Quotient algorithms generalize to finitely L-presented groups

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1 Finite (L)-presentations

A finite presentation is an expression of the form $\langle S \mid R \rangle$, where S is a finite alphabet and R is a finite subset of the free group F_S on S. This finite presentation defines the group F_S/K, where K is the normal subgroup of F_S generated by R. Finite L-presentations have been introduced by Bartholdi [2]. A finite L-presentation is an expression of the form $\langle S \mid Q \mid \Phi \mid R \rangle$, where S is a finite alphabet, Q and R are finite subsets of the free group F_S and Φ is a finite set of homomorphisms $\phi : F_S \to F_S$. This finite L-presentation defines the group F_S/K, where K is the normal subgroup of F_S generated by

$$Q \cup \bigcup_{\phi \in \Phi^*} \phi(R),$$

where Φ^* is the monoid generated by Φ. Let

$$R_i = Q \cup \bigcup_{\phi \in \Phi^i} \phi(R),$$

where Φ^i contains all products of length at most i in the generators of Φ. Then R_i is a finite subset of F_S and $R_0 \leq R_1 \leq \ldots \leq R$ with $R = \cup_{i \in \mathbb{N}_0} R_i$ holds.

2 Some quotient algorithms

Quotient algorithms have been invented as tools to investigate groups given by finite presentations. Among the known quotient algorithms is the p-quotient algorithm by Havas, Newman and O'Brien [4, 6], the nilpotent quotient algorithm by Nickel [7] and the polycyclic quotient algorithm by Lo [5]. All of these methods allow to determine polycyclic presentations for certain quotients of a given finitely presented group.

All these three quotient algorithms have in common that they proceed by induction along a certain normal series. They inially start with the trivial quotient G/G defined by the trivial polycyclic presentation. In the induction step they assume that they are given a consistent polycyclic presentation for a certain quotient G/N and they determine a consistent polycyclic presentation for a certain quotient G/M with $M \leq N$ and N/M

abelian. For this purpose they construct a module presentation A/T for N/M; they determine the group structure of G/M from this. The module presentation is obtained by attaching tails to the consistent polycyclic presentation of G/N and then evaluating finitely many consistency relations and the elements of R. The generators of T correspond one-to-one to the finitely many consistency relations and the finitely many elements in R. We refer also to [1] for a description of the construction of such a module presentation in the case of the polycyclic quotient algorithm.

In the case of the p-quotient algorithm, A is a free $\mathbb{Z}/p\mathbb{Z}$-module of finite rank. The group structure of A/T can then be computed by Gaussian elimination. In the case of the nilpotent quotient algorithm, A is a free $\mathbb{Z}$-module of finite rank. The group structure of A/T can then be computed by a Smith normal form algorithm. In the case of the polycyclic quotient algorithm, A is a free $\mathbb{Z}H$-module of finite rank for $H = G/N$. The group structure of A/T can then be computed by a Groebner basis algorithm for integral group rings of polycyclic groups, see [5].

3 The generalization

Recently, Bartholdi, Eick & Hartung [3] observed that the nilpotent quotient algorithm by Nickel [7] generalizes to finitely L-presented groups. Similar ideas had been used already by Nickel [8] to determine Engel groups. Further, also the p-quotient algorithm had been applied to groups which are not finitely presented by Newman & O'Brien [6]. The aim of this manuscript is to exhibit a short and unified proof for the following.

1 Lemma: *The p-quotient algorithm by Newman & O'Brien [6], the nilpotent quotient algorithm by Nickel [7] and the polycyclic quotient algorithm by Lo [5] generalize to finitely L-presented groups.*

Proof: All three quotient algorithms depend in the induction step on the determination of the group structure of A/T. In the case of a finitely L-presented group, the submodule T is not necessarily finitely generated, but it has a series $T_0 \leq T_1 \leq \ldots \leq T$ with $\cup_{i \in \mathbb{N}_0} T_i = T$. Each submodule T_i is finitely generated by generators corresponding one-to-one to the finitely many consistency relations and the finitely many elements of R_i. As A is a noetherian module in all three cases, it follows that ascending chains of submodules terminate and hence there exists an i with $T_i = T_{i+k}$ for all $k \in \mathbb{N}_0$. Note that $T_i = T_{i+1}$ implies that $T_i = T_{i+k}$ for all $k \in \mathbb{N}_0$ by the definition of R_i. We now determine the group structure of the quotients $A/T_0, A/T_1, \ldots$ as in the case of finitely presented groups until $A/T_i \cong A/T_{i+1}$. In this case we have obtained the group structure of A/T and thus completed the induction step. •

Note that the proof of Lemma 1 is constructive and translates to an algorithm. This algorithm differs from the orginal algorithm for finitely presented groups by adding a single while-loop to construct the ascending series $T_0, T_1, \ldots$ until $A/T_i \cong A/T_{i+1}$.

References

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