1g_2^{-1} = g_1w^{-1}, \]
\[ g_2g_2g_2^{-1} = g_2, \]

(5) \[ g_1^{-1}wg_1 = w^{-1}, \]
\[ g_2^{-1}wg_2 = w, \]
\[ g_2wg_2^{-1} = w, \]
\[ g_2^{-1}wg_1 = w^{-1}, \]
\[ g_2^{-1}wg_2 = w, \]
\[ g_2wg_2^{-1} = w. \]

This list of relations contains various redundant relations, but forms a consistent polycyclic presentation for \( \tau(G) \).

### 5.3 A consistent polycyclic presentation for \( \nu(G) \)

Now we are in a situation where the methods of Section 2, in particular, Section 2.2, can be applied. We use the consistent polycyclic presentation of \( \tau(G) \) in place of \( F_n/R \) and the finite presentation of \( \nu(G) \) in place of \( F_m/S \). Note that the epimorphism \( \varphi : \nu(G) \rightarrow \tau(G) \) has the form required in Section 2. We obtain the following.

16 Theorem: \( \nu(G) \cong \tau(G) \).

**Proof:** Recall that \( \tau(G) \cong F_m/[K,F_m]S \) by definition, where \( K/S \) is the kernel of \( \varphi \). By Theorem 14, the group \( \nu(G) = F_m/S \) is a central extension of \( \tau(G) = F_m/K \). Thus \( [K,F_m] \leq S \) follows and \( \tau(G) = F_m/[K,F_m]S = F_m/S = \nu(G) \) as desired.

Hence computing a consistent polycyclic presentation of \( \nu(G) \) is a straightforward application of the algorithm in Section 2.2. In summary, this algorithm takes the given consistent polycyclic presentation for \( \tau(G) \), extends it by adding the a new (central) generator \( t_i \) for every relator \( r_i \) of \( \tau(G) \) and modifies every relator \( r_i \) of \( \tau(G) \) to \( r_it_i \). It then evaluates the consistency relations and the relators of \( \nu(G) \) in this new presentation and applies Lemma 2.2 (b).

The following remark can be used to reduce the number of new generators added and the number of relators evaluated in this process.

17 Lemma: It is redundant to add new generators corresponding to the relations in (1) and (2) of the definition of \( \tau(G) \). If these generators are not introduced, then it is redundant to evaluate the relators (1) and (2) in the definition of \( \nu(G) \).

**Proof:** The relators in (1) and (2) of the definition of \( \nu(G) \) coincide with the relators in (1) and (2) of the definition of \( \tau(G) \). Thus if we add new generators to the relators in (1) and (2) of the definition of \( \tau(G) \) and then evaluate the relators in (1) and (2) of the definition of \( \nu(G) \), then we obtain the corresponding generators itself as result. This, in turn, yields that the corresponding generators are eliminated in the process of building the quotient described in Lemma 2.2 (b). Thus the result follows.
18 Example: (Continuation of Example 2, 10 and 15)
We sketch the determination of $\nu(G)$ as the central extension $\tau(G)$ of $\tau(G)$. This is a fairly elaborate computation and we omit details here. The result is a consistent polycyclic presentation for $\nu(G)$ on generators $g_1, g_2, \overline{g}_1, \overline{g}_2, w, t_1, t_2, t_3$ so that the subgroup $T = \langle t_1, t_2, t_3 \rangle$ is elementary abelian of order 8 and the extension of $\tau(G)$ by $T$ is described by the following relations:

\[
\begin{align*}
g_1^2 &= 1, \quad g_2^n = g_2^{-1}, \\
\overline{g}_1^2 &= 1, \quad \overline{g}_2^n = \overline{g}_2^{-1}, \\
g_1^{-1} g_1 g_1 &= g_1 t_1, \\
g_2^{-1} g_1 g_2 &= g_1 w, \\
g_1^{-1} g_2 g_1 &= \overline{g}_2 w^{-1} t_2, \\
g_2^{-1} g_2 g_2 &= \overline{g}_3 t_3, \\
g_2 g_1 g_2^{-1} &= \overline{g}_1 w^{-1}, \\
g_2 g_3 g_2^{-1} &= \overline{g}_3 t_3, \\
g_1^{-1} w g_1 &= w^{-1}, \\
g_2^{-1} w g_2 &= w, \\
g_2 w g_2^{-1} &= w, \\
\overline{g}_1^{-1} w \overline{g}_1 &= w^{-1}, \\
\overline{g}_2^{-1} w \overline{g}_2 &= w, \\
\overline{g}_2 w \overline{g}_2^{-1} &= w.
\end{align*}
\]

This yields that $G \otimes G \cong \langle w, t_1, t_2, t_3 \rangle \leq \nu(G)$ and thus $G \otimes G \cong \mathbb{Z} \times C_2^3$. Further, we obtain that $\nabla(G) \cong T \cong C_2^3$ and $J(G) \cong T \cong C_2^3$ by computing kernels of the natural homomorphisms.

5.4 A summary of the algorithm
We summarise our method to compute a consistent polycyclic presentation for $G \otimes G$ from a consistent polycyclic presentation of $G$ as follows.

(a) Determine a consistent polycyclic presentation for $G \wedge G$.
(b) Determine a consistent polycyclic presentation for $\tau(G)$.
(c) Determine a consistent polycyclic presentation for $\nu(G)$.
(d) Determine an induced polycyclic presentation for the subgroup $[G, \overline{G}]$ of $\nu(G)$.

Step (a) is achieved by an application of the central extension method of Section 2. If $G = F_n / R$ is a consistent polycyclic presentation of $G$, then we compute a consistent polycyclic presentation for $G^* = F_n / [R, F_n]$ and obtain $G \wedge G$ by identifying it with $(G^*)'$. Step (b) is then a direct application of the method developed in Section 5.2. Step (c) is achieved by another application of the central extension method of Section 2 to compute $\tau(G)$, which by Theorem 16 is isomorphic to $\nu(G)$. Step (d) is a standard application of methods for polycyclically presented groups.
5.5 The commutative diagram

As a final comment in this section, we summarise how we can now determine consistent polycyclic presentations for all groups in the commutative diagram \( \Gamma(G/G') \) together with the corresponding homomorphisms in this diagram. As above, let \( F_n/R \) be a consistent polycyclic presentation for \( G \).

Then \( G \wedge G \) is identified with \( F_n/[R, F_n] = (G^\ast)' \). This allows one to define the homomorphism \( \gamma : G \wedge G \to G' : a[R, F_n] \mapsto aR \) whose kernel is the Schur multiplicator \( M(G) \) of \( G \). Note that this yields a second method to compute \( M(G) \). However, this second method is usually not more effective than the method of Section 3.1.

Further, the group \( \nu(G) \) is determined as a central extension of \( \tau(G) \) and thus we also obtain the corresponding natural epimorphism \( \alpha : G \otimes G \to G \wedge G \). Further, this allows one to compute \( \nabla(G) \) as the kernel of \( \alpha \) (or of \( \pi \)).

Finally, we obtain the natural homomorphism \( \alpha \circ \gamma : G \otimes G \to G' \). This allows one to determine \( J(G) \) as the kernel of this homomorphism. By construction, we can also read off the natural epimorphism \( J(G) \to M(G) \).

6 Implementation and sample runtimes

All algorithms outlined in this paper are implemented as part of the Polycyclic package [9] of GAP [21]. In this section we report on some sample applications of our implementations. We mainly concentrate on computations of the nonabelian tensor square and the nonabelian exterior square. We also include reports on the computation of \( \nu(G) \) and \( \tau(G) \), as these groups form the main steps towards computing \( G \otimes G \).

The following considerations help us understand the performance of our algorithms. Let \( n \) be the number of generators of the group \( G \) given by a polycyclic presentation. Then \( G \) has \( \frac{n(n-1)}{2} \) conjugate relations of the type \( g_j^{-1}g_i g_j \). In computing the central extension \( G^\ast \) of \( G \) a new generator is introduced for each such conjugate relation. The consistency test yields an integer matrix with \( \frac{n(n-1)}{2} \) columns and \( O(n^3) \) rows (cf. Section 2.1). Even for small values of \( n \), this can produce integer matrices, for which computing the Smith Normal Form together with the matrix \( Q \) of column transformations is computationally challenging. As the entries of \( Q \) appear as exponents in the polycyclic presentation for \( G^\ast \) and hence influence the performance of subsequent computations with \( G^\ast \), it would be useful to have strategies for keeping the entries in \( Q \) small.

The number of generators of the group \( \tau(G) \) is \( 2n \) plus the number of generators for \( G \wedge G \), which we assume to be roughly \( n \) for the purpose of this analysis. Therefore, computing \( \tau(G)^\ast \) introduces roughly \( 9n^2/2 \) new generators and the consistency test produces an integer matrix with roughly the same number of columns.

If \( G \) is nilpotent, one could use strategies to reduce the number of new generators significantly as is done, for example, in the nilpotent algorithm [17]. Our current implementation for the central extension method does not use any such strategy. Hence it can be more effective in this case to apply the nilpotent quotient algorithm as implemented in the NQ package [18] of GAP to the defining presentation of \( \nu(G) \). This approach has been used in [15] and [3]. A similar strategy has also been used in [11]. In the following, we mainly
concentrate on examples of polycyclic, but not necessarily nilpotent groups.
The following table gives an overview of the GAP-commands and the groups and homomorphisms they compute.

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau(G)$</td>
<td>NonAbelianExteriorSquarePlus</td>
</tr>
<tr>
<td>$\nu(G)$</td>
<td>NonAbelianTensorSquarePlus</td>
</tr>
<tr>
<td>$G \rtimes G$</td>
<td>NonAbelianExteriorSquare</td>
</tr>
<tr>
<td>$G \otimes G$</td>
<td>NonAbelianTensorSquare</td>
</tr>
<tr>
<td>$M(G)$</td>
<td>SchurMultiplier</td>
</tr>
<tr>
<td>$G \rtimes G \rightarrow G$</td>
<td>NonAbelianExteriorSquareEpimorphism</td>
</tr>
<tr>
<td>$G \otimes G \rightarrow G \rtimes G$</td>
<td>NonAbelianTensorSquareEpimorphism</td>
</tr>
</tbody>
</table>

The Polycyclic package manual contains more details and the description of functions that compute the homomorphisms of the commutative diagram ($\ast$). With these, the groups in the commutative diagram ($\ast$) can be determined explicitly.

All runtimes listed below have been obtained on a computer with an Intel Pentium M 1500 MHz processor.

### 6.1 Very small groups

In [4] the nonabelian tensor squares of the nonabelian groups of order at most 30 are determined. We can now redo this computation in 6 seconds. Our results agree with the results of [4]. The computation of the nonabelian tensor squares of all nonabelian solvable groups of order at most 100 takes 7 minutes with our implementation. The groups of order at most 100 are available in GAP as part of the SmallGroups library.

### 6.2 Larger finite solvable groups

In this section we consider two finite solvable groups and determine runtimes for our algorithms with respect to these groups. Consistent polycyclic presentations in GAP-readable form for these groups are available in the Format package of GAP (see the files grp/FI22.gi and grp/UPP.gi).

Let $G$ be the solvable maximal subgroup of $Fi_{22}$ of order $2^8 \cdot 3^9$ and derived length 6. Then $|G'| = 2^7 \cdot 3^9$ and

- $|G \rtimes G| = 2^{31} \cdot 3^{11}$ (computed in 3 sec.)
- $|\tau(G)| = 2^{24} \cdot 3^{29}$ (computed in 9 sec.)
- $|\nu(G)| = 2^{25} \cdot 3^{29}$ (computed in 11 min.)
- $|G \otimes G| = 2^9 \cdot 3^{14}$ (computed in 11 min.)

Let $G = U(4, 7) \rtimes C_3$, where $U(4, 7)$ denotes the group of upper triangular matrices in $GL(4, 7)$. Then $G$ is of order $2^{12} \cdot 3^{13} \cdot 7^{18}$, derived length 4 and $|G'| = 2^8 \cdot 3^8 \cdot 7^{18}$. Further,

- $|G \rtimes G| = 2^{30} \cdot 3^{30} \cdot 7^{18}$ (computed in 2 min.)
- $|\tau(G)| = 2^{54} \cdot 3^{36} \cdot 7^{54}$ (computed in 6 min.)
- $\nu(G)$ and $G \otimes G$ could not be computed.

These examples demonstrate that the main bottleneck of our algorithm is the computation of $\nu(G)$ as a central extension of $\tau(G)$.
6.3 Polycyclic space groups

For a prime $p$, consider the natural irreducible action of the cyclic group $C_p$ on $\mathbb{Z}^{p-1}$ and define $G_p = \mathbb{Z}^{p-1} \rtimes C_p$. Then $G_p$ is a polycyclic group of Hirsch length $p - 1$, which has interesting arithmetical properties. Consistent polycyclic presentations for these groups can be constructed readily using the facilities of the Polycyclic package of GAP. Note that the action of the group $C_p$ on $\mathbb{Z}^{p-1}$ can be obtained as companion matrix of the $p$-th cyclotomic polynomial over $\mathbb{Q}$. The groups can be constructed from that using the function SplitExtensionPcpGroup.

In the following tables we list the abelian invariants of the factors of the derived series for the nonabelian tensor squares, the nonabelian exterior squares and the Schur multiplicators of these groups. In the last column we add the runtime used to compute the nonabelian tensor square.

We write the obtained abelian invariants $(x, x, \ldots, y, y, \ldots)$ in collected form $(x^a, y^b, \ldots)$, where the invariant $x$ occurs $a$ times, the invariant $y$ occurs $b$ times, etc.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$G \otimes G$</th>
<th>$G \wedge G$</th>
<th>$M(G)$</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_2$</td>
<td>$(2^4, 0)$</td>
<td>$(0)$</td>
<td>$(0)$</td>
<td>0.01 sec.</td>
</tr>
<tr>
<td>$G_3$</td>
<td>$(3^4, 0^2), (0)$</td>
<td>$(3, 0^2), (0)$</td>
<td>$(0)$</td>
<td>0.08 sec.</td>
</tr>
<tr>
<td>$G_5$</td>
<td>$(5^4, 0^4), (0^2)$</td>
<td>$(5, 0^4), (0^2)$</td>
<td>$(0^2)$</td>
<td>0.8 sec.</td>
</tr>
<tr>
<td>$G_7$</td>
<td>$(7^4, 0^6), (0^3)$</td>
<td>$(7, 0^6), (0^3)$</td>
<td>$(0^2)$</td>
<td>5 sec.</td>
</tr>
</tbody>
</table>

The dihedral groups of order 6, 8 and 12 and the infinite dihedral group have 2-dimensional faithful integral representations:

$$D_6 \cong \langle \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ -1 & -1 \end{array} \right) \rangle \quad D_8 \cong \langle \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \rangle$$

$$D_{12} \cong \langle \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array} \right) \rangle \quad D_{\infty} \cong \langle \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) \rangle$$

For each of these dihedral groups $G$ let $\mathbb{Z}^2 \rtimes G$ be the split extension of $\mathbb{Z}^2$ by $G$, where $G$ acts via the above representation. For $D_8$ and $D_{\infty}$ there also exist non-split extensions, which we denote by $\mathbb{Z}^2.D_8$ and $\mathbb{Z}^2.D_\infty$, respectively. All of the resulting extensions are polycyclic space groups (or almost crystallographic in the case of $D_{\infty}$). Consistent polycyclic presentations for these groups can again be constructed using the facilities of the Polycyclic package of GAP. The split extensions can be constructed readily using the function SplitExtensionPcpGroup. For the non-split extensions we refer to the Polycyclic manual for the description of the construction of such groups.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$G \otimes G$</th>
<th>$G \wedge G$</th>
<th>$M(G)$</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}^2 \rtimes D_6$</td>
<td>$(3^2, 6), (0^2), (2)$</td>
<td>$(3^2), (0^2), (2)$</td>
<td>$(2)$</td>
<td>0.73 sec.</td>
</tr>
<tr>
<td>$\mathbb{Z}^2 \rtimes D_8$</td>
<td>$(2^8, 4^2), (0^2)$</td>
<td>$(2^2, 4^2), (0^2)$</td>
<td>$(2^3)$</td>
<td>1.08 sec.</td>
</tr>
<tr>
<td>$\mathbb{Z}^2.D_8$</td>
<td>$(2^4, 4^2), (0^2)$</td>
<td>$(2^2, 4), (0^2)$</td>
<td>$(2)$</td>
<td>0.66 sec.</td>
</tr>
<tr>
<td>$\mathbb{Z}^2 \rtimes D_{12}$</td>
<td>$(2^2, 6^2), (0^2), (2)$</td>
<td>$(3, 6), (0^2), (2)$</td>
<td>$(2^3)$</td>
<td>0.64 sec.</td>
</tr>
<tr>
<td>$\mathbb{Z}^2 \rtimes D_{\infty}$</td>
<td>$(0^3, 2^8), (0)$</td>
<td>$(0^2, 2^4), (0)$</td>
<td>$(2^3, 0)$</td>
<td>0.44 sec.</td>
</tr>
<tr>
<td>$\mathbb{Z}^2.D_{\infty}$</td>
<td>$(0^3, 2^3), (0)$</td>
<td>$(0^2, 2), (0)$</td>
<td>$(0)$</td>
<td>0.37 sec.</td>
</tr>
</tbody>
</table>
References


