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Special Presentations for Finite Soluble Groups and Computing (Pre-)Frattini Subgroups

Bettina Eick

ABSTRACT. In the first part of this note a survey of the properties of special composition generating sequences of finite soluble groups (see [**1**]) is given. As application of special composition generating sequences we introduce an algorithm to compute a prefrattini subgroup of a finite soluble group. From that we derive an algorithm to compute the the Frattini subgroup of a finite soluble group. Additionally we investigate relationships between the computed prefrattini subgroup and the classes of subgroups displayed by the special composition generating sequences such as maximals subgroups and a Hall system together with its system normalizer. Finally timings of the GAP -implementation of the described algorithms are included.

This note is concerned with algorithms for finite soluble groups. A finite group is soluble if and only if its composition factors are of prime order. This characterisation has been used traditionally to compute with finite soluble groups.

However, a finite soluble group has other characterisations as well. For example, a finite group is soluble if and only if it contains a Hall system (see [**2**], pages 220 ff). And a finite group is soluble if and only if its chief factors are elementary abelian. Furthermore the complements to the complementable chief factors of a finite soluble group are just its maximal subgroups. Hall systems and maximal subgroups have been playing an important role in the theory of finite soluble groups. Accordingly a recent development in computation with finite soluble groups has been introduced by J. Cannon and C. R. Leedham-Green (see [**1**]). Its aim is not only to use the composition series structure, but also to allow one to read off from a sequence of generators as much as possible of a Hall system and of the maximal subgroups of this group. The resulting generating sequences will be called **special composition generating sequences** or **special CGSs** for short. A survey of their properties is given in Section 1.

The Hall systems of a finite soluble group determine the characteristic conjugacy class of system normalizers (see [**2**], pages 235 ff). Cannon and Leedham-Green suggest in [**1**] to allow one to read off a system normalizer from a special CGS additionally. Algorithms yielding such an CGS have been described in [**1**] and [**4**].

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The maximal subgroups of a finite soluble group determine also a characteristic conjugacy class of subgroups, the **prefrattini subgroups** as introduced by W. Gaschütz in [**5**], which are in many ways similar to the system normalizers. As an application of the special CGSs, we introduce an algorithm to compute a prefrattini subgroup of a finite soluble group in Subsections 2.1 and 2.2. Furthermore it will be observed that this algorithm might be used to modify a special CGS such that one could read off a prefrattini subgroup additionally (and even additionally to a system normalizer). In Subsection 2.3 the relationship between the introduced types of subgroups, i.e. a Hall system and its system normalizer, maximal subgroups and the computed prefrattini subgroup, will be investigated.

Then the computed prefrattini subgroup will be used to derive an algorithm to compute the Frattini subgroup of a finite soluble group in Subsection 2.4.

Finally in Section 3 we describe some examples of finite soluble groups and use them to obtain timings of the implementation of the described algorithms in the computer algebra system GAP (see [**10**]).

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1. Special Composition Generating Sequences

This section contains a survey of the properties of a special CGS. First we recall the definition of an (arbitrary) CGS.

1.1. Composition Generating Sequences.

Let G be a finite soluble group throughout the note, and let

$$G = C_1 \rhd C_2 \rhd \ldots \rhd C_n \rhd C_{n+1} = \{1\}$$

be a composition series of G. The prime indices for this series are denoted by $p_i := |C_i : C_{i+1}|$ for $1 \leq i \leq n$. For each index $i \in \{1, \ldots, n\}$ we choose an element $a_i \in C_i \setminus C_{i+1}$. The resulting sequence $\mathcal{A} :=(a_1, \ldots, a_n)$ is called a **composition generating sequence** or **CGS** of G. Then $C_i = \langle a_i, \ldots, a_n \rangle$ for $1 \leq i \leq n$. The series $C_1, \ldots, C_{n+1}$ is called the composition series **determined** by $\mathcal{A}$.

The group G has order $p_1 \cdots p_n$ and can be written as $G = \{a_1^{e_1} \cdots a_n^{e_n} \mid 0 \leq e_i < p_i\}$. The CGS $\mathcal{A}$ defines a finite presentation of G on generators corresponding to $a_1, \ldots, a_n$ which is called a CGS presentation or power-commutator presentation and is used to compute within G. For more detailed information see [**8**].

In most cases a finite soluble group has several different composition series, each determined by several different CGSs. The idea of the special CGSs is to make a good choice for both the composition series and the CGS determining it.

1.2. Preliminaries to special CGSs.

First we recall some well-known facts about maximal subgroups of finite nilpotent groups. Let N be a finite nilpotent group. Then N is the direct product of its Sylow subgroups, say $N = P_1 \times \ldots \times P_l$. The Frattini subgroup $\phi(N)$ of N, i.e., the intersection of the maximal subgroups of N, is given by $\phi(N) = \phi(P_1) \times \ldots \times \phi(P_l)$. The Frattini subgroup of the finite p_i-group P_i is determined by $\phi(P_i) = P_i' P_i^{p_i}$, and so $\phi(N)$ may be computed efficiently. Clearly, the maximal subgroups of N correspond one to one to the maximal subgroups of $N/\phi(N)$. Since $N/\phi(N)$ is a direct product of elementary abelian p_i-groups, its maximal subgroups may be determined using linear algebra.

This yields an efficient algorithm to compute the maximal subgroups of a finite nilpotent group. Part of the motivation to introduce the special CGSs is to find a way to extend this method to finite soluble groups.

1.3. A special composition series. In this subsection we define a composition series of a finite soluble group G which will be used to generalize the ideas of Subsection 1.2.

First "slice" G into nilpotent factors. Define $G_1 := G$ and let G_{i+1} be the smallest normal subgroup of G_i with a nilpotent factor group G_i/G_{i+1}. The resulting series

$$G = G_1 \rhd G_2 \rhd \ldots \rhd G_m \rhd G_{m+1} = \{1\}$$

is called the **lower nilpotent** series of G. Clearly, this is a series of characteristic subgroups of G. The group G is viewed now as a tower of its nilpotent factors where each of the factors has an operator group on top. It will be helpful later on that we have chosen the factors to be maximal nilpotent.

Now we refine the factors in order to exhibit their Frattini subgroups according to Subsection 1.2. For each index $i \in \{1, \ldots, m\}$ we define the characteristic subgroup $G_{i,*}$ of G by $G_{i,*}/G_{i+1} := \phi(G_i/G_{i+1})$. The factor $G_i/G_{i,*}$ is called the **local head** (or **l-head** for short) and, as observed in Subsection 1.2, it is a direct product of elementary abelian groups. The remaining factors $G_{i,*}/G_{i+1}$ of G are called **l-tails**. The lower nilpotent series refined by the subgroups $G_{i,*}$ is called the **nilpotent head series** of G.

The nilpotent head series of G may be refined further, if necessary, to obtain a normal series with elementary abelian factors. In the original algorithm of Cannon and Leedham-Green the lower p-central series of each direct factor of the nilpotent factors is used for the refinement, and so a unique series of characteristic subgroups with elementary abelian factors is obtained. However, for the algorithms developed in this note any refinement of the nilpotent head series to a normal series with elementary abelian factors may be used.

A special CGS determines a composition series which refines the normal series with elementary abelian factors that has been obtained. Such a composition series is called a **special composition series** of G.

1.4. Exhibiting subgroups. Now the aim is to define a CGS of G that determines a special composition series and additionally allows one to read off information about maximal subgroups and a Hall system of G.

DEFINITION 1.1. The CGS $\mathcal{A}$ of G **exhibits** the subgroup $U \leq G$ if $U \cap \mathcal{A}$ is a CGS of U.

The following theorem leads to some of the subgroups which will be exhibited by a special CGS.

THEOREM 1.2. *For each $i \in \{1, \ldots, m\}$ the l-head $G_i/G_{i,*}$ has a complement K_i in G, and all complements to the l-head $G_i/G_{i,*}$ are conjugate in G.*

PROOF. This may be observed from the fact that G_{i-1}/G_i is a maximal nilpotent factor of G_{i-1}, and therefore G_{i-1}/G_i acts fixed-point-freely on $G_i/G_{i,*}$. Then a well-known result (e.g. see [**9**], Lemma 1.b) yields the above theorem. ☐

We want to exhibit an l-head complement K_i for each index $1 \leq i \leq m$ and a p-complement S^p of G for each prime p dividing $|G|$ by a CGS which determines a special composition series of G.

Since it is in general not possible to exhibit an arbitrary set of subgroups by a CGS, we have to prove the following observation first.

LEMMA 1.3. *There exists a CGS $\mathcal{A}$ of G which determines a special composition series such that $\mathcal{A}$ exhibits simultaneously a p-complement S^p for each prime dividing $|G|$ and an l-head complement K_i for $1 \leq i \leq m$.*

PROOF. This will be proved by induction on the length m of the lower nilpotent series.

Suppose $m = 1$, i.e., G is nilpotent. Without loss of generality, we may assume that G is a finite p-group. We choose any special composition series of G and a CGS determining it. Then the subgroup $\phi(G) = G_{1,*} = K_1(G)$ is exhibited by the CGS, and trivially the p-complement $\{1\}$ of G is exhibited as well. Thus in this case there exists a CGS of G as required.

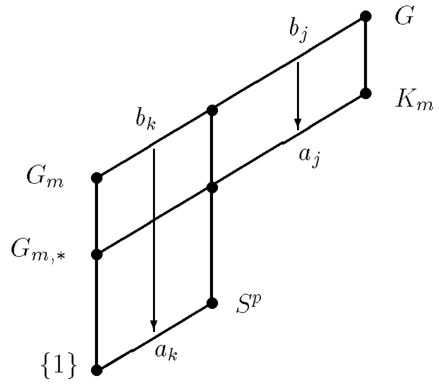
Now suppose $m > 1$. By the inductive hypothesis let $\mathcal{B} :=(b_1, \ldots, b_h)$ be a CGS of G/G_m which determines a special composition series of G/G_m and thus exhibits a set of p-complements of G/G_m as well as a complement to each l-head of G/G_m. Again without loss of generality we assume that G_m is a finite p-group and we choose a special CGS $(a_{h+1}, \ldots, a_n)$ of G_m as above.

For any choice of elements a_j in the coset b_j for $1 \leq j \leq h$ the concatenated sequence $(a_1, \ldots, a_n)$ is a CGS of G which determines a special composition series of G. Now we want to choose the elements $a_j \in b_j$ in such a way that the concatenated sequence $(a_1, \ldots, a_n)$ exhibits a set of p-complements as well as a complement to each l-head of G.

For $1 \leq i \leq m-1$ there exists an l-head complement K_i to $G_i/G_{i,*}$ in G such that K_i/G_m is the l-head complement in G/G_m exhibited by $\mathcal{B}$. Similarly, since G_m is assumed to be a p-group, for any prime $q \neq p$ there exists a q-complement S^q such that S^q/G_m is the q-complement of G/G_m exhibited by $\mathcal{B}$. Thus for any choice of elements $a_j \in b_j$ the l-head complements K_i of G for $1 \leq i \leq m-1$ and the q-complement S^q for any $q \neq p$ will be exhibited by the resulting concatenated sequence.

It remains to consider an l-head complement to $G_m/G_{m,*}$ in G and a p-complement in G. Let S^p be a p-complement of G such that $S^p G_m/G_m$ is exhibited by $\mathcal{B}$. Let H_m be a complement in G to the l-head $G_m/G_{m,*}$. Since G_m is assumed to be a p-group, any p-complement T of H_m is a p-complement of G. Since all p-complements of G are conjugate, $T^g = S^p$ for an element $g \in G$. Thus $K_m := H_m^g$ is an l-head complement to $G_m/G_{m,*}$ in G which contains S^p. The sequence $\mathcal{B}$ is a CGS of $K_m G_m/G_m$. If we choose each element $a_j \in b_j \cap K_m$, then the resulting concatenated sequence will exhibit K_m, since $K_m \cap G_m = G_{m,*}$ is exhibited by the special CGS of G_m. For each element $b_k \in S^p G_m/G_m$ we require additionally that $a_k \in b_k \cap S^p \leq b_k \cap K_m$. Then the resulting concatenated sequence will also exhibit S^p, since $S^p \cap G_m = \{1\}$. This situation is illustrated by the picture below.

So we have observed that we are able to find a CGS $(a_1, \ldots, a_n)$ of G determining a special composition series and exhibiting a set of p-complements as well as a set of l-head complements of G.



□

The proof of Lemma 1.3 shows in particular that the set of exhibited subgroups has to be chosen such that for each exhibited p-complement S^p and each exhibited l-head complement K_i the intersection $K_i \cap S^p$ is a p-complement of K_i, i.e., such that the p-complements **reduce** into the l-head complements (see [**2**], pages 225 ff). Now we are able to define the properties of a special CGS.

DEFINITION 1.4. A **special CGS** of G is a CGS of G that determines a special composition series of G (and therefore exhibits the refined nilpotent head series) and exhibits an l-head complement for each l-head of G and a p-complement of G for each prime dividing $|G|$.

An algorithm to compute a special CGS of a finite soluble group from an arbitrary CGS is described in [**1**] and has been partially improved in [**4**]. It is implemented in GAP and timings are given below in Section 3 (see [**3**] also).

1.5. Computing Hall systems. In this section it will be shown that a Hall subgroup or even a Hall system of a finite soluble group can be obtained using a special CGS without additional computation.

A set of p-complements S^p of G is exhibited by the special CGS $\mathcal{A}$ and therefore explicitly given. The subgroup $\cap_{r \notin \pi} S^r$ for a set π of primes is a Hall π-subgroup of G. The set of Hall subgroups obtained by all possible intersections of this type forms a Hall system of G. The following lemma yields that these subgroups are in fact all exhibited by a special CGS, and so can be obtained without computation.

LEMMA 1.5. *Let U and V be two subgroups of G which are exhibited by the CGS $\mathcal{A}$. Then $U \cap V$ is exhibited by $\mathcal{A}$ as well.*

PROOF. See [**4**]. □

Since in a finite soluble group any two Hall π-subgroups and any two Hall systems are conjugate, it is also possible to obtain all Hall subgroups or all Hall systems by this method.

1.6. Computing maximal subgroups. The next aim is to show how a special CGS may be used to compute all maximal subgroups of G. It is not in general possible to exhibit a set of conjugacy class representatives of maximal subgroups by a CGS; e.g., consider the elementary abelian group with 4 elements. Thus this part is slightly more difficult. On the other hand we will show that a set of l-head complements is already very close to a set of conjugacy class representatives of maximal subgroups.

Let M be a maximal subgroup of G and let G_{i+1} be the largest subgroup in the lower nilpotent series with $G_{i+1} \leq M$. Then $G_{i,*}/G_{i+1} = \phi(G_i/G_{i+1}) \leq \phi(G/G_{i+1}) \leq M/G_{i+1}$, since M/G_{i+1} is a maximal subgroup of G/G_{i+1}. This yields $G_{i,*} \leq M$. Define $L := G_i \cap M$. Since $|G : M| = |G_i : L|$, the subgroup L has p-power index in G_i for some prime p. Recall that $G_i/G_{i,*}$ is a direct product of elementary abelian groups. To investigate the possible subgroups L we may assume without loss of generality that $G_i/G_{i,*}$ is a p-group, i.e., an $\mathbb{F}_pG$-module. From this point of view then, the factor $L/G_{i,*}$ is a maximal $\mathbb{F}_pG$-submodule of $G_i/G_{i,*}$.

Clearly, G_i/L is complemented in G by M. By the same argument as in the proof of Theorem 1.2, all complements to G_i/L in G are conjugate. However, one particular complement to G_i/L in G is given by K_iL where K_i is the exhibited l-head complement to $G_i/G_{i,*}$. Thus M is conjugate to K_iL. Since K_i is already explicitly given by a special CGS, it remains to compute all maximal $\mathbb{F}_pG$-submodules of the l-head $G_i/G_{i,*}$ to obtain a set of conjugacy class representatives of the maximal subgroups M with $G_{i+1} \leq M$ and $G_i \not\leq M$. This may be done using Meat-Axe methods (e.g. by the Smash share package of GAP , see [**7**]). Clearly, by applying this approach to each l-head, one obtains a set of conjugacy class representatives of all maximal subgroups.

Like the exhibited l-head complements, this particular set of conjugacy class representatives of maximal subgroups has the additional property that the exhibited Hall system reduces into each subgroup of this set.

2. Computing (Pre-)Frattini subgroups

Clearly, the Frattini subgroup of a finite soluble group may be computed by its definition using a special CGS to compute all maximal subgroups and then intersecting them all. However, a much more efficient method can be developed which also depends on special CGSs. This method makes use of a certain class of subgroups of G as introduced in [**5**]. In the next section this class of subgroups will be investigated and an algorithm to compute such a subgroup will be derived.

2.1. Prefrattini subgroups.

Definition 2.1. A subgroup $W \leq G$ is called a **prefrattini subgroup** of G if W covers each non-complementable chief factor of G and if each maximal subgroup of G is conjugate to one that contains W.

The following theorem recalls some important properties of prefrattini subgroups of finite soluble groups which have been introduced and proved in [**5**]. They are necessary to find an efficient method to compute prefrattini subgroups as well as they lead to the computation of the Frattini subgroup from a prefrattini subgroup.

Theorem 2.2.

i) *The prefrattini subgroups of G form a characteristic conjugacy class of G.*
ii) *Let $N \trianglelefteq G$. If W is a prefrattini subgroup of G, then WN/N is a prefrattini subgroup of G/N.*
iii) *$\phi(G)$ is the intersection of all prefrattini subgroups of G.*

Statement iii) will be the key to the computation of $\phi(G)$ in Subsection 2.4. The aim now is to develop an algorithm to compute a prefrattini subgroup of G.

Suppose that G is given by a special CGS. We will use the homomorphism principle over the lower nilpotent series, i.e., for $i \in \{1, \ldots, m\}$ we wish to obtain a prefrattini subgroup of G/G_{i+1} from a prefrattini subgroup of G/G_i.

First we consider G/G_2. In a finite nilpotent group all maximal subgroups are normal. Therefore any prefrattini subgroup of a finite nilpotent group is contained in the intersection of all maximal subgroups by definition. This yields with Theorem 2.2 iii) that the prefrattini subgroup is equal to the Frattini subgroup in this case. Thus the prefrattini subgroup of G/G_2 is given by $G_{1,*}/G_2$.

Now let $i > 1$. Suppose that W/G_i is a prefrattini subgroup of G/G_i. We want to derive a prefrattini subgroup of G/G_{i+1} using W/G_i. The following notation will be used throughout.

Definition 2.3. Let A be an elementary abelian normal subgroup of H. Then the H**-radical** of A, denoted by $Rad_H(A)$, is defined as the intersection of all subgroups of A, which are maximal with respect to being H-normal.

The following lemma recalls a well-known fact about the radical (see [**2**], page 107).

Lemma 2.4. *Let A be an elementary abelian normal subgroup of H which has a complement in H. Then $Rad_H(A) = A \cap \phi(H)$.*

The next lemma yields a method to obtain a prefrattini subgroup of $G/G_{i,*}$ using W/G_i.

Lemma 2.5. *Let A be an elementary abelian normal subgroup of H and let K be a complement to A in H. If X is a prefrattini subgroup of K, then $X Rad_H(A)$ is a prefrattini subgroup of H.*

Proof. Let V be a prefrattini subgroup of H. By Lemma 2.4 we obtain that $Rad_H(A) \leq \phi(H)$. Thus Theorem 2.2 iii) yields that $Rad_H(A)$ is contained in each prefrattini subgroup of H, i.e., $Rad_H(A) \leq V$. By Theorem 2.2 ii) then one obtains that $V/Rad_H(A)$ is a prefrattini subgroup of $H/Rad_H(A)$. However, a prefrattini subgroup of $H/Rad_H(A)$ is given by $X Rad_H(A)/Rad_H(A)$ (see [**6**], Lemma 3.3). So $V/Rad_H(A)$ is conjugate to $X Rad_H(A)/Rad_H(A)$, which yields that V is conjugate to $X Rad_H(A)$. Therefore $X Rad_H(A)$ is a prefrattini subgroup of H. □

We apply Lemma 2.5 with $G/G_{i,*} = H$ and $G_i/G_{i,*} = A$. Since G has a special CGS, an l-head complement K_i to $G_i/G_{i,*}$ is exhibited. Then the prefrattini subgroup W/G_i of G/G_i given by generators of the special CGS of G/G_i can be read off as a prefrattini subgroup $\tilde{W}/G_{i,*}$ of $K_i/G_{i,*}$. Let R_i be the subgroup of G defined by $R_i/G_{i,*} := Rad_{G/G_{i,*}}(G_i/G_{i,*})$. Then the factor $R_i/G_{i,*}$ will be denoted by **l-head radical** and in Subsection 2.2 we will introduce some methods to compute these factors. By Lemma 2.5, a prefrattini subgroup of $G/G_{i,*}$ is given by $\tilde{W}R_i/G_{i,*}$.

Since $G_{i,*}/G_{i+1} \leq \phi(G/G_{i+1})$, the factor $G_{i,*}/G_{i+1}$ is contained in each prefrattini subgroup of G/G_{i+1} by Theorem 2.2 iii). Therefore $\tilde{W}R_i/G_{i+1}$ is a prefrattini subgroup of G/G_{i+1} and no further computation is necessary.

This argument yields an algorithm to compute a prefrattini subgroup of G, which is essentially reduced to the computation of the l-head radicals $R_i/G_{i,*}$ for $1 \leq i \leq m$. By Theorem 2.2 i) the prefrattini subgroups of G form a conjugacy class of subgroups of G which yields a method to compute all prefrattini subgroups of G.

The prefrattini subgroup of G as computed by the above algorithm may be exhibited by a special CGS of G by choosing the refinement of the nilpotent head series in such a way that it contains the subgroups R_i for $1 \leq i \leq m$.

As described in [**1**] and [**4**], the system normalizer D of the exhibited Hall system may be exhibited by any special CGS which determines a composition series refining a chief series of G. Thus it is possible to exhibit D and W simultaneously by a special CGS.

2.2. L-head radicals. Since $G_i/G_{i,*}$ is a direct product of elementary abelian p-groups, its maximal G-normal subgroups correspond to the maximal $\mathbb{F}_pG$-submodules of its p-factors. Maximal submodules can be computed by Meat-Axe methods as in Subsection 1.6. So the l-head radical $R_i/G_{i,*}$ may be computed by its definition, i.e., by computing the maximal G-normal subgroups of $G_i/G_{i,*}$ and then intersecting them (as vector spaces over finite fields).

In the remainder of this section we introduce some variations of the computation of the l-head radicals which are more efficient and do not rely on the Meat-Axe methods. In some special cases these methods even yield the computation of the radical. Again we assume here that G is given by a special CGS.

The following is known as Maschke's Theorem (see [**2**], page 39).

THEOREM 2.6. *Let A be an elementary abelian normal p-subgroup of H. If p does not divide $|H : C_H(A)|$, then $Rad_H(A) = \{1\}$.*

We may assume without loss of generality that the l-head $G_i/G_{i,*}$ is an elementary abelian p-group. A Sylow system of G/G_i is exhibited by the induced CGS of G on G/G_i, which yields in particular that every element of the induced CGS on G/G_i is either a p-element or a p'-element. Therefore p does not divide $|G/G_{i,*} : C_{G/G_{i,*}}(G_i/G_{i,*})|$ if and only if all p-elements in the induced CGS on $G/G_{i,*}$ act trivially on $G_i/G_{i,*}$. This property may be checked easily. Clearly the generators of $G_i/G_{i,*}$ act trivially on $G_i/G_{i,*}$.

If the p-Sylow subgroup of $G/G_{i,*}$ does not act trivially on $G_i/G_{i,*}$, the following lemma may yield a reduction of the computation of the l-head radical. Let the core of a subgroup be defined as in [**2**], page 19.

LEMMA 2.7. *Let A be an elementary abelian normal p-subgroup of H which is complemented by K in H. Let L be a p-subgroup of K. Then $Core_K([L, A]) \leq Rad_H(A)$.*

PROOF. If L is a p-subgroup of K, then LA is a p-subgroup of H. Furthermore $[L, A] \leq (LA)' \leq \phi(LA)$. This yields that $Core_K([L, A])$ is a normal subgroup of H which is contained in the Frattini subgroup of the subgroup LA of H. Thus $Core_K([L, A]) \leq A \cap \phi(H) = Rad_H(A)$ by Lemma 2.4. □

Since the computation of a core can be rather time-consuming, we will use the following special case in the algorithm.

COROLLARY 2.8. *Let A be an elementary abelian normal p-subgroup of H which is complemented by K. Suppose that L is a normal p-subgroup of K. Then $[L, A] \leq Rad_H(A)$. In particular if L is a normal p-Sylow subgroup of K, then $[L, A] = Rad_H(A)$.*

PROOF. By Lemma 2.7 and Theorem 2.6. □

The factor $K_2/G_{2,*}$ of the l-head complement K_2 is nilpotent and so its Sylow subgroups are all normal. Therefore the l-head radical $R_2/G_{2,*}$ can be computed by Corollary 2.8 efficiently.

In the case that we want to compute the l-head radical $R_i/G_{i,*}$ for $i > 2$ we proceed as follows. Let K_i be the exhibited l-head complement to $G_i/G_{i,*}$. The lower nilpotent series of G/G_i induces a lower nilpotent series of the factor $K_i/G_{i,*}$ which can be read off directly from the induced CGS of $K_i/G_{i,*}$. The p-Sylow subgroup of the smallest nontrivial subgroup of this induced nilpotent series is a normal p-subgroup $L/G_{i,*}$ of $K_i/G_{i,*}$ and so we apply Corollary 2.8. This yields a subgroup C by $C/G_{i,*} := [L/G_{i,*}, G_i/G_{i,*}]$ with $R_i \geq C \geq G_{i,*}$ and it remains to compute $R_i/C = Rad_{G/C}(G_i/C)$. We check whether the p-Sylow subgroup of G/C acts trivially on G_i/C. Clearly, this is always the case, if $p \nmid |G : G_{i-1}|$. If the p-Sylow subgroup of G/C does act trivially, then $R_i = C$ holds by Theorem 2.6. Otherwise we use the Meat-Axe methods to compute R_i/C.

2.3. Relation to Hall systems and maximal subgroups. If G is a finite nilpotent group, then the prefrattini subgroup of G is the intersection of all maximal subgroups of G. If G is not nilpotent, then this statement is false in general. However, if G is a finite soluble group, then a prefrattini subgroup of G can be written as the intersection of a certain set of maximal subgroups of G. The following lemma yields such a set of maximal subgroups.

Let G be a finite soluble group given by a special CGS throughout.

LEMMA 2.9. *Let W be the prefrattini subgroup computed by the algorithm introduced in Subsection 2.1, and let $\{M_1, \ldots, M_r\}$ be the set of conjugacy class representatives of maximal subgroups as computed by the algorithm in Subsection 1.6. Then $W = \bigcap_{i=1}^{r} M_i$.*

PROOF. Clear by the algorithms. ☐

In several papers a connection between Hall systems and prefrattini subgroups has been investigated. For example in [**6**] the following characterisation of prefrattini subgroups can be found.

THEOREM 2.10. *U is a prefrattini subgroup of G if and only if*

i) *U avoids all complementable chief factors of G and covers all other chief factors and*
ii) *U permutes with a Hall system of G, i.e., $UX = XU$ for each subgroup X in the Hall system (see [**2**], page 230).*

The following lemma yields the relationship between the exhibited Hall system, its system normalizer and the computed prefrattini subgroup W.

LEMMA 2.11. *Let W be the prefrattini subgroup computed by the algorithm introduced in Subsection 2.1. Then the following hold.*

i) *W permutes with the exhibited Hall system of G.*
ii) *W permutes with the system normalizer D of the exhibited Hall system.*

PROOF. Consider part i). By Lemma 2.9 the prefrattini subgroup W is the intersection of the maximal subgroups M_i of G. Let X be a Hall π-subgroup from the exhibited Hall system and let $|G : M_i|$ be a power of the prime p. If $p \in \pi$, then $XM_i = G = M_iX$ and so X permutes with M_i. If on the other hand $p \notin \pi$, then $X \leq M_i$, since the exhibited Hall system reduces into M_i. Thus the maximal subgroups M_i permute with the exhibited Hall system and by Proposition (4.29) of [**2**], page 232, then the prefrattini subgroup does as well.

Part ii) is left as an Exercise as in [**2**], page 424. □

2.4. Computing the Frattini subgroup. Let G be a finite soluble group. Then by Theorem 2.2 the Frattini subgroup of G may be obtained as core of a prefrattini subgroup of G under operation of G. Thus if G is given by a special CGS, then the computation of the Frattini subgroup of G is reduced to the computation of the l-head radicals of G followed by the computation of the core of the corresponding prefrattini subgroup of G. A reduction in the computation of the core may be obtained by the observation that the normal subgroup R_m is always contained in the Frattini subgroup of G.

The core of a subgroup is computed by its definition, i.e. we compute conjugates to the subgroup in question and intersect them until we obtain a normal subgroup. To intersect subgroups of groups given by a special CGS we use an efficient method depending on the special CGS as described in [**3**]. This method is also implemented in GAP .

As the timings below show, this approach yields a fast method to obtain the Frattini subgroup of a finite soluble group. In fact this method is usually faster than the computation of the conjugacy class representatives of maximal subgroups of G, since the usage of Meat-Axe methods is reduced to fewer cases than in the algorithm to compute conjugacy class representatives of maximal subgroups.

3. Examples and timings

In [**5**] Gaschütz asked whether or not prefrattini subgroups are characterised by their cover-avoidance property as described in Theorem 2.10 i). This question has been answered in [**6**] by J. D. Gillam, who constructed a family of finite soluble groups which have more than one conjugacy class of subgroups avoiding every complemented chief factor and covering the other chief factors.

We describe the special CGS structure of this family of examples briefly. Let $p \geq 3$ and q be primes with $p \mid q-1$ and $q \not| \, p^i - 1$ for $1 \leq i < q-1$. Let $H = C_p \wr C_p$ of order p^{p+1}, let $K = C_q^p$ and let $L = C_p^{p(q-1)}$. Then the examples in [**6**] are of the form $G = H \ltimes (K \ltimes L)$, and the lower nilpotent series of G is given by $G_3 = L$ and $G_2 = K \ltimes L$. Furthermore L is a minimal normal subgroup of G. The three subgroups H, K and L of G are exhibited by a special CGS together with $\phi(H)$.

A prefrattini subgroup of G is given by $\phi(H)$. On the other hand each complement to L in $\phi(H)L$ avoids each complemented chief factor of G and covers the other chief factors of G. One may observe, e.g. using GAP , that there exist more than one conjugacy class of complements to L in $\phi(H)L$.

Since H does not centralize L, the algorithm designed in Subsections 2.1 and 2.2 needs to use Meat-Axe methods to compute a prefrattini subgroup of G. Therefore these groups are rather bad cases for this algorithm but are also not large. In the following list of examples there is a group of this family included.

EXAMPLES 3.1. Here we introduce some examples of finite soluble groups. The groups are sorted according to their composition length.

To understand the timings introduced below it is important to have some knowledge about the nilpotent head series of these groups. Therefore the orders of the groups are given in a special factorisation describing the indices of this series. Thus the sizes of the factors of the lower nilpotent series are seperated by $|$. If the l-tail of a nilpotent factor is not trivial, then the size of the l-head is seperated by : from the size of the l-tail.

i) $MFi22$ is a maximal subgroup of the Fischer group Fi_{22} of factorised order $2 \mid 3^2 \mid 2^4 : 2^3 \mid 3^6 : 3$.

ii) *Gill* is the group from the family of groups as construced by Gillam and described above for the primes $p = 3$ and $q = 7$. It is of factorised order $3^2 : 3^2 \mid 7^3 \mid 3^{18}$.

iii) Let C_r be the cyclic group with r Elements and let S_3 be the symmetric group on three points. With S_3 acting in its natural permutation representation we obtain the group $(C_{35} \wr S_3) \wr C_4$ of factorised order $2^2 \cdot 5 \cdot 7 : 2^4 \mid 3^4 \cdot 5^3 \cdot 7^3 \mid 5^8 \cdot 7^8$.

iv) Let U be the subgroup of upper triangular matrices in $GL(4, 7)$. Then $U \wr C_3$ is a group of factorised order $2^4 \cdot 3^5 : 3^8 \mid 2^8 \mid 7^9 : 7^9$.

v) *Dark* is a group constructed according to Dark's construction (see [**2**], page 630 ff) of factorised order $3^2 \mid 7 \mid 2^3 \mid 3^7 \mid 31^8 \mid 5^{24}$.

The timings given for the implementation of the above algorithms in GAP were obtained on a 486 PC using GAP 3.4.2 started with 6 mb of workspace, and include garbage collection. All timings are given in seconds.

The algorithms listed are the ones to compute a special CGS (as described in [**1**] and partially improved in [**4**]), to compute conjugacy class representatives of maximal subgroups, to compute a prefrattini subgroup and to compute the Frattini subgroup. The computation of maximal submodules relies here on the methods of [**7**].

To give some idea of the results the number of conjugacy classes of maximal subgroups, the size of the computed prefrattini subgroup and the size of the Frattini subgroup are included in the table.

	special	maximals		prefrattini		Frattini	
group	time	number	time	size	time	size	time
$MFi22$	2.695	8	1.136	$2^3 \cdot 3$	1.098	3	1.231
Gill	3.573	6	10.583	3^2	4.583	1	4.591
$(C_{35} \wr S_3) \wr C_4$	28.596	13	5.204	2^4	0.145	1	0.234
$U \wr C_3$	6.891	224	20.079	$3^8 \cdot 7^9$	0.056	$3^2 \cdot 7^9$	5.896
Dark	42.983	9	16.176	1	0.503	1	0.540

These times show, that the computation of a prefrattini subgroup is usually much faster than the computation of conjugacy class representatives of maximal subgroups. This is due to the reduced usage of the Meat-Axe. For example the large group *Dark* contains only one factor of size 3^7 for which the Meat-Axe has to be used in the computation of a prefrattini subgroup, while the Meat-Axe has to be used for all but the first l-head of *Dark* in the computation of conjugacy class representatives of maximal subgroups.

One may observe by these times, that the group *Gill* is a bad case for the computation of a prefrattini subgroup. This is due to the fact that the factor of order 3^{18} which has to be processed by the Meat-Axe appears to be a difficult case for the Meat-Axe.

The computation of the Frattini subgroup from a prefrattini subgroup is usually very fast. The example $U \wr C_3$ is a bad case for this computation which is derived from the fact that the prefrattini subgroup of $U \wr C_3$ is rather large and not normal.

However, this method to compute the Frattini subgroup is usually much faster than the computation of the conjugacy class representatives of maximal subgroups. This yields that the method to compute the Frattini subgroup as described in Subsection 2.4 is more efficient than the computation of the intersection of all maximal subgroups.

Finally it remains to remark that the computation of a special CGS depends not only on the group, but also on the arbitrary CGS which is used as input. E.g., the input CGS for *Dark* appears to be a rather bad case for this algorithm, while the input CGS for $U \wr C_3$ is a good case.

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Lehrstuhl D für Mathematik, RWTH Aachen, 52056 Aachen, Germany
E-mail address: `Bettina.Eick@math.rwth-aachen.de`