Some new simple Lie algebras in characteristic 2

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Abstract

We describe an algorithm for computing automorphism groups and testing isomorphisms of finite dimensional Lie algebras over finite fields. The algorithm is particularly effective for simple or almost simple Lie algebras. We show how it can be used in a computer search for new small-dimensional simple Lie algebras over the field with two elements.

1 Introduction

Algorithms for computing automorphism groups and for testing isomorphisms of Lie algebras over finite fields are available for certain classes of Lie algebras: Schneider [9] describes a method for nilpotent Lie algebras and Eick [3] introduces a method for solvable Lie algebras. The method by Schneider [9] has been used to classify the nilpotent Lie algebras of dimension 9 over the field with two elements $\mathbb{F}_2$.

The first part of this paper contains an algorithm to test isomorphism and to compute automorphism groups for arbitrary finite dimensional Lie algebras over finite fields. The algorithm is based on 'nice' presentations for the considered Lie algebras. There are two variations exhibited: a deterministic and a Monte-Carlo version. The Monte-Carlo version may fail to detect an isomorphism (with some probability), but it is significantly more effective than the deterministic version.

The classification of simple Lie algebras in positive characteristic has recently been completed for all algebraically closed fields of characteristic at least 5, we refer to the book by Strade [10] for details. The classifications in characteristic 2 and 3 are still open and computational methods to support these classifications would be welcome. Vaughan-Lee [11] used computational methods to determine up to isomorphism all simple Lie algebras of dimension at most 9 over $\mathbb{F}_2$.

In the second part of this paper we apply the new isomorphism test in a computer-based randomized search for new simple Lie algebras of dimension at most 20 over $\mathbb{F}_2$. Our approach does not yield a complete classification, but it found 2 new simple Lie algebras in dimension 15 and 4 new simple Lie algebras in dimension 16. Descriptions of the new Lie algebras and a table of all known simple Lie algebras of dimension at most 20 and their isomorphisms are included below.

The paper is structured as follows. Section 2 recalls some well known basic facts. Section 3 describes the new isomorphism test for Lie algebras over finite fields. Section 4 discusses the computation of the full automorphism group and its subgroup generated by exponential automorphisms. Section 5 outlines the search for new simple Lie algebras and contains the new simple Lie algebras.

2 Preliminaries

Many algorithms for Lie algebras are described in the book by de Graaf [2]. The book mainly focusses on Lie algebras in characteristic 0. The characteristic $p > 0$ case is less well developed so far. The aim of this section is to describe some basic algorithms which we need later. We assume throughout that all arising Lie algebras are given by structure constants tables; that is, for a Lie algebra L we are given a basis $b_1, \ldots, b_n$ and elements $\alpha_{i,j,k} \in \mathbb{F}$ with

$$b_i b_j = \sum_{k=1}^{n} \alpha_{i,j,k} b_k.$$

In this setting it is straightforward to compute a basis for a subalgebra of L given by generators, to check whether a subalgebra is an ideal and to compute structure constants tables for sub- and quotient algebras. Further, we can readily determine its derivation Lie algebra $Der(L)$ as a subalgebra of the matrix algebra $M_{n\times n}(\mathbb{F})$, see the book by de Graaf [2] for details.

2.1 Ideals and constituents

There exists a highly effective algorithm to check whether a given Lie algebra L over a finite field $\mathbb{F}$ is simple or, if not, then to compute the simple constituents of L. For this purpose let m_j denote the $n \times n$-matrix over $\mathbb{F}$ whose entry at position (i, k) is the structure constant $\alpha_{i,j,k}$. Then m_j is a matrix representation for the right adjoint (or, equivalently, the multiplication from the right) of b_j in L. Let M be the associative algebra generated by the matrices $m_1, \ldots, m_n$. Then M acts by multiplication from the right on $V = \mathbb{F}^n$.

1 Lemma: *The ideals of L correspond one-to-one to the M-submodules of V.*

Proof: By construction, the M-submodules of V correspond one-to-one to the right ideals of L. As L is a Lie-algebra, right ideals are two-sided ideals and thus the result follows. •

Using the so-called 'MeatAxe', see [6] for background, we can readily check whether the associative algebra M acts irreducibly (and thus L is simple) or determine a non-trivial M-submodule of V (and thus a non-trivial ideal I of L). In the latter case we can then apply the method recursively to L/I and I and thus finally obtain a series

$$L = L_1 > L_2 > \ldots > L_r > L_{r+1} = \{0\},$$

with $L_{i+1} \trianglelefteq L_i$ and L_i/L_{i+1} simple. This series allows one to read off the simple constituents of L directly.

2.2 Presentations and isomorphisms

We briefly discuss some features of presentations of finite dimensional Lie algebras. First, we recall how presentations of Lie algebras can be used to check whether a map is an epimorphism. We omit the proof of the following theorem as it is well-known.

2 Theorem: *Let L and $\overline{L}$ be two Lie algebras over an arbitrary field $\mathbb{F}$. Let $\langle g_1, \ldots, g_d \mid z_1, \ldots, z_k \rangle$ be a finite presentation for L and let $\{h_1, \ldots, h_d\}$ be a generating set for $\overline{L}$. If $z_i(h_1, \ldots, h_d) = 0$ for $1 \le i \le k$, then the map $g_i \mapsto h_i$ for $1 \le i \le k$ extends to a Lie algebra epimorphism $L \to \overline{L}$.*

Theorem 2 is most useful if a 'nice' presentation for the Lie algebra L is available. We note that if $g_1, \ldots, g_d$ is an arbitrary generating set for L, then any finite presentation of L (for example the presentation corresponding to its structure constants table) can be rewritten to a finite presentation of L on the generators $g_1, \ldots, g_d$ by adjusting the relations of the presentation to the new generators.

3 Isomorphism testing

In this section we describe an approach for testing whether two Lie algebras L_1 and L_2 over a finite field $\mathbb{F}$ are isomorphic.

3.1 Types, bins and efficient generating sets

We introduce some notation. Let L be a Lie algebra over the finite field $\mathbb{F}$ and let $L \to Der(L) : g \mapsto \overline{g}$ denote the homomorphism from L to $Der(L)$ via the adjoint action.

Types:

- The type $t(g) \in \mathbb{F}[x]$ for $g \in L$ is the minimal polynomial of its adjoint action $\overline{g}$.
- The type of $(g_1, \ldots, g_d)$ is defined via $t(g_1, \ldots, g_d) = (t(g_1), \ldots, t(g_d))$.

Bins: Let $U \subseteq L$.

- The bin of $t \in \mathbb{F}[x]$ in U is defined to be $b_U(t) = \{g \in U \mid t(g) = t\}$.
- $b_U(t_1, \ldots, t_d) = b_U(t_1) \times \ldots \times b_U(t_d)$ the cartesian products of sets.
- The bin of $g \in L$ in U is $b_U(g) = b_U(t(g))$.
- $b_U(g_1, \ldots, g_d) = b_U(t(g_1), \ldots, t(g_d))$.

We say that a generating set $g_1, \ldots, g_d$ for L is *d-efficient* if there exists no d-element generating set $h_1, \ldots, h_d$ for L with $|b_L(h_1, \ldots, h_d)| < |b_L(g_1, \ldots, g_d)|$. Further, a generating set for L is *efficient*, if it is d-efficient and d is the minimal cardinality for a generating set for L. The following outlines an algorithm to determine or approximate a d-efficient generating set for a Lie algebra L over a finite field $\mathbb{F}$. It takes as input the Lie algebra L, the number d and a function $f : \mathbb{N} \to \mathbb{N}$.

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Algorithm: DEfficientGeneratingSet($L, d, f$)
• Determine the elements of $L$ and their types.
• Let $t_1, \ldots, t_r$ denote the different types of elements in $L$.
• Sort the elements of $L$ into their bins $b_L(t_1), \ldots, b_L(t_r)$.
• Let $s_i$ denote the number of elements in $b_L(t_i)$ for $1 \leq i \leq r$.
• Let $v_1 < \ldots < v_s$ be the different values of $d$-fold products $s_{i_1} \cdots s_{i_d}$.
• For $i$ from 1 to $s$ do:
    - Loop over all $d$-tuples $(s_{i_1}, \ldots, s_{i_d})$ with $i_1 \leq \ldots \leq i_d$ and $s_{i_1} \cdots s_{i_d} = v_i$.
    - For each such $(s_{i_1}, \ldots, s_{i_d})$ loop over $f(v_i)$ different, randomly chosen elements
      $(g_1, \ldots, g_d)$ in the corresponding bin $b_L(t_{i_1}, \ldots, t_{i_d})$.
    - For each $(g_1, \ldots, g_d)$ check whether it generates $L$;
    - If so, then return $(g_1, \ldots, g_d)$ and stop.
• Return fail.
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If f is the identity function, then this algorithm is a deterministic routine to compute a d-efficient generating set for L. The algorithm returns such a generating set or fail; in the latter case there exists no d-element generating set for L. The smaller $f(m)$ is, the more effective is the algorithm. However, if $f(m) < m$, then the algorithm may return a generating set which is not d-efficient or it may return fail even if there exists a d-element generating set in L. Our experiments suggest that the algorithm is effective and has a high chance of success if $f(m) = \lfloor \sqrt{m} \rfloor$ is used. It then performs well for Lie algebras with up to 2^{20} elements and $d \leq 4$.

3.2 The isomorphism test algorithm

Assume that we are given two Lie algebras L_1 and L_2 over a finite field $\mathbb{F}$. The aim is to check whether L_1 and L_2 are isomorphic. Our approach to this problem splits into 3 steps. The third step depends on a predefined number $m \in \{d, \ldots, |L_2|\}$.

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Step 1: Elementary checks
(a) Check whether $\dim(L_1) = \dim(L_2)$ holds.
(b) Check further invariants, e.g. $\dim(Der(L_1)) = \dim(Der(L_2))$.
If either of these equalities is not satisfied, then return false.

Step 2: Precomputations in $L_1$
(a) Determine (or approximate) an efficient generating set $(g_1, \ldots, g_d)$ for $L_1$.
(b) Determine a presentation $P$ for $L_1$ on $g_1, \ldots, g_d$.

Step 3: Computations in $L_2$
(a) Choose a random subset $U$ of $L_2$ of size $m$.
(b) Determine the bin $b := b_U(t(g_1), \ldots, t(g_d))$.
(c) Loop over the elements $(h_1, \ldots, h_d)$ in $b$:
        - Check if $(h_1, \ldots, h_d)$ generates $L_2$.
        - Check if $(h_1, \ldots, h_d)$ satisfies the relators of $P$.
        - If both are satisfied, then return the homomorphism induced by $L_1 \to L_2 : g_i \mapsto h_i$.
(d) Return false.
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This algorithm either returns an isomorphism from L_1 onto L_2 or false. If $m \geq |L_1|$, then the algorithm returns false only if the two Lie algebras are not isomorphic and hence it is a deterministic isomorphism test in this case. Otherwise it is possible that the algorithm returns false even though L_1 and L_2 are isomorphic. The smaller m is, the more time- and space efficient is the algorithm and the larger is the probability that it does not detect an isomorphism between the Lie algebras.

4 Automorphism groups

In this section we discuss the determination of the automorphism group $Aut(L)$ of a Lie algebra L defined over a finite field $\mathbb{F}$.

4.1 Computing the automorphism group

We can determine or approximate $Aut(L)$ with the following variation of the method in Section 3.2. The algorithm requires as input a Lie algebra L and a number $m \in \{d, \ldots, |L|\}$. If $m = |L|$, then the algorithm is a deterministic method to compute $Aut(L)$. If $m < |L|$, then the algorithm determines a subgroup of $Aut(L)$.

Step 1: Skipped

Step 2: Precomputations in L (similar to Section 3.2)

(a) Determine (or approximate) an efficient generating set $(g_1, \ldots, g_d)$ for L.
(b) Determine a presentation P for L on $g_1, \ldots, g_d$.

Step 3: Computations in L (variation on Section 3.2)

(a) Initialize A as the trivial subgroup of $Aut(L)$.
(b) Determine the bin $b := b_L(g_1, \ldots, g_d)$.
(c) Loop over m randomly chosen elements $(h_1, \ldots, h_d)$ in b:
  - Check if $(h_1, \ldots, h_d)$ generates L.
  - Check if $(h_1, \ldots, h_d)$ satisfies the relators of P.
  - If both is satisfied, then
    - Let $\alpha : L \to L : g_i \mapsto h_i$.
    - Reset A to $\langle A, \alpha \rangle$.

(d) Return A.

The method can be improved if a subgroup $\overline{A}$ of $Aut(L)$ is known *a priori*. This can be used in Step (3c) of the algorithm by initializing A with $\overline{A}$. Further, instead of looping over all elements in the bin b, it is sufficient to loop over orbits under the action of $\overline{A}$. The next section exhibits a useful candidate for a subgroup $\overline{A}$.

4.2 Exponential automorphisms

We say that $d \in Der(L)$ is *p-nilpotent* if $d^p = 0$ holds. We define for a p-nilpotent derivation d its exponential matrix

$$exp(d) = \sum_{i=0}^{p-1} \frac{1}{i!} d^i.$$

We call a p-nilpotent derivation d an *annihilator* if it satisfies for all $x, y \in L$ and for all $i, j \geq 0$ with $i + j \geq p$ the equation $d^i(x)d^j(y) = 0$. Let $Ann(L) \subseteq Der(L)$ denote the subset of annihilator derivations in $Der(L)$.

3 Lemma: *If $d \in Ann(L)$, then $exp(d)$ is an automorphism of L.*

Proof: We have to show that $exp(d)(xy) = exp(d)(x)exp(d)(y)$ for all $x, y \in L$. This follows from expanding both sides of the equation using the definition of $exp(d)$ and that d is an annihilator derivation. •

We define $Exp(L)$ as the subgroup of $Aut(L)$ generated by $\{exp(d) \mid d \in Ann(L)\}$. Note that every element $exp(d)$ has order p or 1. Thus $Exp(L)$ is a subgroup of $Aut(L)$ generated by automorphisms of order p.

5 Searching new simple Lie algebras

In this section we describe our search for new simple Lie algebras over the field $\mathbb{F}_2$. We first describe the list of known simple Lie algebras and we recall how scalar extensions can be used to construct new simple Lie algebras from given ones. Then we outline our search and exhibit the results of it.

5.1 Known simple Lie algebras

Searching through the literature, we found the following known simple Lie algebras over the field $\mathbb{F}_2$.

- A, B, C, D, E, F, G describe the simple constituents of the classical Lie algebras as determined by Hiss [4] and Hogeweij [5].
- W, S, H, K describe the simple constituents of the Lie algebras of Cartan type; these have been determined computationally from the Lie algebras of Cartan type.
- P describes the Hamiltonian type Lie algebras as determined by Lin [8];
- Q describes the Contact type Lie algebras as described by Zhang and Lin [12];
- Kap_i describes the ith series of Lie algebras determined by Kaplansky [7] for $1 \leq i \leq 4$.
- Bro_i describes the ith series of Lie algebras determined by Brown [1] for $1 \leq i \leq 3$.
- V_7, V_8 and V_9 are simple Lie algebras determined by Vaughan-Lee [11].

5.2 Extending scalars

Let L be a Lie algebra over a field $\mathbb{F}$ and let $\mathbb{E}$ be an extension field of $\mathbb{F}$. Then L and $\mathbb{E}$ are vector spaces over $\mathbb{F}$. The tensor product $L \otimes_{\mathbb{F}} \mathbb{E}$ is thus a vector space of dimension $\dim(L)[\mathbb{E} : \mathbb{F}]$ over $\mathbb{F}$. It is a Lie algebra via the multiplication $(l_1 \otimes e_1)(l_2 \otimes e_2) = l_1 l_2 \otimes e_1 e_2$. We note that $Der(L) \otimes \mathbb{E}$ embeds into $Der(L \otimes \mathbb{E})$ via the linear map α determined by

$$\alpha : Der(L) \otimes \mathbb{E} \to Der(L \otimes \mathbb{E}) \quad \text{with} \quad \alpha((d \otimes e))(l \otimes k) = d(l) \otimes ke,$$

and $Aut(L) \times Gal(\mathbb{E}/\mathbb{F})$ embeds into $Aut(L \otimes \mathbb{E})$ via the group homomorphism β defined by

$$\beta : Aut(L) \times Gal(\mathbb{E}/\mathbb{F}) \to Aut(L \otimes \mathbb{E}) \quad \text{with} \quad \beta((\gamma, \delta))(l \otimes k) = \gamma(l) \otimes \delta(k).$$

5.3 A computer-based search

For an arbitrary Lie algebra L over $\mathbb{F}_2$, we can readily construct the constituents of randomly chosen subalgebras of L by choosing two (or more) random elements in L, taking the subalgebra U of L they generate and determining the simple constituents of U. This method can be used to produce many simple Lie algebras in short time. The isomorphism test of Section 3.2 can then be used to check for any newly determined simple Lie algebra over $\mathbb{F}_2$ whether it is isomorphic to one of the known simple Lie algebras.

As a result we obtain the following table of simple Lie algebras over $\mathbb{F}_2$ up to dimension 20. The table exhibits the names of the Lie algebras if available (and thus also isomorphisms among the known simple Lie algebras), the dimension $\dim(Der(L))$, its restrictedness, the order of $Aut(L)$, the number of elements of $Ann(L)$ and the order of $Exp(L)$.

dim	*nr*	*names*	*der*	*res*	*aut*	*ann*	*exp*
3	1	$W(2)$	5	–	6	4	6
6	1	$W(2) \otimes \mathbb{F}_4$	10	–	120	16	60
7	1	$W(3)$	10	–	16	16	16
7	2	V_7, $P(1,2)$	10	–	4	2	2
8	1	A_2, $W(1,1)$, $Q(1,1,1)$	8	+	336	1	1
8	2	V_8	8	+	432	1	1
9	1	$W(2) \otimes \mathbb{F}_8$, V_9	15	–	1512	64	504
10	1	$Kap_3(5)$	14	–	720	16	720
12	1	$W(2) \otimes \mathbb{F}_{16}$	20	–	16320	256	4080
14	1	$W(3) \otimes \mathbb{F}_4$	20	–	1536	256	256
14	2	$V_7 \otimes \mathbb{F}_4$	20	–	96	4	4
14	3	$S(2,2)$	20	–	1536	80	256
14	4	$P(1,1,1,1)$, $Kap_1(4)$	20	–	1152	14	1152
14	5	A_3, B_3, C_3, G_2, $S(1,1,1)$, $H(1,1,1,1)$	21	+	1451520	64	1451520
14	6	$Bro_2(1,1)$	20	–	10752	8	64
15	1	$W(2) \otimes \mathbb{F}_{32}$	25	–	163680	1024	32736
15	2	$W(4)$	19	–	2048	1152	2048
15	3	$Kap_3(6)$, $Kap_2(4)$,	20	–	23040	32	23040
15	4	$P(2,1,1)$	19	–	64	6	16
15	5	$P(3,1)$	19	–	512	48	64
15	6	$P(2,2)$	19	–	256	20	64
15	**7**	**New**	19	–	32	4	8
15	**8**	**New**	19	–	192	10	192
16	1	$W(1,1) \otimes \mathbb{F}_4$, $A_2 \otimes \mathbb{F}_4$, $V_8 \otimes \mathbb{F}_4$	16	+	241920	1	1
16	2	$W(2,1)$, $Q(2,1,1)$	17	–	2048	24	32
16	**3**	**New**	17	–	1536	4	8
16	**4**	**New**	17	–	384	4	8
16	**5**	**New**	17	–	512	8	8
16	**6**	**New**	17	–	360	1	1
18	1	$W(2) \otimes \mathbb{F}_{64}$	30	–	1572480	4096	262080
20	1	$Kap_3(5) \otimes \mathbb{F}_4$	28	–	1958400	256	979200

The new Lie algebras in dimension 15 arose as constituents of subalgebras of $Kap_1(5)$. The new Lie algebras in dimension 16 arose as constituents of subalgebras of the Lie algebra of 8×8-matrices with trace 0 over $\mathbb{F}_2$.

5.4 Explicit generators for the new Lie algebras

Lie algebra number 7 of dimension 15:

$$\left(\begin{array}{ccccccccccccccc}
1 & . & 1 & . & . & . & . & 1 & 1 & 1 & . & . & 1 & . & 1 \\
. & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & 1 & 1 & . & 1 \\
1 & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . & . & . & 1 & 1 \\
. & . & . & . & 1 & . & . & 1 & . & . & . & 1 & 1 & . & . \\
. & . & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & 1 & . & 1 \\
. & . & . & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & . & 1 & 1 \\
1 & . & . & 1 & . & . & 1 & 1 & 1 & . & 1 & 1 & . & . & . \\
. & . & . & 1 & 1 & 1 & . & . & 1 & 1 & 1 & . & . & 1 & . \\
. & 1 & 1 & . & 1 & . & 1 & . & . & . & . & . & . & . & . \\
. & . & 1 & 1 & . & . & 1 & . & 1 & . & 1 & 1 & . & . & 1 \\
. & . & 1 & . & 1 & . & . & . & 1 & 1 & . & . & 1 & 1 & . \\
. & . & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & . & 1 & 1 \\
1 & . & . & 1 & 1 & . & 1 & . & . & . & . & . & . & . & . \\
. & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & . & 1 & 1 & . & 1 \\
. & . & 1 & 1 & 1 & . & 1 & 1 & 1 & . & . & . & 1 & 1 & .
\end{array}\right)
\left(\begin{array}{ccccccccccccccc}
1 & 1 & 1 & . & . & 1 & 1 & 1 & . & 1 & . & . & 1 & . & 1 \\
. & 1 & . & 1 & 1 & 1 & 1 & 1 & . & . & 1 & . & . & 1 & . \\
. & . & 1 & . & . & 1 & 1 & 1 & 1 & 1 & . & . & 1 & 1 & 1 \\
. & . & 1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & . & . & . & . \\
. & . & 1 & . & . & 1 & 1 & 1 & 1 & 1 & . & . & 1 & 1 & 1 \\
. & 1 & . & 1 & . & . & 1 & . & 1 & 1 & . & . & . & 1 & 1 \\
. & . & . & . & 1 & . & 1 & 1 & 1 & 1 & . & . & . & 1 & 1 \\
. & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . & . & 1 & . & 1 \\
. & . & . & . & . & 1 & . & 1 & . & 1 & . & . & 1 & 1 & . \\
1 & 1 & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & . \\
1 & 1 & 1 & . & . & 1 & . & . & . & . & 1 & . & . & . & 1 \\
. & . & . & . & . & . & . & 1 & 1 & . & . & . & 1 & . & . \\
. & 1 & . & 1 & . & . & 1 & . & 1 & 1 & . & . & . & 1 & 1 \\
1 & . & 1 & 1 & 1 & 1 & . & 1 & . & . & 1 & . & . & . & 1 \\
. & . & 1 & . & 1 & . & . & 1 & . & 1 & . & . & . & 1 & .
\end{array}\right)$$

Lie algebra number 8 of dimension 15:

$$\begin{pmatrix}
1 & . & 1 & 1 & 1 & . & . & 1 & 1 & 1 & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & 1 & . & . & . & 1 & 1 & . & . & . & . \\
. & 1 & . & . & . & . & 1 & 1 & 1 & . & . & . & . & . & . \\
. & 1 & . & . & . & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & . \\
. & . & . & 1 & 1 & 1 & 1 & 1 & . & . & . & . & . & . & . \\
. & 1 & . & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & 1 & . & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & . \\
1 & . & . & 1 & . & . & 1 & 1 & . & 1 & . & . & . & 1 & . \\
1 & 1 & . & . & 1 & . & 1 & . & 1 & 1 & . & 1 & 1 & 1 & . \\
. & 1 & 1 & 1 & . & . & 1 & . & . & 1 & 1 & . & . & 1 & . \\
. & . & 1 & 1 & 1 & . & . & . & . & 1 & 1 & 1 & 1 & 1 & .
\end{pmatrix}
\begin{pmatrix}
1 & 1 & . & . & . & . & 1 & 1 & . & . & 1 & . & . & . & 1 \\
. & . & . & 1 & 1 & 1 & 1 & . & . & 1 & 1 & 1 & 1 & . & . \\
1 & . & . & . & . & 1 & 1 & . & . & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . \\
1 & 1 & . & . & . & 1 & . & . & 1 & . & . & 1 & . & 1 & 1 \\
1 & . & . & 1 & . & . & 1 & 1 & . & 1 & 1 & . & 1 & . & 1 \\
. & . & . & . & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . & . \\
. & . & 1 & . & . & . & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . \\
. & . & 1 & 1 & . & 1 & 1 & 1 & 1 & 1 & . & . & . & 1 & . \\
. & . & 1 & . & . & . & . & 1 & . & . & . & . & . & 1 & 1 \\
. & . & 1 & 1 & 1 & 1 & . & 1 & 1 & . & 1 & . & 1 & . & . \\
1 & 1 & . & . & 1 & . & . & . & . & 1 & 1 & . & 1 & 1 & 1 \\
. & . & 1 & . & . & 1 & 1 & 1 & . & . & 1 & . & 1 & 1 & 1 \\
. & . & . & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & 1 & . & 1 \\
1 & 1 & 1 & . & . & 1 & 1 & . & 1 & . & 1 & 1 & . & . & .
\end{pmatrix}$$

Lie algebra number 3 of dimension 16:

$$\begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . \\
. & 1 & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & 1 & 1 & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . \\
. & 1 & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & 1 & . & . & 1 & . & 1 & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . \\
1 & . & . & . & . & . & 1 & 1 & . & 1 & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{pmatrix}
\begin{pmatrix}
1 & . & . & 1 & . & . & . & 1 & . & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . \\
1 & . & 1 & . & . & . & 1 & 1 & . & 1 & . & . & 1 & . & . & . \\
. & 1 & . & 1 & . & . & . & . & 1 & . & 1 & . & 1 & . & . & . \\
1 & 1 & 1 & . & . & . & . & . & 1 & . & 1 & 1 & . & 1 & 1 & . \\
. & . & . & 1 & 1 & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & 1 & . & . & . & 1 & . & 1 & . & . \\
. & . & 1 & . & . & . & . & . & 1 & . & . & 1 & . & . & . & 1 \\
. & 1 & 1 & 1 & . & . & . & 1 & . & . & . & 1 & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . \\
. & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & 1 & . & 1 & . & . & 1 & 1 & . & 1 & . & 1 & . & 1 & 1 & . \\
. & . & 1 & . & 1 & . & 1 & . & . & 1 & . & 1 & 1 & . & . & .
\end{pmatrix}$$

Lie algebra number 4 of dimension 16:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & . & . & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . \\
. & . & 1 & . & . & 1 & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . \\
1 & . & . & 1 & 1 & 1 & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & 1 & 1 & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . \\
. & . & 1 & . & . & 1 & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . \\
1 & 1 & . & . & 1 & 1 & . & 1 & 1 & . & 1 & 1 & 1 & 1 & . & . \\
1 & 1 & . & 1 & 1 & 1 & 1 & 1 & . & . & . & . & . & . & . & . \\
. & 1 & 1 & . & 1 & 1 & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . \\
. & 1 & 1 & 1 & 1 & . & 1 & 1 & . & . & 1 & 1 & . & . & . & . \\
. & 1 & 1 & . & 1 & 1 & 1 & . & 1 & . & 1 & 1 & 1 & 1 & . & . \\
. & 1 & 1 & . & . & . & 1 & . & 1 & . & . & . & 1 & 1 & . & . \\
1 & 1 & . & . & 1 & 1 & . & 1 & 1 & . & . & . & . & . & 1 & 1 \\
1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . & . & 1 & 1 \\
. & 1 & . & 1 & . & 1 & . & 1 & . & 1 & 1 & 1 & . & . & 1 & . \\
. & 1 & . & 1 & . & . & . & 1 & 1 & . & 1 & 1 & 1 & 1 & . & . \\
. & 1 & . & 1 & . & . & 1 & . & . & 1 & 1 & 1 & 1 & 1 & . & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & . & . & 1 & 1 & . & . & 1 & 1 & . & . & . & 1 & 1 \\
1 & . & 1 & . & . & 1 & . & . & 1 & 1 & 1 & . & . & 1 & 1 & 1 \\
. & 1 & 1 & . & 1 & . & 1 & 1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 \\
. & . & 1 & 1 & . & . & . & . & . & 1 & . & 1 & 1 & . & . & . \\
1 & . & . & . & . & 1 & 1 & . & . & 1 & . & 1 & 1 & 1 & . & . \\
. & . & . & 1 & . & 1 & . & 1 & 1 & . & 1 & 1 & 1 & . & . & . \\
1 & . & . & . & . & 1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \\
1 & 1 & . & . & 1 & 1 & . & . & 1 & 1 & 1 & . & 1 & . & . & . \\
1 & 1 & 1 & . & . & 1 & 1 & . & 1 & . & . & 1 & 1 & . & 1 & 1 \\
. & . & . & . & 1 & . & 1 & 1 & 1 & . & 1 & . & 1 & 1 & . & . \\
. & . & 1 & 1 & 1 & . & 1 & 1 & . & . & . & 1 & 1 & 1 & . & . \\
1 & 1 & 1 & . & . & 1 & . & . & . & 1 & . & . & . & . & 1 & . \\
. & . & . & 1 & 1 & 1 & . & 1 & . & . & 1 & 1 & . & . & . & . \\
. & . & . & . & 1 & . & 1 & 1 & 1 & . & . & 1 & . & . & 1 & 1 \\
1 & 1 & 1 & 1 & . & 1 & . & . & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
. & . & 1 & . & 1 & . & . & . & 1 & 1 & 1 & 1 & 1 & . & 1 & .
\end{pmatrix}$$

Lie algebra number 5 of dimension 16:

$$\begin{pmatrix}
1 & 1 & . & 1 & . & 1 & . & . & 1 & 1 & . & 1 & . & 1 & 1 & 1 \\
. & 1 & 1 & . & 1 & . & . & 1 & . & 1 & 1 & 1 & . & 1 & 1 & 1 \\
1 & . & . & 1 & . & 1 & 1 & . & . & . & . & . & . & . & . & . \\
. & 1 & 1 & . & . & 1 & . & . & 1 & 1 & . & 1 & . & 1 & 1 & 1 \\
. & 1 & . & . & . & . & 1 & . & 1 & 1 & 1 & 1 & 1 & 1 & . & . \\
. & 1 & 1 & . & . & 1 & 1 & . & . & . & 1 & . & . & 1 & 1 & 1 \\
. & 1 & . & . & . & . & . & . & . & . & 1 & . & . & 1 & 1 & 1 \\
1 & . & . & 1 & . & 1 & . & . & 1 & 1 & 1 & 1 & . & . & . & . \\
. & 1 & 1 & 1 & 1 & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . & . & . \\
. & . & . & 1 & . & 1 & 1 & 1 & 1 & . & . & . & . & 1 & . & . \\
1 & . & 1 & . & 1 & . & 1 & . & . & . & 1 & . & . & 1 & . & . \\
. & . & . & . & . & 1 & . & 1 & 1 & . & . & . & . & . & 1 & . \\
1 & 1 & 1 & 1 & . & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 & 1 & 1 & . \\
1 & 1 & . & 1 & . & 1 & 1 & . & . & . & . & . & 1 & 1 & . & . \\
1 & . & . & 1 & . & . & 1 & 1 & 1 & . & 1 & . & 1 & . & . & 1 \\
1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & . & . & . & 1 & . & 1 & 1
\end{pmatrix}
\begin{pmatrix}
. & . & 1 & . & . & 1 & . & . & 1 & . & . & . & 1 & 1 & 1 & 1 \\
1 & . & 1 & . & . & . & 1 & 1 & 1 & . & . & . & . & . & 1 & 1 \\
. & 1 & 1 & 1 & 1 & . & . & . & . & 1 & . & 1 & 1 & . & . & . \\
. & . & 1 & . & 1 & 1 & . & . & 1 & . & . & . & 1 & 1 & 1 & 1 \\
. & 1 & . & 1 & 1 & 1 & 1 & 1 & . & 1 & . & . & . & 1 & . & . \\
. & 1 & . & 1 & 1 & . & . & 1 & . & 1 & 1 & 1 & . & 1 & . & . \\
. & 1 & . & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & . & 1 & . & . \\
1 & 1 & 1 & 1 & 1 & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 \\
1 & . & 1 & . & 1 & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . & . & . \\
1 & . & 1 & 1 & . & . & . & 1 & 1 & . & 1 & 1 & . & . & 1 & 1 \\
. & . & . & . & 1 & 1 & . & 1 & . & . & . & 1 & . & . & 1 & 1 \\
1 & 1 & . & . & . & 1 & . & . & 1 & . & . & 1 & 1 & 1 & . & 1 \\
. & . & . & 1 & 1 & . & 1 & . & 1 & . & . & 1 & 1 & . & 1 & . \\
. & . & 1 & 1 & 1 & . & . & 1 & . & 1 & . & 1 & . & 1 & . & . \\
. & 1 & 1 & . & . & . & 1 & 1 & 1 & . & . & 1 & 1 & 1 & 1 & . \\
. & . & 1 & . & . & 1 & . & . & 1 & 1 & 1 & . & 1 & . & 1 & .
\end{pmatrix}$$

Lie algebra number 6 of dimension 16:

$$\begin{pmatrix}
. & 1 & 1 & 1 & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . \\
1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . & 1 & 1 & . \\
1 & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
1 & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & 1 & . & 1 & . & . & . & 1 & . & 1 & . & . & 1 & . \\
. & 1 & . & . & 1 & . & 1 & . & . & 1 & . & 1 & . & . & 1 & . \\
1 & . & . & 1 & . & . & . & 1 & . & 1 & . & 1 & . & . & 1 & . \\
. & 1 & . & . & 1 & 1 & . & . & . & . & . & 1 & . & . & . & . \\
. & 1 & . & . & . & . & 1 & . & . & 1 & . & 1 & . & . & 1 & . \\
1 & . & 1 & . & . & . & . & . & . & 1 & 1 & 1 & . & . & 1 & . \\
. & 1 & 1 & . & 1 & 1 & . & 1 & 1 & 1 & 1 & 1 & . & 1 & . & . \\
1 & . & 1 & 1 & 1 & 1 & . & 1 & . & . & 1 & 1 & . & . & . & . \\
1 & 1 & . & 1 & . & 1 & 1 & . & 1 & . & . & . & . & 1 & 1 & . \\
1 & . & . & 1 & 1 & . & 1 & . & . & . & . & 1 & . & . & . & . \\
. & . & . & 1 & 1 & . & 1 & 1 & . & . & . & . & . & . & . & .
\end{pmatrix}
\begin{pmatrix}
. & . & 1 & . & 1 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & 1 & 1 & 1 \\
1 & . & . & . & 1 & 1 & . & 1 & 1 & . & 1 & 1 & 1 & . & . & 1 \\
. & 1 & 1 & . & . & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 & 1 & 1 & . \\
. & . & 1 & . & 1 & . & . & . & . & 1 & 1 & . & 1 & 1 & . & 1 \\
1 & 1 & . & 1 & . & . & . & . & 1 & 1 & . & . & 1 & . & . & . \\
. & . & 1 & 1 & . & . & . & 1 & . & . & 1 & 1 & 1 & 1 & 1 & . \\
. & 1 & 1 & . & 1 & 1 & . & 1 & . & 1 & . & . & 1 & 1 & 1 & 1 \\
. & 1 & 1 & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . \\
. & . & 1 & 1 & 1 & 1 & . & . & 1 & 1 & 1 & . & 1 & . & 1 & . \\
. & 1 & 1 & 1 & 1 & . & 1 & . & . & 1 & . & . & 1 & 1 & 1 & 1 \\
1 & . & 1 & 1 & 1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 & 1 & . & . \\
. & 1 & 1 & 1 & . & . & 1 & . & . & 1 & . & 1 & . & 1 & . & . \\
. & . & . & 1 & 1 & 1 & 1 & 1 & . & . & . & . & 1 & . & . & . \\
. & 1 & 1 & 1 & . & 1 & 1 & . & 1 & 1 & 1 & . & 1 & . & 1 & . \\
1 & 1 & . & . & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . \\
. & . & . & 1 & 1 & . & 1 & . & . & . & . & 1 & . & 1 & 1 & .
\end{pmatrix}$$

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