Metabelian $p$-groups and coclass theory

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Abstract

We show that the coclass tree associated with the metabelian $p$-groups of a fixed coclass is virtually periodic. The proof of this result is obtained by showing that the metabelian $p$-groups of a fixed coclass consist of finitely many coclass families and finitely many other groups. This also implies that the metabelian $p$-groups of a fixed coclass can be classified in the sense that they can be described by finitely many parametrised presentations.

1 Introduction

The coclass of a finite $p$-group $G$ of order $p^n$ and class $c$ is $cc(G) = n - c$. Coclass theory suggests that one might classify and investigate finite $p$-groups by using the coclass as primary invariant. This general approach was first suggested by Leedham-Green and Newman [15]. For background and details we refer to the book by Leedham-Green and McKay [14].

A major goal of coclass theory is to show that the isomorphism types of finite $p$-groups of a fixed coclass can be classified in the sense that they can be described by a finite set of data. A more explicit conjectural description of this major goal is given in [11]. This major goal has been achieved independently by du Sautoy [5] and by Eick and Leedham-Green [10] for the prime 2, but it is still open for odd primes.

Eick and Leedham-Green [10] introduced a construction for certain infinite families of $p$-groups of coclass $r$: the coclass families. (See Section 4 for a definition.) These families underpin the main results in [10] and they have further significant applications in $p$-group theory; see for example [9] or [7]. Our first result here is the following, see Section 4.

1 Theorem: For every prime $p$ and every $r \in \mathbb{N}$, the isomorphism types of the metabelian $p$-groups of coclass $r$ consist of finitely many coclass families and finitely many other groups.

We list some applications of Theorem 1. Let $\mathcal{M}(p,r)$ denote the graph associated with the metabelian $p$-groups of coclass $r$. (See Section 3 for a definition.) A coclass tree $\mathcal{M}$ in $\mathcal{M}(p,r)$ is an infinite subtree of $\mathcal{M}(p,r)$ which has exactly one infinite path starting at its root. Similar to the graph associated with all $p$-groups of coclass $r$, the subgraph
M(p, r) consists of finitely many coclass trees and finitely many other groups. We describe the infinite paths in M(p, r) via their corresponding pro-p-groups. For a coclass tree M in M(p, r) denote with M_i its full subtree consisting of all edges which are descendants of the i-th edge on the maximal infinite path in M. The following is an application of Theorem 1, see Section 4.

2 Corollary: For every prime p and every r ∈ N, the graph M(p, r) is virtually periodic; that is, for every coclass tree M in M(p, r) there exist d, l ∈ N so that M_i ∼= M_{i+d}.

It is proved in [10, Section 9] that each coclass family can be defined by a single parametrised presentation; that is, given a coclass family (G_0, G_1, ...), there exists a parametrised presentation P(x) so that G_i is defined by P(i) for each i ∈ N_0. The presentation P(x) can be obtained as follows. Associated to the coclass family (G_0, G_1, ...) there is a finite p-group R acting on a module T ∼= Z_p^d, where Z_p denotes the p-adic integers, and an e ∈ N_0 so that G_i is an extension of R by M_i = T/p^{i+e}T via a cocycle γ_i. Thus P(x) can be obtained as a presentation for the extension of R with M_x via γ_x. Then P(x) has a fixed finite number of generators. Its relations incorporate the parameter x, since M_x and also γ_x depend on x. Further, the presentation P(x) can be defined so that each evaluated presentation P(i) is a consistent polycyclic presentation. Theorem 1 and Corollary 11 of [10] have the following application.

3 Corollary: Let p be an arbitrary prime and let r ∈ N. Then up to isomorphism, the metabelian p-groups of coclass r can be described by finitely many consistent parametrised presentations.

Thus it is possible to classify the metabelian p-groups of coclass r for any given p and r. The methods of [10] together with the results here would allow one to determine a finite set of consistent parametrised presentations for any given p and r; However, this will only be practical for a few very small cases of p and r.

Miech [17, 18, 19] introduced a classification of the metabelian p-groups of coclass 1. For this purpose Miech determined a certain type of presentation for these groups and then solved the isomorphism problem using these presentations. Corollary 3 can be considered as a weak extension of Miech’s result to arbitrary coclass.

Nebelung [20] exhibited an explicit classification of the finite metabelian 3-groups with commutator factor of type (3, 3). In Section 5 we show how our methods can be used to gain insight into these groups. Further, in Section 5 we show how theoretical and computational tools can be combined to investigate the graphs M(3, 2), M(3, 3) and M(5, 2).

We note that the classification of metabelian p-groups has applications in number theory, see for example [16] and [20]. In fact, this article has been initiated by questions of Mayer on finite metabelian p-groups in M(3, 3) and M(5, 2).

2 Infinite pro-p-groups

An infinite pro-p-group S has finite coclass r if its lower central series quotients S/γ_i(S) are finite p-groups and lim_{i→∞} cc(S/γ_i(S)) = r. In this section we first recall some of the
Let $W$ be an infinite pro-$p$-group of coclass $r$ with a hypercenter $h(S)$ of order $p^s$. Then $S/h(S) \cong W_k$ for $k = r - s$.

**Proof:** Consider the natural epimorphism $\varphi : S \to S/h(S)$. The image $H$ of this epimorphism is a uniserial $p$-adic space group of coclass $k = r - s$ by Lemma 7.4.4 and Theorem 7.4.12 of [14]. It remains to show that $H \cong W_k$. As $S$ is metabelian, the group $H$ is metabelian and $H'$ is abelian. Corollary 7.4.5 of [14] asserts that the uniserial $p$-adic space group $H$ has a maximal abelian normal subgroup $N$ with $N \cong \mathbb{Z}_p^l$ for some $l \in \mathbb{N}$. As $H'$ is abelian, it follows that $H' \leq N$. Thus $P = H/N$ is abelian. As observed in Prop. 10.1.13 of [14] this shows that $P$ is cyclic. By Corollary 9.4.6 of [14] it follows that $H^2(P, N) = \{0\}$. Thus $H$ is a split extension $H \cong N \rtimes P$. Prop. 10.5.1 of [14] now proves that $cc(H) = k = \log_p(|P|)$ and thus $P$ is cyclic of order $p^k$. Using Prop. 10.1.13 of [14] again, we find that the module structure for $N$ as $\mathbb{Z}_pP$-module is uniquely determined and the rank of $N$ as $\mathbb{Z}_p$-module is $l = (p-1)p^{k-1} = d(k)$. In summary, this shows that $H \cong W_k$. 

4. Theorem

2.1 The general case

A detailed structure analysis of the infinite pro-$p$-groups of finite coclass is given in [14]. We recall some of their main features here.

Let $\mathbb{Z}_p$ denote the $p$-adic integers. A group $S$ is called a uniserial $p$-adic pre-space group if there exists a normal subgroup $T$ in $S$ with $S/T$ finite so that $T \cong \mathbb{Z}_p^d$ for some $d \in \mathbb{N}$ and the series $T = T_0 > T_1 > \ldots$ defined by $T_{i+1} = [T_i, S]$ has quotients of order $p$ only.

In this case the integer $d$ is called the dimension of $S$ and the subgroup $T$ is a translation subgroup with point group $S/T$. The group $S$ is called a uniserial $p$-adic space group if it is a uniserial $p$-adic pre-space group and has a trivial centre. In this case there exists a unique maximal translation subgroup $T$ in $S$.

Let $S$ be an infinite pro-$p$-group of finite coclass $r$. Then $S$ is a $p$-adic pre-space group of some dimension $d$. If $h(S)$ denotes the hypercenter of $S$, then $h(S)$ is a finite subgroup of $S$ of order $p^s$ with $s < r$ and $S/h(S)$ is an infinite pro-$p$-group of coclass $r - s$. The quotient $S/h(S)$ is a uniserial $p$-adic space group of the same dimension $d$ as $S$.

2.2 Infinite metabelian pro-$p$-groups

Examples of uniserial $p$-adic space groups which are metabelian can be constructed as follows. For $k \in \mathbb{N}$ let $q = p^k$, let $d = d(k) = (p - 1)p^{k-1}$ and let $T = T(k) = \mathbb{Z}_p^d$. Then the cyclic group $C_q$ of order $q$ acts faithfully on $T$ as primitive $q$-th root of unity. Let $W_k = T \times C_q$ the split extension of $T$ by $C_q$. Note that $T$ is uniserial as $C_q$-module. Thus $W_k$ is a uniserial $p$-adic space group and thus, by [14, Prop. 10.5.1], an infinite pro-$p$-group of finite coclass $k$. The group $W_k$ is metabelian with $W'_k = [T, W_k] = T_1$ and commutator quotient $W_k/W'_k \cong C_q \times C_p$. The following theorem shows that these example groups play a significant role for all metabelian infinite pro-$p$-groups of finite coclass.
Theorem 4 implies the following result on the derived quotient of an infinite metabelian pro-$p$-group of coclass $r$.

**5 Corollary:** Let $S$ be an infinite metabelian pro-$p$-group of coclass $r$ whose hypercenter $h(S)$ has order $p^a$. Let $k = r - s$ and $q = p^k$. Then $S/S'$ has the quotient $C_q \times C_p$. Thus

$$p^{r+1} \geq |S/S'| \geq p^{k+1} \quad \text{and} \quad s + 2 \geq rk(S/S') \geq 2 \quad \text{and} \quad p^r \geq \exp(S/S') \geq p^k.$$  

The number of isomorphism types of infinite pro-$p$-groups of fixed coclass is finite, see Conjecture D and its proof in [14]. Asymptotic bounds for the number of isomorphism types of pro-$p$-groups of coclass $r$ are exhibited in [6]. Thus the number $f(p, r, s)$ of isomorphism types of infinite metabelian pro-$p$-groups of coclass $r$ whose hypercenter has order $p^3$ is also finite. Upper and lower bounds for it are given in the following.

**6 Theorem:** The number $f(p, r, s)$ of isomorphism types of infinite metabelian pro-$p$-groups of coclass $r$ whose hypercenter has order $p^a$ is bounded by

$$\frac{2a(s)}{27} s^3 \leq \log_p f(p, r, s) \leq \frac{b_1(s)}{2} ds^2 + \frac{b_2(s)}{3} s^3,$$

where $d = (p - 1)p^{k-1}$ for $k = r - s$ and $a(s), b_1(s), b_2(s)$ are functions in $s$ which tend to 1 for $s \to \infty$.

**Proof:** First consider the lower bound. If $G$ is an arbitrary metabelian group of order $p^a$, then $W_k \times G$ is an infinite metabelian pro-$p$-group of coclass $r = s + k$ with hypercenter $G$ of order $p^a$. Hence $f(p, k, s)$ is bounded below by the number of isomorphism types of finite metabelian groups of order $p^a$. Higman [12] proved that there are at least $p^{\frac{2}{27}(s^3 - 6s^2)}$ isomorphism types of finite class-2 groups of order $p^a$. This yields the lower bound with $a(s) = (1 - \frac{6}{s})$.

It remains to consider the upper bound. Let $E$ be an infinite metabelian pro-$p$-group of coclass $r$ whose hypercenter has order $p^a$ for some $s > 0$. Then $E$ has a central subgroup $A$ of order $p$ and $E/A$ is an infinite metabelian pro-$p$-group of coclass $r - 1$ whose hypercenter has order $p^{a-1}$. Hence we can consider $E$ as a metabelian extension of $G \cong E/A$ by $A \cong C_p$. An upper bound for the number of isomorphism types of arbitrary extensions of $G$ by $C_p$ is $|H^2(G, C_p)|$. Let $Ext(G', C_p)$ denote the subgroup of $H^2(G', C_p)$ consisting of those cocycle classes which correspond to abelian extensions. Then the metabelian extensions of $G$ by $C_p$ correspond to those cocycle classes in $H^2(G, C_p)$ whose images under the restriction $\rho : H^2(G, C_p) \to H^2(G', C_p)$ are contained in $Ext(G', C_p)$. Let $H^2(G, C_p)$ denote the full preimage of $Ext(G', C_p)$ under $\rho$. Then $H^2(G, C_p)$ is an elementary abelian $p$-group and its order is an upper bound for the number of isomorphism types of metabelian extensions of $G$ by $C_p$.

We determine an upper bound for the rank $c$ of $H^2(G, C_p)$ using the Lyndon-Hochschild-Serre spectral sequence with the normal subgroup $G'$:

$$c \leq rk(H^2(G/G', H^0(G', C_p))) + rk(H^1(G/G', H^1(G', C_p))) + rk(Ext(G', C_p)).$$
To analyse the three summands in this formula, we consider the structure of $G$ in more detail. Let $H$ be the hypercenter of $G$ so that $|H| = p^{x'-1}$ and $G/H \cong W_k$. As $G'$ is abelian and $G'$ maps onto $W_k^d \cong Z_p^d$ for $d = (p - 1)p^{k-1}$, it follows that $G' = F \times T$ for $F = H \cap G' \leq H$ finite and $T \cong Z_p^d$. Let $Q = H/F$ and denote $|Q| = p^x$ and $|F| = p^y$. Then $G/G'$ has order $p^{x+1+k}$ and rank at most $x + 2$.

As $C_p$ is a trivial $G$-module, we see that $H^2(G/G', H^0(G', C_p)) = H^2(G/G', C_p)$ and this group has rank at most $(x + 2)(x + 3)/2$. Next, $H^1(G', C_p) = \text{Hom}(G', C_p)$ has rank $rk(G') \leq y + d$. Hence $H^1(G/G', H^1(G', C_p))$ has rank at most $(y + d)rk(G/G') \leq (y + d)(x + 2)$. Finally, as $G' = F \times T$ with $T$ a free module and $F$ finite of order $p^y$, it follows that $\text{Ext}(G', C_p)$ has rank at most $y$. Thus $c \leq (x + 2)(x + 3)/2 + (y + d)(x + 2) + y$. Recall that $y + x = s$ and $y, x \in \mathbb{N}_0$. Thus $(x + 2)(x + 3)/2 + (y + d)(x + 2) + y = (x + 2)(x + 3)/2 + (y + d) + y \leq (s + 2)(s + d + 2) + s$. Define $k(p, d, s) = (s + 2)(s + d + 2) + s$. Then

$$f(p, r, s) \leq p^{k(p, d, s)} f(p, r - 1, s - 1).$$

Using induction this implies that

$$\log_p f(p, r, s) \leq \sum_{i=1}^{s} k(p, d, i)$$

$$\leq \sum_{i=1}^{s} (i + 2)(d + i + 2) + i$$

$$\leq d \sum_{i=1}^{s} (i + 2) + \sum_{i=1}^{s} (i + 2)^2 + i$$

$$\leq \frac{b_1(s)}{2} ds^2 + \frac{b_2(s)}{3} s^3$$

for functions $b_1(s)$ and $b_2(s)$ that tend to 1 for $s \to \infty$.

### 3 Coclass graphs

Given two finite $p$-groups $G$ and $H$, we say that $H$ is a descendant of $G$ and $G$ is an ancestor of $H$ if $H/\gamma_c(H) \cong G$ holds for some $c \in \mathbb{N}$. The difference $cl(H) - cl(G)$ is the distance of $H$ from $G$. Thus $G$ itself is the unique descendant of $G$ of distance 0. The group $H$ is an immediate descendant of $G$ if it is a descendant of distance 1.

The finite $p$-groups of coclass $r$ can be visualised by a graph $\mathcal{G}(p, r)$: the vertices of $\mathcal{G}(p, r)$ correspond one-to-one to the isomorphism types of finite $p$-groups of coclass $r$ and there is an edge $G \to H$ if the group $H$ is an immediate descendant of $G$. Hence there is a path from $G$ to $H$ in $\mathcal{G}(p, r)$ if $H$ is a descendant of $G$.

The investigation of the graph theoretic structure of coclass graphs $\mathcal{G}(p, r)$ plays a major role in coclass theory. We recall some of the basic features of $\mathcal{G}(p, r)$ in the following before we investigate the full subgraph $\mathcal{M}(p, r)$ of all metabelian groups in $\mathcal{G}(p, r)$. 

5
3.1 The general case

Let $S$ be an infinite pro-$p$-group of finite coclass $r$. Let $t = t(S) \in \mathbb{N}$ be minimal so that $S/\gamma_t(S)$ has coclass $r$ and there exists no infinite pro-$p$-group $\hat{S}$ of coclass $r$ with $\hat{S} \not\cong S$ and $\hat{S}/\gamma_t(\hat{S}) \cong S/\gamma_t(S)$. Then we call the index $t$ the root index of $S$ and we define the coclass tree $T(S)$ associated with $S$ as the full subtree of $G(p, r)$ consisting of the descendants of $S/\gamma_t(S)$.

The finite $p$-groups $S/\gamma_t(S), S/\gamma_{t+1}(S), \ldots$ form an infinite path in the tree $T(S)$ starting at its root; this path is called the main line of $T(S)$. The definition of the root index implies that every infinite path in $T(S)$ is contained in the main line. For each $i \geq t$ denote with $B_i(S)$ the subtree of $T(S)$ consisting of the descendants of $S/\gamma_i(S)$ which are not descendants of $S/\gamma_{i+1}(S)$; then $B_i(S)$ is the $i$-th branch of $T(S)$. With this notation it follows that $T(S)$ consists of the sequence of finite branches connected by the main line of $T(S)$.

It can be proved that the graph $G(p, r)$ consists of finitely many coclass trees and finitely many other groups; see Chapter 10 of [14]. Thus the investigation of $G(p, r)$ reduces to a large extend to an investigation of the coclass trees $T(S)$ for the finitely many infinite pro-$p$-groups $S$ of coclass $r$.

Let $\delta \in \mathbb{N}$ and let $G$ be a subgraph of $G(p, r)$. We define $G_\delta$ as the full subgraph of $G$ consisting of those groups which have distance at most $\delta$ to a main line group in $G(p, r)$.

Hence $G_\delta$ is a pruned version of $G$. We say that $G$ is bounded by $\delta$ if $G = G_\delta$. Note that $G_\delta(p, r)$ consists of the union of the pruned coclass trees $T_\delta(S)$ for $S$ an infinite pro-$p$-group of coclass $r$.

It has been conjectured in [21] and proved independently in [5] and [10] that for every $\delta \in \mathbb{N}$ the graph $G_\delta(p, r)$ is virtually periodic. This implies that the infinite graph $G_\delta(p, r)$ can be described by a finite subgraph.

3.2 The metabelian case

In this section we exhibit first steps towards an investigation of the full subgraph $M(p, r)$ consisting of all metabelian groups in $G(p, r)$. Note that ancestors of metabelian groups in $G(p, r)$ are metabelian and thus the full subgraph $M(S)$ of the metabelian groups in a coclass tree $T(S)$ is a tree.

7 Theorem: Let $S$ be an infinite pro-$p$-group of coclass $r$. The tree $M(S)$ is infinite if and only if $S$ is metabelian.

Proof: Let $t = t(S)$ denote the root index of $S$. If the group $S$ is metabelian, then the groups $S/\gamma_t(S), S/\gamma_{t+1}(S), \ldots$ are metabelian and form an infinite path in $M(S)$. Thus $M(S)$ is infinite in this case.

Conversely, suppose that the tree $M(S)$ is infinite. Consider an arbitrary $j \geq t$. There are only finitely many groups in $T(S)$ which are not descendants of $S/\gamma_j(S)$. As $M(S)$ is infinite, there exists a metabelian group in $T(S)$ which is a descendant of $S/\gamma_j(S)$. Hence $S/\gamma_j(S)$ is metabelian. As $\cap_{j \in \mathbb{N}} \gamma_j(S) = \{1\}$, this implies that $S'' = \{1\}$ and thus $S$ is metabelian.
8 Theorem: There exists $\delta = \delta(p, r)$ so that $\mathcal{M}(p, r)$ is bounded by $\delta$.

Proof: For every $r \in \mathbb{N}$ the graph $G(2, r)$ is bounded by Theorem 11.3.7 and Corollary 11.4.2 of [14]. Hence the result follows in this case and we consider the case $p > 2$ in the remainder of this proof.

All but finitely many groups in $\mathcal{M}(p, r)$ are contained in a coclass tree $\mathcal{M}(S)$ for some infinite pro-$p$-group $S$ of finite coclass $r$. Hence it is sufficient to prove that each tree $\mathcal{M}(S)$ is bounded. If $S$ is not metabelian, then $\mathcal{M}(S)$ is finite and thus bounded. Hence we assume that $S$ is a metabelian infinite pro-$p$-group of coclass $r$.

The group $S$ is a uniserial $p$-adic pre-space group of dimension $d$ for some $d \in \mathbb{N}$. Let $T$ be a translation subgroup of $S$ with point group $P = S/T$ and let $T = T_0 > T_1 > \ldots$ denote the unique $S$-invariant series through $T$. We embed $T \cong \mathbb{Z}_p^d$ into $V = T \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^d$, where $\mathbb{Q}_p$ denotes the $p$-adic rationals. This allows one to extend the series through $T$ to a series $\ldots > T_{-1} > T_0 > T_1 > \ldots$.

By Lemma 10.4.3 of [14] there exists an index $i \in \mathbb{N}$ such that $ST_{-i}$ is a split extension of $T_{-i}$ by $P$. Let $i$ be minimal with this property and define $S^* = ST_{-i}$ and $T^* = T_{-i}$. Then $S^*$ is a uniserial $p$-adic pre-space group of finite coclass with translation subgroup $T^*$ and point group $P$. The group $P$ acts uniserially on $T^*$ with series $T^*_0, T^*_1, \ldots$. The group $S$ embeds into $S^*$ with $T^* \cap S = T = T_0 = T^*_0$. The group $S^*$ is also called the minimal split supergroup of $S$.

We briefly recall the notion of a constructible group, see Definition 8.4.3 of [14] and [11]. We use the version of [11] for our purposes. This considers the case of a split pre-space group first. Let $\gamma : T^* \wedge T^* \rightarrow T^*$ be a $\mathbb{Z}_pP$-module homomorphism with $\gamma(T^* \wedge T^*) = T^*$ and $\gamma(T^*_j \wedge T^*_j) = T^*_k$. Let $j \leq m \leq k$. Then $\gamma$ induces a homomorphism $\gamma_{m} : T^*_j/T^*_m \wedge T^*_j/T^*_m \rightarrow T^*_m/T^*_m$ and this allows one to define a group $T^*_{\gamma, m}$ whose elements are those of $T^*/T^*_m$ and whose multiplication is defined by

$$(a + T^*_m)(b + T^*_m) = (a + b + T^*_m) + \frac{1}{2}\gamma_{m}(a + T^*_j \wedge b + T^*_j).$$

Note that commutators can be computed easily in $T^*_{\gamma, m}$ as

$$[a + T^*_m, b + T^*_m] = \gamma_{m}(a + T^*_j \wedge b + T^*_j).$$

In particular, if $m = j$, then $T^*_{\gamma, m}$ is abelian. If $j < m \leq k$, then $T^*_{\gamma, m}$ has class 2 and derived subgroup $T^*_j/T^*_m$. A constructible group for $S^*$ now has the form $S^*_{\gamma, m} := T^*_{\gamma, m} \rtimes P$. To obtain a constructible group for $S$ we assume that $i \leq j$ and use the natural homomorphism $\varphi : S^*_{\gamma, m} \rightarrow (T^*/T^*_j) \rtimes P$. The image of this homomorphism contains $S/T^*_j = S/T^*_j \rtimes P$. Let $S^*_{\gamma, m}$ be the full preimage of $S/T^*_j$ under $\varphi$. Then $S^*_{\gamma, m}$ is a constructible group for $S$.

We now investigate the derived series of a constructible group. The point group $P$ acts uniserially on the group $T^*_m$. Hence $T^*_{i+1}/T^*_m \leq S^*_{\gamma, m}$. Recall that $T^*$ has rank $l$ as $\mathbb{Z}_p$-module and let $i + 1 = (h-1)l + a$ with $0 \leq a < l$. Then $T^*_{i+1} \geq p^h T^*$ and $h$ is minimal with this property. Further, the group $S^*_{\gamma, m}$ contains

$$\gamma_{m}(T^*_{i+1}/T^*_j, T^*_i/T^*_j) \geq \gamma_{m}(p^h T^*/T^*_j, p^h T^*/T^*_j) = p^{2h}\gamma_{m}(T^*/T^*_j, T^*_{i+1}/T^*_j) = p^{2h} T^*_{i+1}/T^*_m.$$
Hence if \( m \geq j + 2hl \), then \( S''_{\gamma, m} \neq \{1\} \). Note that \( S_{\gamma, m}/(T^*_j/T^*_m) \cong S/T_{j-1} \) is a main line quotient of \( S_{\gamma, m} \). Hence if \( S_{\gamma, m} \) is metabelian, then it has distance at most \( 2hl \) from a main line group. The integers \( l \) and \( i \) depend on \( S \) only and thus \( h \) also depends on \( S \) only.

By [14, Theorem 11.3.9] all but finitely many groups \( G \) in \( \mathcal{M}(S) \) have a normal subgroup \( N \) so that \( \log_p(|N|) \leq f(p, r) \) and \( G/N \) is constructible. Hence all but finitely many groups \( G \) in \( \mathcal{M}(S) \) have distance at most \( f(p, r) + 2hl \) from the main line in \( \mathcal{M}(S) \). This completes the proof.

\[ \bullet \]

4 Coclasse families

Let \( S \) be an infinite pro-\( p \)-group of coclass \( r \) and dimension \( d \) and let \( \delta \in \mathbb{N} \). In [10] a partition of the set of groups in \( \mathcal{T}_S(S) \) into finitely many infinite families, the so-called coclass families, and finitely many other groups is suggested.

The above-mentioned virtual periodicity of \( \mathcal{T}_S(S) \) is compatible with the construction of the coclass families: if \( (G_0, G_1, \ldots) \) is a coclass family associated to \( \mathcal{T}_S(S) \), then the virtual periodicity maps the vertex associated to \( G_i \) to the vertex associated to \( G_{i+1} \) for each \( i \in \mathbb{N}_0 \).

In this section we first recall an explicit construction of the coclass families. Finally, we prove the main results of this paper as stated in Theorem 1 and Corollary 2.

4.1 Construction of coclass families

The group \( S \) is an infinite pro-\( p \)-group of coclass \( r \). Then \( S \) is a uniserial \( p \)-adic pre-space group of some dimension \( d \). Choose \( l \in \mathbb{N} \) and let \( T = \gamma_l(S) \). Assume that \( l \) is large enough so that \( T \) is a translation subgroup and its point group \( R \) has coclass \( r \). As usual, denote with \( T = T_0 > T_1 > \ldots \) the unique maximal \( S \)-invariant series through \( T \).

Let \( \epsilon \in Z^2(R, T) \) so that the extension of \( R \) by \( \epsilon \) is isomorphic to \( S \). For \( n \in \mathbb{N} \) let \( \epsilon_n \) denote the image of \( \epsilon \) under the natural projection \( Z^2(R, T) \to Z^2(R, T/T_n) \). For \( q \in \mathbb{N} \) let \( \sigma^q : Z^2(R, T/T_n) \to Z^2(R, T/T_{n+qd}) \) denote the map induced by multiplication \( T/T_n \to T/T_{n+qd} \), \( t + T_n \to p^q t + p^q T_n \). Note that \( T_{n+qd} = p^q T_n \) so that this is well-defined. Choose \( e \in \mathbb{N} \). For \( n \in \mathbb{N} \) divide \( n - e \) by \( d \) with remainder so that \( n - e = qd + j \) with \( 0 \leq j < d \) is obtained. Then for \( \gamma \in Z^2(R, T/T_{e+j}) \) we denote with

\[ E_{l,e}(\gamma, q) \]

the extension of \( R \) by \( T/T_n \) via \( \epsilon_n + \sigma^q(\gamma) \). The following theorem is implicitly included in the results of [10].

9 Theorem: Let \( S \) be an infinite pro-\( p \)-group of coclass \( r \) and dimension \( d \) and let \( \delta \in \mathbb{N} \). Then there exist \( l \) and \( e \) and subsets \( \Gamma_j \subseteq Z^2(R, T/T_{e+j}) \) for \( 0 \leq j < d \) with the following property: for almost all groups \( G \) in \( \mathcal{T}_S(S) \) there exist a unique \( \gamma \in \Gamma_j \) and a unique \( q \in \mathbb{N} \) so that \( G \cong E_{l,e}(\gamma, q) \).
Proof: Let $R = S/\gamma_l(S)$. Then $R$ is a main line group in $T_δ(S)$. Consider the groups $G$ in $T_δ(S)$ which are descendants of $R$ and have order at least $|R|p^e$. Note that this holds for almost all groups in $T_δ(S)$. Let $|G| = |R|p^n$ with $n \geq e$. Division with remainder by $d$ allows one to write $n - e = qd + j$ with $0 \leq j < d$. It is proved in [10] that $G$ is an extension of $R$ by $T/T_n$ and the cocycle corresponding to this extension can be chosen as $\epsilon_n + \sigma^q(\gamma)$ for some $\gamma$ in a suitably defined set $\Gamma_j \subseteq Z^2(R, T/T_{e+j})$.  

[10] also contains details on the possible choices for the parameters $l$ and $e$. They can be chosen so that they apply to every infinite pro-$p$-group $S$ of coclass $r$. In the following we usually omit these parameters in the notation. Let $\Gamma = \Gamma_0 \cup \ldots \cup \Gamma_{d-1}$ and for $\gamma \in \Gamma$ let

$$\mathcal{E}(\gamma) := \{E_{l,e}(\gamma, q) \mid q \in \mathbb{N}_0\}.$$  

The set $\mathcal{E}(\gamma)$ is called the \emph{coclass family} defined by $\gamma$. Theorem 9 asserts that almost all groups in $T_δ(S)$ are contained in a coclass family and thus almost all groups in $T_δ(S)$ are contained in the finite and disjoint union

$$\bigcup_{\gamma \in \Gamma} \mathcal{E}(\gamma).$$

4.2 Metabelian coclass families

Let $S$ be an infinite pro-$p$-group of finite coclass $r$. Our aim in this section is to show that the graph $\mathcal{M}(S)$ consists of finitely many coclass families and finitely many other groups. As in [10], this implies then that $\mathcal{M}(S)$ is virtually periodic.

If the underlying pro-$p$-group $S$ is not metabelian, then $\mathcal{M}(S)$ is a finite tree by Theorem 7 and the result follows directly. It hence remains to consider the case that $S$ is metabelian. By Theorem 8 we note that all but finitely many groups of $\mathcal{M}(S)$ are contained in $T_δ(S)$ for some $\delta \in \mathbb{N}$. Hence almost all groups in $\mathcal{M}(S)$ are contained in some coclass family.

The following theorem asserts that a coclass family is either completely contained in $\mathcal{M}(S)$ or is disjoint from $\mathcal{M}(S)$.

10 Theorem: Let $S$ be a metabelian infinite pro-$p$-group of coclass $r$, let $\delta \in \mathbb{N}$ and let $(G_0, G_1, \ldots)$ be a coclass family in $T_δ(S)$. If one of the groups $G_i$ is metabelian, then all of the groups $G_0, G_1, \ldots$ are metabelian.

Proof: We continue to use the notation of Theorem 9. Thus let $l$, $e$, $R$, $T$ and $e$ be as defined there. Then $T = \gamma_l(S)$ and $R = S/T$. Note that $S = \gamma_1(S)$ and $S' = \gamma_2(S)$ for the lower central series. As $R = S/T$ has coclass $r$, it follows that $l \geq 2$ and $T \leq S'$. As $S$ is metabelian, it follows that $R'$ acts trivially on $T$. This implies that $R'$ acts trivially on $T/T_n$ for each $n \in \mathbb{N}$.

Let $q \in \mathbb{N}_0$ be arbitrary and consider $G_q = E(\gamma, q)$. Recall that $G_q$ is an extension of $R$ by $T/T_n$ for $n = e + qd + j$ via a cocycle $\delta_q := \epsilon_n + \sigma^q(\gamma)$. Thus $R$ is a quotient of $G_q$. As both $G_q$ and $R$ have coclass $r$, this implies that $\gamma_l(G_q) = T/T_n$ and $G_q'$ is the full preimage of $R'$ under the natural epimorphism $G_q \to R$ associated with the extension structure of
Let \( g, h \in G'_{q} \) and write \( g = (r, t) \) and \( h = (s, u) \) with \( r, s \in R' \) and \( t, u \in T/T_n \). Since \( R' \) acts trivially on \( T/T_n \), we find that:

\[
[g, h] = (r, t)^{-1}(s, u)^{-1}(r, t)(s, u) \\
\quad = (r^{-1}s^{-1}rs, \delta_t(r, s) + \delta_t(r^{-1}, s^{-1}) + \delta_t(r^{-1}s^{-1}, rs) - \delta_t(r, r^{-1}) - \delta_t(s, s^{-1})) \\
\quad =: (1, \hat{\delta}_t(r, s)).
\]

Hence evaluating commutators in extensions induces a map \( \delta_q \mapsto \hat{\delta}_q \). This map is linear and hence compatible with addition and it is compatible with the map \( \sigma \) defining the multiplication with \( p \). Thus

\[ \hat{\delta}_q = \hat{\epsilon}_n + \sigma^q(\hat{\gamma}). \]

The above calculation shows that an extension defined by a cocycle \( \kappa \) is metabelian if and only if \( \hat{\kappa} = 0 \). As \( S \) is metabelian, this implies that \( \hat{\epsilon} = 0 \) and thus \( \hat{\epsilon}_n = 0 \). Further, one of the groups \( G_{q_0} \) is metabelian and thus \( \sigma^{q_0}(\hat{\gamma}) = 0 \) for one value \( q_0 \). As \( \sigma \) is defined via multiplication with \( p \), this implies that \( \sigma^q(\hat{\gamma}) = 0 \) for each \( q \in \mathbb{N}_0 \). Thus \( \hat{\delta}_q = 0 \) and \( G_q \) is metabelian for each \( q \in \mathbb{N}_0 \).

Theorem 10 implies that a coclass family is either completely contained in a tree \( \mathcal{M}(S) \) or disjoint from it. Thus Theorem 10 has the following direct consequence.

**11 Corollary:** Let \( S \) be a metabelian infinite pro-\( p \)-group of coclass \( r \). Then \( \mathcal{M}(S) \) consists of finitely many coclass families and finitely many other groups.

Using the same ideas as in [10], this implies the following.

**12 Corollary:** Let \( S \) be a metabelian infinite pro-\( p \)-group of coclass \( r \) and dimension \( d \). Then \( \mathcal{M}(S) \) is virtually periodic with period \( d \).

As \( \mathcal{M}(p, r) \) consists of finitely many coclass trees \( \mathcal{M}(S) \) and finitely many other groups, these Corollaries directly extend to all of \( \mathcal{M}(p, r) \) and hence provide a proof for Theorem 1 and Corollary 2.

**5 Examples**

The perhaps easiest examples of metabelian coclass graphs are \( \mathcal{M}(2, 1) \) and \( \mathcal{M}(3, 1) \): these graphs satisfy \( \mathcal{M}(2, 1) = G(2, 1) \) and \( \mathcal{M}(3, 1) = G(3, 1) \) and both graphs \( G(2, 1) \) and \( G(3, 1) \) are well-known, see [1]. In this section we investigate some further graphs \( \mathcal{M}(p, r) \).

GAP and its package [8] can be used to explore \( \mathcal{M}(p, r) \) for any given \( p \) and \( r \). For this purpose one can first determine the infinite metabelian pro-\( p \)-groups of coclass \( r \) using the description in Theorem 4: that is, one can construct the groups \( W_k \) for \( 1 \leq k \leq r \) and then determine iteratedly central extensions of \( W_k \). The isomorphism problem for the resulting pro-\( p \)-groups can be solved by reducing it to large finite \( p \)-quotients of them; the isomorphism problem for finite \( p \)-groups can be solved as an application of [22].
Given an infinite metabelian pro-$p$-group $S$ of coclass $r$, Corollary 12 asserts that $\mathcal{M}(S)$ is virtually periodic; that is, if $d$ is the dimension of $S$, then there exists $l \in \mathbb{N}$ so that $\mathcal{M}(S) \cong \mathcal{M}_{l+d}(S)$, or equivalently, $\mathcal{B}M_i(S) \cong \mathcal{B}M_{i+d}(S)$ for all $i \geq l$ and $\mathcal{B}M_i(S) = B_i(S) \cap \mathcal{M}(S)$ the branches of $\mathcal{M}(S)$. The known bounds for $l$ are usually too large to be of practical use. Hence we determine the initial sequence of branches $\mathcal{B}M_i(S), \mathcal{B}M_{i+1}(S), \ldots$ until $\mathcal{B}M_i(S) \cong \mathcal{B}M_{i+d}(S)$ holds for a full sequence of $d$ branches. We then conjecture that we have found the periodic pattern and thus conjecture the full structure of $\mathcal{M}(S)$.

We note that the Gap package [2] allows to visualise finite graphs. Throughout this section, we draw graphs in collected form: if a vertex has a number $n$ attached, then this vertex and its descendants exists $n$ times.

5.1 The case of coclass 1

The graph $\mathcal{M}(p,1)$ consists of a single metabelian coclass tree $\mathcal{M}(S)$ for $S = W_1 = \mathbb{Z}_p^{p-1} \rtimes C_p$ and the single additional group $C_{p^2}$. Note that $\mathcal{M}(p,1)$ is well-known for $p = 2$ and $p = 3$ by [1]. We consider the case $p \geq 5$ in the following.

Corollary 12 asserts that $\mathcal{M}(S)$ is virtually periodic with period $d = p - 1$. Let $\mathcal{M}_i$ denote the full subtree of all descendants of $S/\gamma_i(S)$ in $\mathcal{M}(S)$. Computational methods suggest that the subtree $\mathcal{M}_{p-1}$ is periodic.

Let $\mathcal{B}M_i$ denote the $i$th branch of $\mathcal{M}(S)$; that is, let $\mathcal{B}M_i = B_i(S) \cap \mathcal{M}(S)$ be the full subtree of all descendants of $S/\gamma_i(S)$ which are not descendants of $S/\gamma_{i+1}(S)$ in $\mathcal{M}(S)$. Theorem 1 in [19] implies that each vertex in a branch $\mathcal{B}M_i$ for some $i \geq p - 1$ has distance at most $p - 2$ from the root; that is, the tree $\mathcal{M}_{p-1}$ is bounded by $p - 2$.

Using computational methods, we determined the initial sequence of branches of $\mathcal{M}_{p-1}$ for $p = 5$ and $p = 7$. We exhibit these branches for $p = 5$ in Figure 1. This suggests that $\mathcal{M}(5,1)$ consists of 248 coclass families and finitely many other groups.

![Figure 1: Branches $\mathcal{B}M_4, \ldots, \mathcal{B}M_7$ in $\mathcal{M}(5,1)$](image)

We exhibit the initial sequence of branches of $\mathcal{M}_{p-1}$ for $p = 7$ in Figures 2 and 3. This suggests that $\mathcal{M}(7,1)$ consists of 22661 coclass families and finitely many other groups.
5.2 Metabelian 3-groups with abelian invariants \((3, 3)\)

Nebelung [20] gives an explicit and detailed classification of the metabelian 3-groups with commutator factor of type \((3, 3)\). Here we do not go into that much detail, but it seems useful to exhibit briefly how our methods could apply to the construction of these groups.

As a first step, we investigate the infinite metabelian pro-3-groups \(S\) with commutator factor of type \((3, 3)\). We first show that such groups \(S\) exist for every coclass. Denote with \(W\) the split extension of \(C_3 = \langle g \rangle\) with \(T = \langle t_1, t_2 \rangle \cong \mathbb{Z}_3^2\). Let \(U = \langle g, t_1 t_2, t_2^3 \rangle\) be a normal subgroup of index 3 in \(W\). Let \(W\) act uniserially on \(M \cong \mathbb{Z}_3^2\) with kernel \(U\). Define the following non-split extension \(E\) of \(W\) with \(M\):

\[
E = \langle g, t_1, t_2, m_1, m_2 \mid g^3 = m_2, t_1^g = t_2^{-1} m_2, t_2^g = t_1 t_2^{-1} m_2, m_1^g = m_1, m_2^g = m_2, t_2^t = t_2 m_1^{-1} m_2, m_1^t = m_1^{-1} m_2, m_2^t = m_1^{-1}, m_1^t_2 = m_2^{-1}, m_2^t_1 = m_2^{-1}, m_1^t_2 = m_2^{-1}, m_2^t_1 = m_2^{-1} \rangle.
\]

Note that \(E/E'\) is of type \((3, 3)\) and \(E\) is metabelian. The module \(M\) is uniserial under
the action of $W$ and hence has a unique series of submodules $M = M_0 > M_1 > \ldots$ with quotients $M_i/M_{i+1}$ of order 3. For each $s \in \mathbb{N}_0$ the quotient $Q_s := E/M_s$ is an extension of $W$ by a finite group of order $3^s$. Hence $cc(Q_s) = s + 1$ and $Q_s$ is metabelian with commutator factor group of type $(3, 3)$. Thus we obtain an infinite sequence of infinite metabelian pro-3-groups with abelianisation $(3, 3)$. Computational experiments suggest the following conjecture.

**13 Conjecture:** Let $S$ is an infinite metabelian pro-3-group with abelianisation $(3, 3)$ and hypercenter of order $3^{s+1}$ for $s \geq 0$.

(a) $S$ is an extension of $W$ by $M/M_{s+1}$.

(b) $S$ has a centre $Z(S)$ of order 3 and $S/Z(S) \cong Q_s$.

By Theorem 4, the infinite metabelian pro-3-groups with commutator factor of type $(3, 3)$ can be arranged as tree $T$: the root of $T$ corresponds to $W$ and two groups $G$ and $H$ of $T$ are connected by an edge if $G/Z(G) \cong H$. Then Conjecture 13 suggests that $T$ has a single maximal infinite path and the branches of $T$ from this path have depth 1. The branches are described in more detail in the following conjecture which is again based on computational experiments.

**14 Conjecture:** The number of isomorphism types of infinite metabelian 3-groups with abelianisation $(3, 3)$ and hypercenter of order $3^s$ are:

- 1 if $s = 0$.
- 3 if $s = 1$.
- 4 if $s \geq 2$ and $s$ even.
- 6 if $s \geq 2$ and $s$ odd.

Conjecture 14 implies that $T$ is virtually periodic with period 2. According to this periodic pattern we split the groups in $T$ into 10 infinite families of groups and 4 additional groups: the 4 groups are those with $s \in \{0, 1\}$ and the infinite families correspond to the graph isomorphism associated with the periodic pattern.

Let $S_{s,i}$ denote the $i$th group with hypercentre of order $3^s$ in Conjecture 14; sort these groups so that $S_{s,1}$ is the quotient of $E$. As a next step, we consider the coclass tree for each of these groups. First, $S_{0,1} = W$ and the coclass tree of $W$ is well-known (it is the coclass tree in $G(3, 1)$). Based on computational evidence, we conjecture the structure of the coclass trees for the other groups. Let $\mathcal{M}_{s,i}$ denote the descendant tree of the class $(s + 3)$-quotient of $S_{s,i}$ and let $\mathcal{B}_1(a)$ and $\mathcal{B}_2(a, b, c)$ denote branches exhibited in Figure 4. Note that the branch $\mathcal{B}_1(a)$ corresponds to $a + 1$ groups and the branch $\mathcal{B}_1(a, b, c)$ corresponds to $1 + a + ac + b$ groups.

**15 Conjecture:** Each tree $\mathcal{M}(S_{s,i})$ is periodic.

(a) If $s \geq 2$ is even, then

- $\mathcal{M}(S_{s,i})$ has period 2 with the branches $\mathcal{B}_1(9)$ and $\mathcal{B}_1(14)$.
- It thus consists of 25 coclass families.
Figure 4: $B_1(a)$ and $B_2(a, b, c)$

- $\mathcal{M}(S_{s,2})$ has period 1 with the branch $B_2(1, 4, 9)$;  
  It thus consists of 15 coclass families.
- $\mathcal{M}(S_{s,3})$ has period 2 with branches $B_2(1, 4, 9)$ and $B_2(2, 3, 6)$;  
  It thus consists of 33 coclass families.
- $\mathcal{M}(S_{s,4})$ has period 1 with the branch $B_2(2, 6, 9)$;  
  It thus consists of 27 coclass families.

(b) If $s \geq 2$ is odd, then
- $\mathcal{M}(S_{s,1})$ has period 1 and the branch $B_1(11)$;  
  It thus consists of 12 coclass families.
- $\mathcal{M}(S_{s,i})$ ($i=2,3$) have period 2 and branches $B_2(2, 3, 4)$ and $B_2(1, 2, 6)$;  
  It thus consists of 24 coclass families each.
- $\mathcal{M}(S_{s,i})$ ($i=4,5,6$) have period 2 and branches $B_2(2, 6, 5)$ and $B_2(1, 3, 9)$;  
  It thus consists of 33 coclass families each.

(c) If $s = 1$, then the trees are as in (b), but only the trees $\mathcal{M}(S_{s,i})$ for $i \leq 3$ occur.

Conjecture 15 suggests that there are only finitely many different graph-isomorphism types of coclass trees arising for the infinitely many pro-3-groups associated with this case. This suggests that all but finitely many of the metabelian 3-groups with abelianisation $(3, 3)$ can be described by a finite amount of data: the data needed is the index $(s, i)$ defining the group $S_{s,i}$, the coclass families within $\mathcal{M}_{s,i}$ and the class of the considered group.

5.3 The graph $\mathcal{M}(3, 2)$

The graph $\mathcal{G}(3, 2)$ is described in [11]. It has 16 coclass trees corresponding to the 16 infinite pro-3-groups of coclass 2. By Theorem 4 and the results in [11] it follows readily that there are 7 metabelian pro-3-groups of coclass 2: the group $W_2$ and the 6 isomorphism types of extensions of $W_1$ by a cyclic group of order 3.

Let $S$ be one of the 6 extensions of $W_1$ by a cyclic group of order 3. Then the coclass tree $T(S)$ is bounded and virtually periodic with period 2. It is easy to explore with computational methods. The metabelian subtree $\mathcal{M}(S)$ is similarly easy to explore. Its groups have distance at most 4 from the main line.

The coclass tree $T(W_2)$ is unbounded. Its so-called skeleton is described in [11]. The full tree $T(W_2)$ is conjecturally known due to computational experiments. The subtree of metabelian groups $\mathcal{M}(W_2)$ is bounded and virtually periodic with period 6. It can be investigated computationally and this suggests that it is bounded by 5. Its branches are
moderately thick and hence are omitted here.

5.4 The graph $\mathcal{M}(5, 2)$

The graph $\mathcal{G}(5, 2)$ has not been determined or investigated so far. It can be expected that it is a rather complex. We restrict our attention to the subgraph $\mathcal{M}(5, 2)$. Using Theorem 4 we determine the following infinite metabelian pro-5-groups of coclass 2:

(1) There are 10 central extensions of $W_1$ by the cyclic group of order 5. These have the abelian invariants $(5, 5)$ (7 groups), $(5, 5, 5)$ (1 group) and $(5, 25)$ (2 groups).

(2) The group $W_2 = \mathbb{Z}_5^{20} \rtimes C_{25}$.

All coclass trees $\mathcal{T}(S)$ for $S$ an infinite metabelian pro-5-group of coclass 2 are unbounded. Their subtrees $\mathcal{M}(S)$ are bounded and virtually periodic with period 4 for the groups in (1) and period 20 for $W_2$.

5.5 The graph $\mathcal{M}(3, 3)$

The graph $\mathcal{G}(3, 3)$ has not been determined or investigated so far. We consider $\mathcal{M}(3, 3)$ in the following. Using Theorem 4, we determine the following infinite metabelian pro-3-groups of coclass 3:

(1) There are 42 groups with a hypercenter of order 9; these are extensions of $W_1$ by a group of order 9. Their abelian invariants are $(3, 3)$ (4 groups), $(3, 3, 3)$ (24 groups), $(3, 3, 3, 3)$ (1 group), $(3, 3, 9)$ (3 groups), $(3, 9)$ (7 groups), $(3, 27)$ (2 groups) and $(9, 9)$ (1 group).

(2) There are 10 groups with a hypercenter of order 3; these are extensions of $W_2$ by a group of order 3. Their abelian invariants are $(3, 3, 9)$ (1 group), $(3, 9)$ (6 groups), $(3, 27)$ (2 groups), $(9, 9)$ (1 group).

(3) The group $W_3 \cong \mathbb{Z}_3^{18} \rtimes C_{27}$.

The coclass trees $\mathcal{T}(S)$ are bounded for the groups $S$ in (1), while they are unbounded for the groups in (2) and (3). Their subtrees $\mathcal{M}(S)$ are bounded and virtually periodic with period 2 for the groups in (1), period 6 for the groups in (2) and period 18 for the group in (3).

References


