A nilpotent quotient algorithm for certain infinitely presented groups and its applications

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Abstract

We describe a nilpotent quotient algorithm for a certain type of infinite presentations: the so-called finite L-presentations. We then exhibit finite L-presentations for various interesting groups and report on the application of our nilpotent quotient algorithm to them. As result, we obtain conjectural descriptions of the lower central series structure of various interesting groups including the Grigorchuk supergroup, the Brunner-Sidki-Vieira group, the Basilica group, certain generalized Fabrykowski-Gupta groups and certain generalized Gupta-Sidki groups.

1 Introduction

A finite L-presentation is a certain type of presentation with finitely many generators and possibly infinitely many relators. It is denoted by $\langle S \mid Q \mid \Phi \mid R\rangle$, where S is a finite alphabet, both Q and R are finite subsets of the free group F_S on S and Φ is a finite set of endomorphisms of F_S. If Φ^* is the free monoid generated by Φ, then $\langle S \mid Q \mid \Phi \mid R\rangle$ defines the group

$$F_S/\langle Q \cup \bigcup_{\varphi\in\Phi^*} \varphi(R)\rangle^{F_S}.$$

Well-known examples of finitely L-presented groups are the Grigorchuk group [14] and the Gupta-Sidki group [21]. Both groups play a role in the study of the famous Burnside problems. It is well-known that they are not finitely presented [27, 15, 34]. Many other examples are exhibited in [1].

A major reason to introduce L-presentations was the desire to understand better some examples of so-called self-similar groups, and, in particular, the striking patterns along their lower central series. Self-similar groups have appeared across a wide range of mathematics, answering classical questions in infinite group theory, for example, on torsion and growth, as well as establishing new links with complex dynamics. We refer to the book by Nekrashevych [28] for details.

The first aim in this paper is to describe a nilpotent quotient algorithm for finitely L-presented groups. It takes as input a finite L-presentation of a group G and a positive integer n and it determines a polycyclic presentation for the lower central series quotient $G/\gamma_{n+1}(G)$. In particular, it can determine the abelian invariants of G and its maximal nilpotent quotient if it exists. Our algorithm generalizes the nilpotent quotient algorithm for finitely presented groups by Nickel [29]. An implementation of this algorithm is available in the NQL package [22] of the computer algebra system GAP [36].

The second part of this paper is devoted to various applications of our algorithm. First, the algorithm is applied to a number of well-known finitely L-presented groups such as the Grigorchuk group, the Grigorchuk supergroup, the Brunner-Sidki-Vieira group, the Basilica group, Baumslag's infinitely generated group with trivial multiplicator [10], and the Lamplighter group on two lamp states. For most of these groups, very little had been known so far about their lower central series. Using our algorithm, we compute many terms of the series, and then state detailed conjectures about their lower central series quotients.

We then construct finite L-presentations for an infinite family of groups which generalize the Fabrykowski-Gupta group [12, 13], a group of intermediate word-growth. Using our algorithm, we obtain various insights into the lower central series structure of the groups in this family. In particular, we conjecture that there is an interesting subfamily of groups of finite width.

Finally, we determine finite L-presentations for an infinite family of groups generalizing the Gupta-Sidki group [21]. The groups in this family are finitely generated torsion p-groups for odd primes p. We investigated some of these groups with our algorithm. We obtain that these groups do not exhibit an obvious pattern in their lower central series factors and they could possibly have infinite width.

The algorithm described in the first part of this paper and its applications in the second part have been developed as part of the Diploma project [23].

2 More about L-presentations

First we recall the basic notions used to work with finitely L-presented groups. A finite L-presentation $\langle S \mid Q \mid \Phi \mid R\rangle$ is called *ascending*, if Q is empty, and *invariant*, if

$$K = \langle Q \cup \bigcup_{\varphi\in\Phi^*} \varphi(R)\rangle^{F_S}$$

satisfies $\varphi(K) \subseteq K$ for every $\varphi \in \Phi$. Invariant L-presentations will play an important role for our algorithm. We record some basic observations on L-presentations.

Remark 1.

- *Every ascending L-presentation is invariant.*
- *An invariant L-presentation $\langle S \mid Q \mid \Phi \mid R\rangle$ determines the same group as the ascending L-presentation $\langle S \mid \emptyset \mid \Phi \mid Q \cup R\rangle$.*
- *A finite presentation $\langle S \mid R\rangle$ determines the same group as the finite L-presentations $\langle S \mid R \mid \emptyset \mid \emptyset\rangle$ and $\langle S \mid \emptyset \mid \{\mathrm{id}\} \mid R\rangle$.*

Many of the well-known examples of L-presentations are ascending. A famous example is the Grigorchuk group, see [27] and [15] for details.

Example 2. *The Grigorchuk group can be defined by the following ascending (and hence invariant) finite L-presentation*

$$\langle \{a, c, d\} \mid \emptyset \mid \{\sigma'\} \mid \{a^2, [d, d^a], [d, d^{acaca}]\}\rangle,$$

with

$$\sigma' \colon \left\{ \begin{array}{ccc} a & \mapsto & c^a \\ c & \mapsto & cd \\ d & \mapsto & c \end{array} \right\}.$$

We note that there are other finite L-presentations for the Grigorchuk group known. An example is the following non-ascending, but invariant, L-presentation

$$\langle \{a,b,c,d\} \mid \{a^2,b^2,c^2,d^2,bcd\} \mid \{\sigma\} \mid \{[d,d^a],[d,d^{acaca}]\}\rangle,$$

with

$$\sigma\colon \left\{ \begin{array}{ccc} a & \mapsto & c^a \\ b & \mapsto & d \\ c & \mapsto & b \\ d & \mapsto & c \end{array} \right\}.$$

Part I
The NQL algorithm

In this part we describe a new algorithm for computing nilpotent quotients of finitely L-presented groups. Applications of our algorithm will be exhibited in Part II.

3 Polycyclic and nilpotent presentations

Every finitely generated nilpotent group is polycyclic and hence can be described by a consistent polycyclic presentation. This type of presentation allows effective computations with the considered group and thus it facilitates detailed investigations of the underlying group. In this section we recall the definitions and some of the basic ideas on polycyclic presentations with particular emphasis on finitely generated nilpotent groups. Further information and references can be found in [24, Chapter X].

A *polycyclic presentation* is a presentation on a sequence of generators, $e_1,\ldots,e_n$ say, whose relations have the following form for certain $r_1,\ldots,r_n \in \mathbb{N}\cup\{\infty\}$ and certain $f_{i,j,k}, g_{i,j,k}, h_{i,k} \in \mathbb{Z}$ with $f_{i,j,k}, g_{i,j,k}, h_{i,k} \in \{0,\ldots,r_k-1\}$ if $r_k<\infty$:

$$\begin{array}{rcl} e_i^{e_j} & = & e_{j+1}^{f_{i,j,j+1}}\cdots e_n^{f_{i,j,n}} \text{ for } j<i, \\ e_i^{e_j^{-1}} & = & e_{j+1}^{g_{i,j,j+1}}\cdots e_n^{g_{i,j,n}} \text{ for } j<i \text{ with } r_j=\infty, \text{ and} \\ e_i^{r_i} & = & e_{i+1}^{h_{i,i+1}}\cdots e_n^{h_{i,n}} \text{ for all } i \text{ with } r_i<\infty. \end{array}$$

Let G be the group defined by the above presentation and let $G_i = \langle e_i,\ldots,e_n\rangle \le G$. Then the above relations imply that the series $G = G_1 \trianglerighteq G_2 \trianglerighteq \ldots \trianglerighteq G_n \trianglerighteq G_{n+1} = \{1\}$ is a subnormal series with cyclic factors. We say that this is the *polycyclic series* defined by the presentation. The factors of this polycyclic series satisfy that $[G_i : G_{i+1}] \le r_i$ for $1 \le i \le n$. The polycyclic presentation is called *consistent* if $[G_i : G_{i+1}] = r_i$ holds for $1 \le i \le n$. The consistency of a polycyclic presentation can be checked effectively, see [35, page 424].

Nilpotent presentations are a special type of polycyclic presentations for finitely generated nilpotent groups. Let $G = \gamma_1(G) \ge \gamma_2(G) \ge \ldots$ denote the lower central series of G. Then we say that a polycyclic presentation of G is a *nilpotent presentation* if its polycyclic series refines the lower central series of G. A nilpotent presentation is called *weighted*, if there exists a function $w : \{e_1, \ldots, e_n\} \to \mathbb{N}$ such that $w(e_k) = 1$ if and only if $e_k \notin \gamma_2(G)$, and if $w(e_k) > 1$, then there exists a relation $e_i^{e_j} = e_i e_k$ with $j < i < k$ so that $w(e_j) = 1$ and $w(e_i) = w(e_k) - 1$. For each k with $w(e_k) > 1$ one such relation $e_i^{e_j} = e_i e_k$ is choosen and called the *definition* of e_k.

4 Computing abelian invariants

Let $G = \langle S \mid Q \mid \Phi \mid R\rangle$ be a group given by a finite L-presentation. In this section we describe a method to determine the abelian invariants of G and a corresponding consistent weighted nilpotent presentation of $G/\gamma_2(G) = G/G'$.

Our method is a direct generalization of the well-known approach to determine the abelian invariants of a finitely presented group. We refer to [35] or [24] for further information.

Write $S = \{s_1, \ldots, s_m\}$ and let F be the free group on S. Then every element $w \in F$ is a word in $S \cup S^{-1}$, say $w = s_{i_1}^{e_1} \cdots s_{i_l}^{e_l}$ with $e_i = \pm 1$. Define $a_j = \sum_{i_k = j} e_k \in \mathbb{Z}$ for $1 \leq j \leq m$ and set $\overline{w} = s_1^{a_1} \cdots s_m^{a_m}$. Then $\overline{w}$ can be considered as the collected word corresponding to w. It satisfies $wF' = \overline{w}F'$ and hence $\overline{w}$ is a representative of the coset wF'. Translating to additive notation, we can represent $\overline{w}$ by the vector $a_w = (a_1, \ldots, a_m) \in \mathbb{Z}^m$.

Every endomorphism φ of F satisfies $\varphi(F') \subseteq F'$ and hence induces an endomorphism $\overline{\varphi}$ of F/F'. Translating to additive notation as above, we can represent $\overline{\varphi}$ by a matrix $M_\varphi \in M_m(\mathbb{Z})$ which acts by multiplication from the right on $\mathbb{Z}^m$ as φ acts on F/F'. Thus we obtain a homomorphism $End(F) \to M_m(\mathbb{Z}) : \varphi \mapsto M_\varphi$. These constructions yield the following description of $G/\gamma_2(G)$.

Lemma 3. *$G/\gamma_2(G) \cong \mathbb{Z}^m/U_G$ where $U_G = \langle a_q, a_r M_\varphi \mid q \in Q, r \in R, \varphi \in \Phi^*\rangle$.*

If a subgroup V of $\mathbb{Z}^m$ is given by a finite set of generators, then algorithms for membership testing in V and for computing the abelian invariants of the quotient $\mathbb{Z}^m/V$ together with a corresponding minimal generating set for this quotient are described in [35, Chapter 8]. Both methods mainly rely on Hermite normal form computations for integral matrices. The latter allows to read off a consistent nilpotent presentation for $\mathbb{Z}^m/V$.

To apply these methods in our setting, it remains to determine a finite generating set for the subgroup U_G of $\mathbb{Z}^m$ as defined in Lemma 3. The following straightforward method achieves this aim. Note that this method terminates, since ascending chains of subgroups in $\mathbb{Z}^m$ terminate.

FiniteGeneratingSet(U_G)
    initialise $U := \{a_r \mid r \in R\}$
    initialise $T := \{a_r \mid r \in R\}$
    while $T \neq \emptyset$ do
        choose $t \in T$ and delete t from T
        for φ in Φ do
            compute $s := tM_\varphi$
            if $s \notin \langle U\rangle$ then add s to U and add s to T
        end for
    end while
    return $U \cup \{a_q \mid q \in Q\}$

This completes our algorithm to determine the abelian invariants of G and a consistent nilpotent presentation of $G/\gamma_2(G)$. Note that this presentation can be considered as weighted by assigning the weight 1 to every generator.

5 Computing nilpotent quotients I

The algorithm of Section 4 generalizes readily to a method for determining nilpotent quotients. This is straightforward to describe, but the resulting algorithm is usually not very effective in its applications. We include a description of this generalization here for completeness and we refer to Section 6 for a significantly more effective approach towards computing nilpotent quotients.

Let $G = \langle S \mid Q \mid \Phi \mid R \rangle$ be a group given by a finite L-presentation and consider $n \in \mathbb{N}$. We wish to determine a consistent polycyclic presentation for the quotient $G/\gamma_n(G)$. As above, let F be the free group on S. Then a consistent polycyclic presentation for the group $H = F/\gamma_n(F)$ together with the corresponding natural epimorphism $\epsilon : F \to H$ can be determined using a nilpotent quotient algorithm for finitely presented groups, or directly from theoretical background on free groups.

As $\gamma_n(F)$ is invariant under every endomorphism φ of F, we obtain that φ induces an endomorphism $\overline{\varphi}$ of the quotient $F/\gamma_n(F)$. This endomorphism $\overline{\varphi}$ can be translated to an endomorphism $\tilde{\varphi}$ of H via ϵ. Thus we obtain a homomorphism $End(F) \to End(H) : \varphi \mapsto \tilde{\varphi}$. This setting yields the following description of $G/\gamma_n(G)$.

Lemma 4. *$G/\gamma_n(G) \cong H/(U_G)^H$ where $U_G = \langle \epsilon(q), \tilde{\varphi}(\epsilon(r)) \mid q \in Q, r \in R, \varphi \in \Phi^* \rangle$.*

Let V be a subgroup of H given by a finite set of generators. Then standard methods for polycyclically presented groups facilitate an effective membership test in V, the computation of the normal closure of V and the determination of a consistent polycyclic presentation of H/V^H. We refer to [24, Chapter X] for background.

It only remains to determine a finite generating set for the subgroup U_G of H as described in Lemma 4 to complete our construction for $G/\gamma_n(G)$. Now, as ascending chains of subgroups in polycyclic groups terminate, we can use the same method as in Section 4 to achieve this aim.

The main deficiency of this method is that it needs to compute a consistent polycyclic presentation for the quotient $F/\gamma_n(F)$. This quotient can easily be very large, even if the desired quotient $G/\gamma_n(G)$ is rather small.

6 Computing nilpotent quotients II

Let G be defined by a finite L-presentation and consider $n \in \mathbb{N}$. In this section we describe an effective method to determine a consistent polycyclic presentation for $G/\gamma_n(G)$. First, in Section 6.1, we consider the special case that G is given by an invariant L-presentation. Then, in Section 6.2, we apply this special case method to obtain a method for the general case.

6.1 Invariant finite L-presentations

Let $G = \langle S \mid Q \mid \Phi \mid R \rangle$ be a group given by a finite invariant L-presentation and consider $n \in \mathbb{N}$. We wish to determine a consistent nilpotent presentation for $G/\gamma_n(G)$. Note that the case $n = 1$ is trivial and the case $n = 2$ is covered by Section 4. Hence we assume $n \geq 3$ in the following.

Our overall idea generalizes the method for finitely presented groups described by Nickel [29]. Thus our basic approach is an induction on n. In the induction step, we assume that we have given a consistent weighted nilpotent presentation for $G/\gamma_{n-1}(G)$ and we seek to extend this to $G/\gamma_n(G)$. We discuss this step in more detail in the following.

First, we introduce some more notation. As before, let F be the free group on $S = \{s_1, \ldots, s_m\}$ and set $K = \langle Q \cup \bigcup_{\varphi \in \Phi^*} \varphi(R) \rangle^F$ so that $G = F/K$. Define $K_n := K\gamma_n(F)$ for $n \in \mathbb{N}$. Then it follows that

$$G/\gamma_n(G) \cong F/K_n \text{ for all } n \in \mathbb{N}.$$

As input for the induction step we assume that we have given the following data describing the given nilpotent quotient F/K_{n-1}:

(a) A consistent weighted nilpotent presentation E/T defining a group H, where E is free on the generators $e_1, \ldots, e_\ell$, say;

(b) A homomorphism $\tau : F \to H$ with kernel K_{n-1} which is defined by the images $\tau(s_i) = w_i(e_1, \ldots, e_\ell)$ for $1 \leq i \leq m$;

(c) For every e_j of weight $w(e_j) = 1$, an index $i(j)$ such that $w_{i(j)}(e_1, \ldots, e_\ell) = u_{i(j)} e_j$ and $u_{i(j)}$ is a word in $e_1, \ldots, e_{j-1}$. For every j with $w(e_j) = 1$ one such image $w_{i(j)}$ is choosen and called the *definition* of e_j.

The definition of a weighted nilpotent presentation incorporates that every generator of weight greater than one in H can be written as a word in the generators of weight one. Thus H is generated by elements of weight one. Condition (c) implies that for every generator of weight one we can compute a preimage in F. This yields that the homomorphism τ is surjective, and it follows that

$$H \cong F/K_{n-1}.$$

The induction step now proceeds in two stages. First, we determine a similar set of data as in (a)-(c) describing the nilpotent factor group $F/[K_{n-1}, F]$. For this purpose we use the first part of the algorithm described in [29, Sections 4 and 5]. We briefly summarize this part as follows. Recall that some relations of H and some images $w_{i(j)}$ are definitions of the generators of H. For every other relation of H and for every other image $\tau(s_i)$ we introduce a new abstract generator. We use these to define a presentation for a central extension of H as described in [29]. We then enforce the consistency of this new presentation and thereby obtain a consistent polycyclic presentation for this central extension of H. This part of the algorithm in [29] does not depend on the relators of the input presentation for G and it yields the following data:

(a') A consistent polycyclic presentation E^*/T^* defining a group H^* and having generators $e_1, \ldots, e_\ell, e_{\ell+1}, \ldots, e_{\ell+d}$, say;

(b') A homomorphism $\tau^* : F \to H^*$ with kernel $[K_{n-1}, F]$ which is defined by the images $\tau^*(s_i)$ for $1 \leq i \leq m$. These images have the following form:

- If $i = i(j)$ for some j, then $\tau^*(s_i) = w_{i(j)}(e_1, \ldots, e_\ell)$.
- Otherwise, $\tau^*(s_i) = w_{i(j)}(e_1, \ldots, e_\ell)e_k$ for some $k > \ell$.

It follows from a similar argument as above that τ^* is an epimorphism so that $H^* \cong F/[K_{n-1}, F]$. The central subgroup $K_{n-1}/[K_{n-1}, F]$ of $F/[K_{n-1}, F]$ corresponds via τ^* to the subgroup $M := \langle e_{\ell+1}, \ldots, e_{\ell+d} \rangle$ of the group H^*. Thus H^* is a central extension of M by H.

In the second stage of the induction step, we determine a set of data as in (a)-(c) for F/K_n from the data given in (a')-(b') for $F/[K_{n-1}, F]$. We note that

$$K_n = K\gamma_n(F) = K[K, F][\gamma_{n-1}(F), F] = K[K_{n-1}, F]$$

and thus

$$F/K_n \cong H^*/\tau^*(K).$$

Hence we seek to determine a finite generating set for $\tau^*(K)$ as subgroup of H^*. Once such a finite generating set is determined, we can then use standard methods for computing with polycyclically presented groups to determine a consistent weighted nilpotent presentation for $H^*/\tau^*(K)$ and the data as required in (a)-(c) from the data given in (a')-(b').

We investigate $\tau^*(K)$ in more detail in the following. Recall that $M = \langle e_{\ell+1}, \ldots, e_{\ell+d} \rangle$ is an abelian subgroup of H^*.

Lemma 5. $\tau^*(K) \leq M$.

Proof. This follows directly, as $K \leq ker(\tau)$ and τ^* extends τ. □

Note that M is a finitely generated abelian group by construction. It now remains to determine a finite generating set for $\tau^*(K)$ as a subgroup of M.

Lemma 6. *Every endomorphism $\varphi \in \Phi^*$ induces an endomorphism $\overline{\varphi} \in End(M)$ via τ^* and we obtain a homomorphism $\Phi^* \to End(M) : \varphi \mapsto \overline{\varphi}$.*

Proof. Consider $\varphi \in \Phi^*$. As the given L-presentation is invariant, it follows that $\varphi(K) \subseteq K$ holds. Clearly $\gamma_i(F)$ is invariant under φ for every $i \in \mathbb{N}$. Thus we obtain that $K_{n-1} = K\gamma_{n-1}(F)$ and $[K_{n-1}, F]$ are also invariant under φ. Thus φ induces an endomorphism of $K_{n-1}/[K_{n-1}, F]$ and hence, via τ^*, also of M. □

This implies the following lemma.

Lemma 7. $\tau^*(K) = \langle \tau^*(q), \overline{\varphi}(\tau^*(r)) \mid q \in Q, r \in R, \varphi \in \Phi^* \rangle$.

Proof. This follows directly by translating the defining generating set of K to generators of $\tau^*(K) \leq M$. □

As M is finitely generated abelian, it satisfies the ascending chain condition on subgroups. Thus a finite generating set for $\tau^*(K)$ can be computed from the description given in Lemma 7 using a similar approach to the algorithm 'FiniteGeneratingSet' of Section 4.

We summarize our resulting algorithm for the induction step as follows. Let $\mathcal{Q}(F/L)$ denote the data as in (a)-(c) for a quotient F/L of F.

InductionStep($\mathcal{Q}(F/K_{n-1})$)

Compute the data as in (a')-(b') for the quotient $F/[K_{n-1}, F]$ (see [29]).

Induce every $\varphi \in \Phi$ to $\overline{\varphi} \in End(M)$.

Induce every $g \in Q \cup R$ to $\tau^*(g) \in M$.

Determine a finite generating set for $\tau^*(K)$ by Lemma 7 and 'FiniteGeneratingSet'.

Determine a consistent weighted nilpotent presentation for $H^*/\tau^*(K)$.

Construct and return $\mathcal{Q}(F/K_n)$.

6.2 Arbitrary finite L-presentations

Now let $G = \langle S \mid Q \mid \Phi \mid R \rangle$ be a group given by an arbitrary finite L-presentation and consider $n \in \mathbb{N}$. We wish to determine a consistent polycyclic presentation for $G/\gamma_n(G)$. As above, let F be the free group on S and write $K = \langle Q \cup \bigcup_{\varphi \in \Phi^*} \varphi(R) \rangle^F$. Our method proceeds in the following three steps.

Step 1: We determine an invariant finite L-presentation $\langle S \mid \overline{Q} \mid \Phi \mid R \rangle$ defining a group $\overline{G}$, say, so that $\overline{K} = \langle \overline{Q} \cup \bigcup_{\varphi \in \Phi^*} \varphi(R) \rangle^F$ satisfies $\overline{K} \subseteq K$.

Step 2: We determine the nilpotent quotient of the larger group $\overline{G}$ as $H := \overline{G}/\gamma_n(\overline{G})$ using the method of Section 6.1 together with the epimorphism $F \to H$.

Step 3: We determine the finite set U of images of $Q \setminus \overline{Q}$ in H and obtain $G/\gamma_n(G) \cong H/\langle U \rangle^H$ using standard methods for polycyclically presented groups.

Step 1 requires some further explanation. First note that we could always choose $\overline{Q} = \emptyset$ and thus obtain a fully automatic algorithm. However, the effectiveness of the above method relies critically on finding an L-presentation in Step 1 that yields a possibly "small" subgroup $\langle U \rangle^H$. ("Small" means here that the difference in the numbers of generators of the polycyclic presentation for H and its induced presentation for $H/\langle U \rangle^H$ is small.) Thus it may be of interest to supply a "nice" L-presentation for Step 1 by other means. However, there is at current no general algorithm for finding such a "nice" L-presentation.

7 Comments on the GAP implementation

The algorithm described in Sections 4 and 6 has been implemented in the GAP package NQL [22]. Runtimes for various applications of this algorithm are included in the following sections. They exhibit the scope and the range of its possible applications. We note here that all timings displayed below have been obtained on an Intel Pentium 4 computer with clock speed 2.80 GHz by applying the NQL algorithm with a time limit of two hours. Every application completed within 1 GB of memory. Then the computation has been stopped and the resulting nilpotent quotient together with the total time used to obtain this quotient has been listed.

Part II
Applications

In this part we exhibit a number of applications of our algorithm on various finite L-presentations associated with self-similar groups.

8 Some well-known groups

Various interesting examples of finitely L-presented, but not finitely presented groups are known. For instance:

- G: the Grigorchuk group with its L-presentation in [27]; see also [15].
- $\tilde{G}$: the Grigorchuk supergroup with its L-presentation in [1], Theorem 4.6; see also [6].
- BSV: the Brunner-Sidki-Vieira group [11] with its L-presentation in [1], Theorem 4.4.
- Δ: the Basilica group [17, 18] with its L-presentation in [8], Lemma 11.
- B: the Baumslag group [10] with its L-presentation obtained from [10], Theorem 1.
- L: the Lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ with its L-presentation in [1], Theorem 4.1.

Table 1 briefly describes the results of our algorithm applied to these groups. It lists whether the considered groups have ascending or non-invariant L-presentations, it briefly describes the obtained nilpotent quotients by their classes and the number of generators in their nilpotent presentations and it exhibits the runtimes used to determine these nilpotent quotients.

Table 1 shows that our algorithm has a significantly better performance on ascending L-presentations than on non-invariant ones. In the case of a non-invariant L-presentation, the column 'gens' of Table 1 lists in brackets the number of generators of the nilpotent quotient of the invariant L-presentation of the group $\overline{G}$ used in the method in Section 6.2.

Group	prop	class	gens	time (h:min)
G	asc	80	130	1:53
$\tilde{G}$	asc	47	127	1:56
BSV	asc	34	171	1:27
Δ	asc	39	220	1:47
B	non-inv	11	12 (423)	0:21
L	non-inv	9	10 (253)	0:04

Table 1: Some well-known groups

In the remainder of this section, we outline and discuss the lower central series quotients $\gamma_i(*)/\gamma_{i+1}(*)$ for the groups in Table 1 in more detail. These quotients are abelian groups, so may be described by their abelian invariants: a sequence '$(e_1, \ldots, e_k)$' stands for the group $\mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_k\mathbb{Z}$, where the e_i are prime powers or 0. If the same entry e or the same sequence $(e_1, \ldots, e_k)$ appears in n consecutive places, then it is listed once, in the form $e^{[n]}$ or $(e_1, \ldots, e_k)^{[n]}$, respectively.

The lower central series quotients of the Grigorchuk group G are known by theoretical results of Rozhkov [32], see also [5]. Our computations confirm the following theorem.

Theorem 8 ([32]). *The Grigorchuk group G satisfies*

$$\gamma_i(G)/\gamma_{i+1}(G) = \left\{ \begin{array}{cl} (2,2,2) \text{ or } (2,2) & \text{if } i=1 \text{ or } i=2, \text{respectively}, \\ (2,2) & \text{if } i \in \{2\cdot 2^k+1,\dots,3\cdot 2^k\}, \\ (2) & \text{if } i \in \{3\cdot 2^k+1,\dots,4\cdot 2^k\}, \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

For the Grigorchuk supergroup $\tilde{G}$ we computed $\gamma_i(\tilde{G})/\gamma_{i+1}(\tilde{G})$ for $1 \le i \le 64$. The resulting groups are elementary abelian 2-groups with ranks

$$4, 3^{[2]}, 2, 3^{[2]}, 2^{[2]}, 3^{[4]}, 2^{[4]}, 3^{[8]}, 2^{[8]}, 3^{[16]}, 2^{[16]}.$$

This induces the following conjecture.

Conjecture 9. *The Grigorchuk supergroup $\tilde{G}$ satisfies*

$$\gamma_i(\tilde{G})/\gamma_{i+1}(\tilde{G}) = \left\{ \begin{array}{cl} (2,2,2) & \text{if } i \in \{2\cdot 2^k+1,\dots,3\cdot 2^k\}, \\ (2,2) & \text{if } i \in \{3\cdot 2^k+1,\dots,4\cdot 2^k\}, \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

For the Brunner-Sidki-Vieira group BSV the Jennings series is completely determined in [2]. However, so far only the first two quotients of its lower central series are known [11, Propositions 9 and 10]. We computed $\gamma_i(BSV)/\gamma_{i+1}(BSV)$ for $1 \le i \le 43$ and obtained the following abelian invariants:

$$\begin{array}{l} (0,0), (0), (8), \\ (8), (4,8), (2,8), \\ (2,2,8)^{[2]}, (2,2,4,8)^{[2]}, (2,2,2,8)^{[2]}, \\ (2,2,2,2,8)^{[4]}, (2,2,2,2,4,8)^{[4]}, (2,2,2,2,2,8)^{[4]}, \\ (2,2,2,2,2,2,8)^{[8]}, (2,2,2,2,2,2,4,8)^{[8]}, (2,2,2,2,2,2,2,8)^{[3]} \end{array}$$

This leads us to the following conjecture.

Conjecture 10. *The Brunner-Sidki-Vieira group BSV satisfies*

$$\gamma_i(BSV)/\gamma_{i+1}(BSV) = \left\{ \begin{array}{ll} (2^{[2k]}, 8) & \text{if } i \in \{3\cdot 2^k+1,\dots,4\cdot 2^k\}, \\ (2^{[2k]}, 4, 8) & \text{if } i \in \{4\cdot 2^k+1,\dots,5\cdot 2^k\}, \\ (2^{[2k+1]}, 8) & \text{if } i \in \{5\cdot 2^k+1,\dots,6\cdot 2^k\}, \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

For the Basilica group Δ we computed $\gamma_i(\Delta)/\gamma_{i+1}(\Delta)$ for $1 \le i \le 48$ and obtained the following abelian invariants:

$$\begin{array}{l} (0,0), (0), (4)^{[2]}, (4,4), (2,4) \\ (2,2,4)^{[2]}, (2,2,2,4), (2,2,2,2,4)^{[2]}, (2,2,2,4), \\ (2,2,2,2,4)^{[4]}, (2,2,2,2,2,4)^{[2]}, (2,2,2,2,2,2,4)^{[4]}, (2,2,2,2,2,4)^{[2]}, \\ (2,2,2,2,2,2,4)^{[8]}, (2,2,2,2,2,2,2,4)^{[4]}, (2,2,2,2,2,2,2,2,4)^{[8]}, (2,2,2,2,2,2,2,4)^{[4]}, \end{array}$$

This induces the following conjecture.

Conjecture 11. *The Basilica group Δ satisfies*

$$\gamma_i(\Delta)/\gamma_{i+1}(\Delta) = \left\{ \begin{array}{ll} (2^{[2k+2]}, 4) & \text{if } i \in \{6 \cdot 2^k + 1, \ldots, 8 \cdot 2^k\}, \\ (2^{[2k+3]}, 4) & \text{if } i \in \{8 \cdot 2^k + 1, \ldots, 9 \cdot 2^k\}, \\ (2^{[2k+4]}, 4) & \text{if } i \in \{9 \cdot 2^k + 1, \ldots, 11 \cdot 2^k\}, \\ (2^{[2k+3]}, 4) & \text{if } i \in \{11 \cdot 2^k + 1, \ldots, 12 \cdot 2^k\}, \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

Baumslag's group B, which is infinitely presented but has trivial multiplicator [10], and the Lamplighter group $L = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ are both known to be metabelian. This yields that their lower central series patterns can be deduced theoretically [31]. We include the abelian invariants of $\gamma_i(*)/\gamma_{i+1}(*)$ for these two groups as far as we computed them for completeness:

$$\text{for } B\colon\ (3,0), (3)^{[10]} \qquad \text{for } L\colon\ (2,0), (2)^{[8]}.$$

9 Generalized Fabrykowski-Gupta groups

Recall that a self-similar group G acts on the set of words X^* over an alphabet X so that it preserves the length and 'prefix' relation on X^*; that is, for every $x \in X$ and every $g \in G$ there are $y \in X$ and $h \in G$ so that

$$g(xw) = yh(w), \qquad \text{for all } w \in X^*. \tag{1}$$

We will capture the data in equation (1) defining a self-similar group as follows: $(y, h) = \Psi(g, x)$ for some function $\Psi\colon G \times X \to X \times G$. Clearly, it suffices to specify Ψ on $S \times X$ for some generating set S of G.

This construction is used in [16, Example 5.1] to define a group Γ_d for every integer $d \geq 3$ as follows. Let $X = \mathbb{Z}/d\mathbb{Z} = \{0, \ldots, d-1\}$ be an alphabet and let $\Gamma_d = \langle a, r \rangle$ act on X^* via

$$\Psi(a, x) = (x+1, 1), \quad \Psi(r, d-1) = (d-1, r), \quad \Psi(r, 0) = (0, a), \quad \Psi(r, x) = (x, 1) \text{ else.}$$

Then $\Gamma_d = \langle a, r \rangle$ is called the *generalized Fabrykowski-Gupta group*. In the case $d = 3$ it is the Fabrykowski-Gupta group, which was shown in [12, 13] to be a group of intermediate word growth.

In this section we investigate the lower central series structure of the groups Γ_d. For this purpose we first construct finite ascending L-presentations for them. Then we apply our algorithm to these L-presentations. As a result we find that these groups exhibit a very interesting lower central series structure if d is a prime-power and they have a maximal nilpotent quotient otherwise.

9.1 Finite ascending L-presentations for Γ_d

A systematic construction of finite L-presentations for all self-similar groups arising as the iterated monodromy group of a quadratic, complex polynomial is given in [7]. The determination of finite ascending L-presentations for the groups Γ_d presented here mainly relies on [1, Theorem 3.1]. It generalizes the case $d = 3$ described in that paper, correcting at the same time typographical and/or copying errors. The obtained L-presentations are accessible in the forthcoming GAP package FR, see [4].

Theorem 12. *For any $d \geq 3$, the generalized Fabrykowski-Gupta group Γ_d admits a finite ascending L-presentation with generators α, ρ. Writing $\sigma_i = \rho^{\alpha^i}$ for $0 \leq i < d$ and reading indices modulo d, its iterated relations are*

$$R = \left\{ \alpha^d, \left[\sigma_i^{\sigma_{i-1}^n}, \sigma_j^{\sigma_{j-1}^m}\right], \sigma_i^{-\sigma_{i-1}^{n+1}} \sigma_i^{\sigma_{i-1}^n \sigma_{i-1}^{\sigma_{i-2}^m}} \;\middle|\; \begin{array}{c} 1 \leq i, j \leq d \\ 2 \leq |i-j| \leq d-2 \\ 0 \leq m, n \leq d-1 \end{array} \right\},$$

and its only endomorphism φ is defined by $\varphi(\alpha) = \rho^{\alpha^{-1}}$ and $\varphi(\rho) = \rho$. (Note that some relators in R are redundant, since the elements $\sigma_1, \ldots, \sigma_d$ are conjugate. For example, one may fix $i = 1$.)

The proof of this theorem follows the strategy of [1, Theorem 3.1], which proceeds as follows. We first consider the finitely presented group

$$\Gamma = \langle \alpha, \rho \mid \alpha^d, \rho^d \rangle$$

mapping naturally onto Γ_d by 'greek$\mapsto$latin'. We then consider the subgroup $\Delta = \langle \rho^{\alpha^i} : 0 \leq i < d \rangle$ of Γ, and the homomorphism $\Phi : \Delta \to \Gamma \times \cdots \times \Gamma$ with d copies of Γ, defined by

$$\Phi(\rho^{\alpha^i}) = (1, \ldots, \rho, \alpha, \ldots, 1) \text{ with the } \rho \text{ at position } i.$$

We compute a presentation of $\Phi(\Delta)$; the kernel of Φ is generated by the set R of Φ-preimages of relators in that presentation. Finally, we seek a section $\varphi : \Gamma \to \Delta$ of the projection of Φ on its first coordinate. We then have, for all $x \in \Gamma$,

$$\Phi(\varphi(x)) = (x, *, \ldots, *),$$

where the $*$ stand for unimportant elements of Γ. The following result now allows to read off a finite ascending L-presentation for Γ_d.

Scholium 13. *An L-presentation of Γ_d is given by generators α, ρ; endomorphism φ; and iterated relations R.*

In the remainder of this section, we apply this strategy to determine a finite L-presentation for Γ_d and thus prove Theorem 12. A presentation of $\Phi(\Delta)$ can be determined by the Reidemeister-Schreier method. Consider first the presentation

$$\Pi = \langle \alpha_1, \ldots, \alpha_d, \rho_1, \ldots, \rho_d \mid \alpha_i^d, \rho_i^d, [\alpha_i, \alpha_j], [\alpha_i, \rho_j], [\rho_i, \rho_j] \text{ for } i \neq j \rangle;$$

this is a presentation of $\Gamma \times \cdots \times \Gamma$, and $\Phi(\Delta)$ is the subgroup $\langle \sigma_i := \rho_i \alpha_{i+1} \rangle$. Here and below indices are all treated modulo d. We rewrite this presentation as

$$\Pi = \langle \alpha_1, \ldots, \alpha_d, \sigma_1, \ldots, \sigma_d | \alpha_i^d, \sigma_i^d, [\alpha_i, \alpha_j], [\alpha_i, \sigma_j], [\sigma_i \alpha_{i+1}^{-1}, \sigma_j \alpha_{j+1}^{-1}] \text{ for } i \neq j \rangle.$$

Next we rewrite the last set of relations either as $[\sigma_i, \sigma_j]$, if $2 \leq |i-j| \leq d-2$, or as $\sigma_i^{\alpha_i} = \sigma_i^{\sigma_{i-1}}$, in the other cases.

We choose as Schreier transversal all d^d elements $\alpha_1^{n_1} \cdots \alpha_d^{n_d}$. The Schreier generating set easily reduces to $\{\sigma_{i,n} := \sigma_i^{\alpha_i^n}\}$. The Schreier relations are all $[\sigma_{i,n}, \sigma_{j,m}]$ for $2 \leq |i-j| \leq d-2$, all $\sigma_{i,n}^d$, and all $\sigma_{i,n+1} = \sigma_{i,n}^{\sigma_{i-1,m}}$.

In particular, we can use this last relation (with $m = 0$) to eliminate all generators $\sigma_{i,n}$ with $n \neq 0$, replacing them by $\sigma_i^{\sigma_{i-1}^n}$. We obtain $\Phi(\Delta) = \langle \sigma_1, \dots, \sigma_d | \sigma_1^d, \dots, \sigma_d^d, R \rangle$, with

$$R = \left\{ \left[\sigma_i^{\sigma_{i-1}^n}, \sigma_j^{\sigma_{j-1}^m} \right] \text{ whenever } 2 \leq |i-j| \leq d-2, \text{ and } \sigma_i^{-\sigma_{i-1}^{n+1}} \sigma_i^{\sigma_{i-1}^n \sigma_{i-1}^{\sigma_{i-2}^m}} \text{ for all } i \right\}.$$

Note that φ satisfies $\Phi(\varphi(x)) = (x, \alpha^*, \dots, \alpha^*)$ for all $x \in \Gamma$, where the $*$ are unimportant integers, and thus clearly induces a monomorphism of Γ_d. In this way, we obtain the L-presentation of Theorem 12 for Γ_d.

9.2 The lower central series structure of Γ_d

Table 2 summarizes the results of our algorithm applied to Γ_d for some small d. This table has the same format as Table 1 and contains an additional column noting whether our algorithm found a maximal nilpotent quotient.

Group	max quot	class	gens	time (h:min)
Γ_3	no	71	112	1:50
Γ_4	no	66	146	1:55
Γ_5	no	53	60	1:58
Γ_6	yes	3	4	0:00
Γ_7	no	44	50	1:37
Γ_8	no	52	116	1:47
Γ_9	no	58	84	1:54
Γ_{10}	yes	5	6	0:00
Γ_{11}	no	33	35	1:48
Γ_{12}	yes	6	7	0:00
Γ_{14}	yes	7	8	0:00
Γ_{15}	yes	5	6	0:00
Γ_{18}	yes	15	16	0:06
Γ_{20}	yes	6	7	0:02
Γ_{21}	yes	7	8	0:04

Table 2: Fabrykowski-Gupta groups Γ_d for some small d

Table 2 immediately suggests that Γ_d has a maximal nilpotent quotient if and only if d is not a prime-power. Based on this observation we were able to prove the following theorem, proven in Section 9.3.

Theorem 14. *If d is not a prime-power, then Γ_d has a maximal nilpotent quotient, and its nilpotent quotients are isomorphic to the nilpotent quotients of $C_d \wr C_d$, where C_d is the cyclic group of order d.*

Next, we consider the case that d is a prime. For $d = 3$, there is a theoretical description of the lower central series factors of Γ_3, see [3]. Our computations confirm the following theorem.

Theorem 15 ([3, Corollary 3.14]). *The Fabrykowski-Gupta group* Γ_3 *satisfies*

$$\gamma_i(\Gamma_3)/\gamma_{i+1}(\Gamma_3) = \left\{ \begin{array}{cl} (3,3) \text{ or } (3) & \text{if } i = 1 \text{ or } 2 \text{ respectively}, \\ (3,3) & \text{if } i \in \{3^k+2, \dots, 2\cdot 3^k+1\}, \\ (3) & \text{if } i \in \{2\cdot 3^k+2, \dots, 3^{k+1}+1\}, \end{array} \right\} \text{ with } k \in \mathbb{N}_0.$$

For the primes $d = 5, 7, 11$, we list the lower central series factors $\gamma_i(\Gamma_d)/\gamma_{i+1}(\Gamma_d)$ obtained by our algorithm in the following.

- Γ_5: $(5,5), (5)^{[3]}, (5,5), (5)^{[13]}, (5,5)^{[5]}, (5)^{[30]}$.
- Γ_7: $(7,7), (7)^{[5]}, (7,7), (7)^{[33]}, (7,7)^{[4]}$.
- Γ_{11}: $(11,11), (11)^{[9]}, (11,11), (11)^{[22]}$.

Thus, if d is a prime, then the group Γ_d seems to have a very slim lower central series. It seems very likely that these groups exhibit a lower central series pattern similar to that of Γ_3, and we would like to understand and prove this, especially in relation to the classification of groups of "finite width" in [25]. However, for this purpose a larger computed sequence would be helpful. We only formulate the following conjecture.

Conjecture 16. *If d is an odd prime, then Γ_d is a group of width* 2.

Finally, we consider the case that d is a prime-power, say $d = p^n$ with $n > 1$. All the obtained lower central series factors $\gamma_i(\Gamma_d)/\gamma_{i+1}(\Gamma_d)$ are p-groups in this case and, except for some initial entries, they are elementary abelian. Again, we would like to find and prove a general pattern for these factors.

- Γ_4: $(4,4), (4), (2,2)^{[4]}, (2,2,2)^{[3]}, (2,2)^{[13]}, (2,2,2)^{[12]}, (2,2)^{[32]}$.
- Γ_8: $(8,8), (8), (4)^{[4]}, (2,2), (2), (2,2)^{[2]}, (2,2,2), (2,2), (2,2,2)^{[2]}, (2,2,2,2), (2,2,2)^{[8]}, (2,2)^{[23]}, (2,2,2)^{[5]}, (2)$.
- Γ_9: $(9,9), (9)^{[2]}, (3)^{[5]}, (3,3)^{[6]}, (3,3,3), (3,3)^{[17]}, (3)^{[26]}$.

It is possible that these groups Γ_d are all of finite width, but it seems that their width grows with the exponent n.

9.3 Proof of Theorem 14

Let $d \geq 6$ be a composite number. Recall that $\Gamma_d = \langle a, r \rangle$ acts on the free monoid X^* generated by $X = \{0, \dots, d-1\}$. The monoid X^* can be considered as a d-regular tree T where the vertices correspond to the words in X^* and two vertices v and w are connected if either $w = vx$ or $v = wx$ for some $x \in X$. The action of Γ_d on X^* induces an automorphism of T. For a vertex v of the tree T, let $|v|$ denote the distance of v from the root. Then the *n-th level-stabilizer* $\mathrm{Stab}_{\Gamma_d}(n)$ is

$$\mathrm{Stab}_{\Gamma_d}(n) = \bigcap_{|v|=n} \mathrm{Stab}_{\Gamma_d}(v).$$

Since the full subtree T_v of T with root v is isomorphic to the tree T, there is an embedding, called the *decomposition*,

$$\phi\colon \mathrm{Stab}_{\Gamma_d}(1) \to \mathrm{Aut}(T) \times \cdots \times \mathrm{Aut}(T), \quad h \mapsto (h@0, \dots, h@d-1),$$

where $h@v$ denotes the action of h restricted to T_v. It is easy to see that the action of Γ_d on T yields that $\mathrm{Stab}_{\Gamma_d}(1)$ embeds into $\Gamma_d \times \cdots \times \Gamma_d$ so that the projection to each coordinate is onto. In the following, we identify elements of Γ_d with their image under ϕ. For instance, the generator r is contained in $\mathrm{Stab}_{\Gamma_d}(1)$ and has the decomposition

$$(a, 1, \dots, 1, r).$$

In [16, Example 5.1] it was shown that Γ_d is a *regular branch group over* Γ'_d; that is, the group Γ_d acts transitively on each level of T, the subgroup Γ'_d has finite index in Γ_d and satisfies $\phi(\Gamma'_d) \geq \Gamma'_d \times \cdots \times \Gamma'_d$. The following scholium follows from [16, Theorem 4]; see also [6, Proposition 3.8 and Proposition 3.9].

Scholium 17. *Consider $N \trianglelefteq \Gamma_d$. If N is not contained in $\mathrm{Stab}_{\Gamma_d}(m)$ for some $m \in \mathbb{N}$, then N contains $\mathrm{Stab}_{\Gamma_d}(m+3)$.*

Proof. Since N is not contained in $\mathrm{Stab}_{\Gamma_d}(m)$, there exists an element $g \in N$ which acts non-trivially on the m-th level $X^m = \{x \in X^* \mid |x| = m\}$ of T. Let $u \in X^m$ be such that $u^g \neq u$. Let $k \in \Gamma'_d$ be given. As Γ_d is a regular branch group over Γ'_d, there exists an element $u * k \in \Gamma'_d$ which acts on T_u as k and trivially elsewhere; that is, $u * k$ has the m-fold iterated decomposition

$$(1, \dots, 1, k, 1, \dots, 1), \text{ with } k \text{ at position } u.$$

Consider the element $g' = [g, u * k] \in N$. Then g' acts on T_u as k, it acts on T_{u^g} as a conjugate k' of k^{-1}, and it acts trivially elsewhere. Thus $g' \in N$ has the m-fold iterated decomposition

$$(1, \dots, 1, k, 1, \dots, 1, k', 1, \dots, 1)$$

with k and k' at position u and u^g, respectively. Let $\ell \in \Gamma'_d$ be given and let $u * \ell$ be as above. Then $[g', u * \ell] \in N$ and it has the m-fold iterated decomposition

$$(1, \dots, 1, [k, \ell], 1, \dots, 1), \text{ with } [k, \ell] \text{ at position } u.$$

As $\Gamma''_d = \langle [k, \ell] \mid k, \ell \in \Gamma'_d \rangle$ and Γ_d acts transitively on each level of T, the subgroup N contains the product $\Gamma''_d \times \cdots \times \Gamma''_d$ of d^m copies of Γ''_d.

It is clear that Γ'_d contains $\mathrm{Stab}_{\Gamma_d}(2)$. As described in [16, Example 7.1], the subgroup Γ''_d contains $\Gamma'_d \times \cdots \times \Gamma'_d$ with d copies of Γ'_d. Since Γ'_d contains $\mathrm{Stab}_{\Gamma_d}(2)$, we conclude that Γ''_d contains $\mathrm{Stab}_{\Gamma_d}(3)$. Thus N contains $\mathrm{Stab}_{\Gamma_d}(m+3)$. ☐

This information can now be used to prove a first approximation to Theorem 14 as follows. The Fabrykowski-Gupta group Γ_d maps onto $C_d \wr C_d$ by restricting the action of Γ_d to X^2. The wreath product $C_d \wr C_d$ is not nilpotent, since d is not a prime-power, see [9]. Hence, for every $n \in \mathbb{N}$, the subgroup $\gamma_n(\Gamma_d)$ is not contained in $\mathrm{Stab}_{\Gamma_d}(2)$, and thus $\gamma_n(\Gamma_d)$ contains $\mathrm{Stab}_{\Gamma_d}(5)$. As $\mathrm{Stab}_{\Gamma_d}(5)$ has finite index in Γ_d, it follows that Γ_d has a maximal nilpotent quotient, namely the maximal nilpotent quotient of $\Gamma_d/\mathrm{Stab}_{\Gamma_d}(5)$.

Theorem 18. *If d is not a prime-power, then $\gamma_k(\Gamma_d)$ contains $\mathrm{Stab}_{\Gamma_d}(2)$ for any $k \geq 1$. In particular, the nilpotent quotients of Γ_d and $C_d \wr C_d$ are isomorphic.*

Proof. It is easy to see that $\mathrm{Stab}_{\Gamma_d}(2) = \Gamma'_d \times \cdots \times \Gamma'_d$. Let $q > 1$ be a prime-power p^μ such that $d = qe$ and $(q, e) = 1$. We will show by induction on k that $\gamma_k(\Gamma_d)$ contains $\mathrm{Stab}_{\Gamma_d}(2)$ and an element which has the decomposition

$$(a^{\alpha_0} r^{\beta_0}, a^{\alpha_1} r^{\beta_1}, \ldots, a^{\alpha_{d-1}} r^{\beta_{d-1}}), \tag{2}$$

with $(\beta_0, q) = 1$ and $\alpha_{d-1} = 0$. This is clearly true for $k = 1$.

Assume that $\gamma_k(\Gamma_d)$ contains $\mathrm{Stab}_{\Gamma_d}(2) = \Gamma'_d \times \cdots \times \Gamma'_d$. For $\ell \in \mathbb{N}$, consider the ℓ-fold iterated commutator $g_{q,\ell} = [r^e, {}_\ell a]$. Then $g_{q,\ell}$ is contained in $\gamma_{\ell+1}(\Gamma_d)$ and it decomposes, modulo an element of $\Gamma'_d \times \cdots \times \Gamma'_d$, as in (2) with $\beta_i = \alpha_{i+1 \pmod d}$ and the α_i's are the coefficients of $e(X-1)^\ell \pmod{(d, X^d - 1)}$ written as $\alpha_0 X^0 + \cdots + \alpha_{d-1} X^{d-1}$.

Suppose that $\ell \geq k$ is a prime-power $p^\nu > q$. Then the coefficient of X^i in the polynomial $e\,(X-1)^{p^\nu}$ vanishes modulo d whenever i is not a multiple of p. The coefficients of X^0 and X^{p^ν} are $\pm e$ and e, respectively, while all other coefficients are divisible by p. Note that the coefficients of X^0 and X^{p^ν} do not cancel modulo $X^d - 1$ as d is not a prime-power. In particular, we have that either $(\alpha_0, q) = 1$ while α_{d-2} vanishes or $(\alpha_{p^\nu \pmod d}, q) = 1$ while $\alpha_{p^\nu - 2 \pmod d}$ vanishes. Up to a cyclic permutation of indices, implemented by conjugation by a, the decomposition of $g_{q,\ell}$ modulo $\Gamma'_d \times \cdots \times \Gamma'_d$ has the form as in (2).

Hence, for any $k \geq 1$, the lower central series term $\gamma_k(\Gamma_d)$ contains an element g'_q which decomposes as in (2) so that $(\beta_0, q) = 1$ and $\alpha_{d-1} = 0$. Therefore, $\gamma_{k+1}(\Gamma_d)$ contains the element $[g'_q, r]$ which has a decomposition of the form

$$([r^{\beta_0}, a], 1, \ldots, 1).$$

By induction, $\gamma_k(\Gamma_d)$ contains $\Gamma'_d \times \cdots \times \Gamma'_d$ and therefore, $\gamma_{k+1}(\Gamma_d) = [\gamma_k(\Gamma_d), \Gamma_d]$ contains $\gamma_3(\Gamma_d) \times 1 \times \cdots \times 1$. Thus, in particular, $\gamma_{k+1}(\Gamma_d)$ contains an element which decomposes as $([r, a]^{\beta_0}, 1, \ldots, 1)$. Since q was arbitrarily chosen, $\gamma_{k+1}(\Gamma_d)$ contains for all $q|d$ an element of the form $([a, r]^{n_q}, 1, \ldots, 1)$, such that $\gcd\{n_q\} = 1$. Hence, $\gamma_{k+1}(\Gamma_d)$ contains $([a, r], 1, \ldots, 1)$ and so it contains $\Gamma'_d \times \cdots \times \Gamma'_d = \mathrm{Stab}_{\Gamma_d}(2)$. □

The class $cl(d)$ of the maximal nilpotent quotient of $C_d \wr C_d$ was determined in [33]: If $d = p_1^{\beta_1} \cdots p_n^{\beta_n}$ is the prime-power factorization, then $cl(d) = \max_{1 \leq i \leq n} cl(p_i^\beta)$. A formula for the class of $C_{p^\beta} \wr C_{p^\beta}$ for p a prime was determined in [26].

10 Generalized Gupta-Sidki groups

The Gupta-Sidki group has originally been introduced in [21] and has become famous for its role in connection with the Burnside problems. In this section we describe a generalization of this group to a family of groups GS_p for all odd primes p: we define $GS_p = \langle a, t \rangle$ by its action on X^* via

$$\Psi(a, x) = (x + 1, 1), \qquad \Psi(t, 0) = (0, t), \quad \Psi(t, x) = (x, a^x) \text{ if } x > 0.$$

If $p = 3$, this is the original Gupta-Sidki group, which was shown in [20] to be an infinite, finitely generated, 3-torsion group. In the following we first construct finite non-invariant L-presentations for the groups GS_p. Then we investigate these with our algorithm. We currently do not know any finite, invariant L-presentations for these groups; knowing them would greatly speed up computations.

10.1 Finite non-invariant L-presentations for GS_p

With a similar notation as in Section 9.1, we consider $\Gamma = \langle \alpha, \tau | \alpha^p, \tau^p \rangle$, the normal closure Δ of τ in GS_p, and the map $\Phi : \Delta \to \Gamma \times \cdots \times \Gamma$, with p copies of Γ, defined by

$$\Phi(\tau^{\alpha^i}) = (\dots, \alpha^{p-1}, \tau, \alpha, \alpha^2, \dots) \text{ with the } \tau \text{ at position } i.$$

In the group

$$\Gamma \times \cdots \times \Gamma = \Pi = \langle \alpha_1, \dots, \alpha_p, \tau_1, \dots, \tau_p | \alpha_i^p, \tau_i^p, [\alpha_i, \alpha_j], [\alpha_i, \tau_j], [\tau_i, \tau_j] \text{ for } i \neq j \rangle,$$

we consider now the subgroup $\Phi(\Delta) = \langle \sigma_i := \tau_i \alpha_{i+1} \cdots \alpha_{i+k}^k \cdots \alpha_{i-1}^{-1} \rangle$. We rewrite the presentation of Π as

$$\Pi = \langle \alpha_1, \dots, \alpha_p, \sigma_1, \dots, \sigma_p | \alpha_i^p, \sigma_i^p, [\alpha_i, \alpha_j], [\alpha_i, \sigma_j], [\sigma_i \alpha_j^{j-i}, \sigma_j \alpha_i^{i-j}] \text{ for } i \neq j \rangle.$$

We choose as Schreier transversal all p^p elements $\alpha_1^{n_1} \cdots \alpha_p^{n_p}$. The Schreier generating set easily reduces to $\{\sigma_{i,n} := \sigma_i^{\alpha_i^n}\}$. The Schreier relations become $\sigma_{i,m+i}^{-1} \sigma_{j,n+i}^{-1} \sigma_{i,m+j} \sigma_{j,n+j}$ and $\sigma_{i,n}^p$. Furthermore, an easy calculation gives

$$\left[\sigma_i^{(j-k)e} \sigma_j^{(k-i)e}, \sigma_k^{(i-j)e} \sigma_i^{(j-k)e}\right] = \sigma_{i,(j-i)(i-k)e}^{-2(j-k)e} \sigma_i^{2(j-k)e}. \tag{3}$$

For all $\ell > 0$, we may choose arbitrarily j, k such that i, j, k are all distinct and $(j-i)(i-k)/2(j-k) \equiv \ell \pmod p$, and use equation (4) to express $\sigma_{i,\ell}$ in terms of $\sigma_i, \sigma_j, \sigma_k$, namely

$$\sigma_{i,\ell} = \sigma_i \left[\sigma_i^{1/2} \sigma_j^{(k-i)/2(j-k)}, \sigma_k^{(i-j)/2(j-k)} \sigma_i^{1/2}\right]^{-1}. \tag{4}$$

Finally, we may also use equation (4) to construct an endomorphism φ; we summarize:

Theorem 19. *The subgroup $D_p = \langle t \rangle^G$ of the Gupta-Sidki p-group admits a finite ascending L-presentation with generators $\sigma_1, \dots, \sigma_p$ generating a free group Δ; iterated relations*

$$R = \left\{\sigma_i^p;\ \sigma_{i,m+i}^{-1} \sigma_{j,n+i}^{-1} \sigma_{i,m+j} \sigma_{j,n+j}\right\};$$

and an endomorphism $\varphi : \Delta \to \Delta$, defined by

$$\varphi(\sigma_i) = \sigma_{1,i} \text{ as given in equation (4).}$$

It is not possible to extend φ to an endomorphism of Γ. However, the extension of a finitely L-presented group by a finite group is again finitely L-presented; in the present case, it is a simple matter to construct a finite L-presentation for the split extension $GS_p = D_p \rtimes_\zeta \mathbb{Z}/p\mathbb{Z}$, where the automorphism ζ of D_p cyclically permutes the generators, from the L-presentation of D_p.

10.2 The lower central series structure of GS_p

As a preliminary step, we discuss two different strategies to determine nilpotent quotients of GS_p with our algorithm. First, we can apply our algorithm to the non-invariant L-presentation of GS_p. This is straightforward, but usually yields only very limited results, as our algorithm is not effective on non-invariant L-presentations.

For a second, more effective approach we use the structure of GS_p as exhibited in Section 10.1. Every GS_p is of the form $GS_p \cong D_p \rtimes C_p$, where D_p is generated by $\{\sigma_1, \ldots, \sigma_p\}$ and the cyclic group $C_p = \langle \alpha \rangle$ acts by permuting these elements cyclically. An ascending L-presentation for D_p was determined in Section 10.1. Now we can apply our algorithm to the ascending L-presentation of D_p and determine $D_p/\gamma_c(D_p)$ for some c. Then, defining $H_p = D_p/\gamma_c(D_p) \rtimes C_p$, we obtain $GS_p/\gamma_i(GS_p) \cong H_p/\gamma_i(H_p)$ for all $i \leq c$.

Table 3 summarizes runtimes and a brief overview on the results of our algorithm applied to GS_p for $p = 3, 5, 7$. The table uses the same notation as Table 1. Instead of a column 'prop' it has a column 'strategy' which lists the used strategy and hence also determines whether our algorithm was applied to an ascending or non-invariant L-presentation. Note that we applied the nilpotent quotient algorithm for 2 hours in all cases. Thus the runtimes for GS_3 with strategy 1 show that the first 5 quotients are fast to obtain, while the sixth quotient takes over 2 hours and hence did not complete. Further, Table 3 shows that strategy 2 is more successful on GS_3 than strategy 1; a feature that we also observed for other GS_p.

Group	strategy	class	gens	time (h:min)
GS_3	1	5	8 (215)	0:02
GS_3	2	25	51	1:44
GS_5	2	9	22	1:09
GS_7	2	6	13	0:59

Table 3: The Gupta-Sidki groups GS_p for some small primes p

Next, we discuss the obtained results for the lower central series of GS_p and H_p in more detail. Our computational results for GS_3 agree with the following theoretical description of $\gamma_i(GS_3)/\gamma_{i+1}(GS_3)$ from [3].

Theorem 20 ([3, Corollary 3.9]). *Set $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2}$ for $n \geq 3$. Then, for $n \geq 2$, the rank of $\gamma_n(GS_3)/\gamma_{n+1}(GS_3)$ is the number of ways of writing $n-1$ as a sum $k_1\alpha_1 + \cdots + k_t\alpha_t$ with all $k_i \in \{0, 1, 2\}$.*

For $p > 3$ a prime number, no theoretical description of the lower central series factors of GS_p is available; though we are interested in obtaining one. In the following we outline our computed results for the ranks of $\gamma_i(H_p)/\gamma_{i+1}(H_p)$. These are isomorphic to $\gamma_i(GS_p)/\gamma_{i+1}(GS_p)$ for all $i \leq c$, where c is the class listed in Table 3, and they are epimorphic images if $i > c$. This position is indicated by a bar | in the list below; therefore, the values before the bar are proven while those after it are conjectural. Since the quotients are elementary abelian p-groups, only the rank of the quotient is given in the following list:

- H_5: $2, 1, 2^{[2]}, 3, 2, 3^{[2]}, 4 \mid 4^{[3]}, 3^{[3]}, 4^{[4]}, 3, 4^{[2]}, 6^{[3]}, 5, 4, 2^{[3]}, 1^{[3]}$.

- H_7: $2, 1, 2^{[2]}, 3^{[2]}, 4 \ \mid \ 3, 4^{[2]}, 5^{[6]}, 4, 3^{[5]}, 2^{[3]}, 1^{[2]}$.

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