On efficient presentations for
infinite sequences of 2-groups with fixed coclass

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Abstract

Eick & Leedham-Green introduced infinite sequences of finite $p$-groups of fixed coclass. They showed that the groups in such an infinite sequence can be defined by a single parametrised presentation. Our aim is to construct short or even efficient parametrised presentations for such infinite sequences of finite 2-groups.

Keywords: Finite $p$-groups, coclass theory, deficiency zero.
MSC2010: 20D15, 20F05.

1 Introduction

Presentations of groups are a central topic in group theory. They arise naturally from Topology and they often provide highly compact descriptions for their associated groups. Of particular interest are short presentations. The deficiency provides a measure for the size of a presentation: for a finite presentation $\langle X \mid R \rangle$ this is defined as $|X| - |R|$. The deficiency $\text{def}(G)$ of a group $G$ is the maximum of the deficiencies of the finite presentations defining $G$. It is well-known that the deficiency of a finite group is non-positive. Of special interest is the boundary case: the finite groups of deficiency zero. We refer to the book by Johnson [10] for background and a general introduction into the theory of group presentations.

The Schur multiplicator $M(G)$ of a finite group $G$ provides a first invariant towards the search of groups of deficiency zero: Schur observed that $M(G)$ is a finite abelian group generated by at most $-\text{def}(G)$ elements. Thus a group of deficiency zero has trivial Schur multiplicator. There are effective algorithms available to determine the Schur multiplicator of a given finite group [9, 5]. In contrast to this, it is often rather difficult to determine the deficiency of a finite group, see Havas, Newman & O’Brien [8] for further details.

Swan [14] proved that not every finite group with trivial Schur multiplicator has deficiency zero. This induces the question which finite groups with trivial Schur multiplicator have deficiency zero. The following question is still open.

Question 1 Is there a finite $p$-group with trivial Schur multiplicator and deficiency strictly smaller than zero?

Coclass theory provides detailed new insight into the finite $p$-groups with trivial Schur multiplicator. For odd primes $p$, Eick [1] proved that among the infinitely many finite $p$-groups of coclass $r$ there are at most finitely many with trivial Schur multiplicator. This does not hold for the even prime. Eick & Leedham-Green [4] showed that almost all finite
2-groups of coclass \( r \) fall into finitely many infinite coclass sequences. Eick [1] exhibited that for every coclass \( r \) there are infinite coclass sequences of 2-groups with trivial Schur multiplicator. Eick & Feichtenschlager [3] proved that the Schur multiplicators of the groups in an infinite coclass sequence can be determined as formula. Coclass theory also provides some insight into the deficiencies of finite \( p \)-groups. The proof of the coclass conjectures implies that every finite \( p \)-group of coclass \( r \) is an extension of a group of class at most 2 with at most \((p - 1)p^r\) generators by a finite \( p \)-group whose order is bounded by a function in \( p \) and \( r \), see Theorems 11.2.7 and 11.2.10 in [11]. As finite presentations for extensions can be build up from presentations of the underlying groups, this yields the following.

**Theorem 1** There exists a bound \( b = b(p, r) \in \mathbb{N} \) so that \( \text{def}(G) \geq b \) for every finite \( p \)-group \( G \) of coclass \( r \).

The results of Theorems 11.2.7 and 11.2.10 in [11] can also be used to obtain a (crude) lower bound for the deficiencies of almost all finite \( p \)-groups of coclass \( r \). Our first aim here is to determine improved lower bounds for the finite 2-groups contained in the infinite coclass sequences, see Corollary 8 below. Our improved bounds also imply the following general (and again rather crude) lower bound.

**Theorem 2** Let \( m \) be a lower bound for the deficiencies of the (finitely many) infinite pro-2-groups of coclass \( r \). Then \( \text{def}(G) \geq m - 2^{-1}(2^{r-1} + r + 3) - 1 \) holds for almost all 2-groups of coclass \( r \).

Then we consider the infinite coclass sequences of 2-groups with small coclass and trivial Schur multiplicator with the aim to determine or at least to get a close estimate on their deficiencies. We ask the following question.

**Question 2** Given an infinite coclass sequence of finite 2-groups with trivial Schur multiplicator, does there exist a parametrised presentation of deficiency zero defining the groups in the infinite sequence?

Finally, we note that the well-known theorem by Golod and Shafarevic [7] implies that a finite \( p \)-group with deficiency zero has at most 3 generators. The metacyclic groups provide infinitely many examples of deficiency zero groups with 2 generators. Problem 5 in [8] asks whether there are infinitely many 3-generator \( p \)-groups with deficiency zero. A positive solution for this problem is given by Fouladi, Jamali & Orfi [6]. There are infinite coclass sequences of finite 2-groups with 3 generators and trivial Schur multiplicator. These provide a wealth of natural candidates for further solutions to Problem 5 in [8]. We investigate some of them with a view towards determining or bounding their deficiency.

## 2 Coclass theory

In this section we recall some of the fundamental features of coclass theory as far as we need them later. In particular, we recall the construction of the infinite coclass sequences associated with coclass theory.
2.1 Infinite pro-$p$-groups of finite coclass

The structure of the infinite pro-$p$-groups of finite coclass has been studied extensively and it is well-understood. We recall some of its essential features in this section and refer to [11] for further information and proofs.

Let $S$ be an infinite pro-$p$-group of finite coclass $r$. Then the lower central series quotients $S_i := S/\gamma_i(S)$ are infinite $p$-groups. The group $S$ can be constructed as the inverse limit of the groups $S_i$. Further, there exists a $j \in \mathbb{N}$ so that $\gamma_j(S) \cong \mathbb{Z}_p^d$, where $\mathbb{Z}_p$ denotes the $p$-adic numbers and $d \in \mathbb{N}$, and $S_i$ has coclass $r$ for all $i \geq j$. We fix one such $j$ and denote $T := \gamma_j(S)$ as translation subgroup of $S$ with point group $P := S/T$. We usually use additive notation for $T$ and multiplicative notation for the lower central series members $\gamma_j(S)$.

The point group $P$ acts uniserially on the translation subgroup $T$; that is, the series defined by $T_0 = T$ and $T_{i+1} = [T_i, S]$ for $i \geq 1$ satisfies $[T_i : T_{i+1}] = p$ for all $i \in \mathbb{N}_0$ and the subgroups $T_0, T_1, \ldots$ are the only $P$-invariant subgroups in $T$. This implies that $pT_i = T_{i+d}$, since $pT_i$ is $P$-invariant of index $p^d$ in $T_i$.

If $T$ was chosen as $T = \gamma_j(S)$ for $j$, then every subgroup $\gamma_{j+i}(S)$ for $i \in \mathbb{N}_0$ is also a possible choice for $T$. Thus there are infinitely many possible choices for $T$. All these choices have the same rank $d$ over $\mathbb{Z}_p$ and we call $d$ the dimension of $S$.

2.2 Infinite sequences of 2-groups

We briefly recall the construction in [4] of the groups in an infinite sequence $\mathcal{G}$ of 2-groups of coclass $r$. For background and details see also [2].

Each infinite coclass sequence is associated with an infinite pro-2-group $S$ and two parameters $l$ and $e$. The first parameter is used to obtain a translation subgroup $T$ for $S$ via $T = \gamma_l(S)$. Let $P$ be the point group corresponding to $T$. A fundamental theorem (see Section 3.1 in [4]) then asserts that for every $i > e$ there is an isomorphism

$$\rho_i : H^2(P, T/2^iT) \to H^2(P, T) \oplus H^2(P, T).$$

This isomorphism leads to the following definition for infinite coclass sequences. Let $\epsilon \in H^2(P, T)$ be a fixed element which defines $S$ as extension of $T$ by $P$.

**Definition 3** Let $\beta \in H^3(P, T)$ and $i \in \mathbb{N}$. Define $G_i$ as the extension of $P$ by $T/2^{e+i}T$ via the cocycle class $\rho_{e+i}^{-1}(\epsilon \oplus \beta)$. Then $\mathcal{G}_\beta = (G_0, G_1, \ldots)$ is the infinite coclass sequence associated with $\beta$.

The infinite sequences of 2-groups associated with $\beta = 0$ play a special role. They are also called the main line sequences. Their groups are lower central series quotients of the infinite pro-2-group $S$. We say that $k$ is the distance of a group $G$ from the main line, if $k$ is minimal with $G/\gamma_{d(G)+1-k}(G)$ a main line group. The construction in [4] implies that all groups in an infinite sequence have the same distance. We consider a relation between different sequences. For any group $G$, let $\gamma(G)$ denote the last non-trivial subgroup in the lower central series of $G$.

**Remark 4** If $\mathcal{G} = (G_0, \ldots)$ is an infinite sequence, then $\mathcal{G}/\gamma = (G_0/\gamma(G_0), \ldots)$ is also an infinite sequence. Further, If $\mathcal{G}$ has distance $k > 1$, then $\mathcal{G}/\gamma$ has distance $k - 1$. 
3 Parametrised presentations and their deficiency

The construction of the infinite coclass sequences as recalled in Section 2.2 yields that each group in an infinite sequence is an extension of a fixed finite 2-group \( P \) by a module \( T/2^{e+j}T \) by a certain cocycle. This implies (as already proved in [4]) that the groups \( G_j \) in an infinite coclass sequence \( \mathcal{G} = (G_0, G_1, \ldots) \) can be defined by a single parametrised presentation \( \langle X \mid R_j \rangle \). The relations \( R_j \) of this parametrised presentation depend on \( j \), but their number \( |R_j| \) is independent of \( j \). We define the deficiency of \( \langle X \mid R_j \rangle \) by \( |X| - |R_j| \). The deficiency \( \text{def}(\mathcal{G}) \) of the infinite sequence is then defined as the maximum of the deficiencies of the finite parametrised presentations for \( \mathcal{G} \). Thus \( \text{def}(\mathcal{G}) \) is a lower bound for the deficiency of every group in the sequence \( \mathcal{G} \).

In this section we describe the construction of short parametrised presentations for infinite coclass sequences of groups and we determine bounds on their deficiency. Throughout, we consider an infinite coclass sequence \( \mathcal{G} = (G_0, G_1, \ldots) \) associated with the infinite pro-2-group \( S \). Let \( \langle X \mid R \rangle \) be a pro-2-presentation for \( S \) with \( X = \{x_1, \ldots, x_n\} \) and \( R = \{r_1, \ldots, r_m\} \).

3.1 Parametrised presentations for main line sequences

**Lemma 5** If \( \mathcal{G} = (G_0, G_1, \ldots) \) is a main line sequence associated with \( S \), then there exists a word \( t \) in \( X \) such that \( G_j \cong \langle X \cup R_j \cup \{t^2\} \rangle \) for \( j \in \mathbb{N}_0 \).

**Proof:** For \( j \in \mathbb{N}_0 \), the group \( G_j \) is a quotient of \( S \); that is, \( G_j \cong S/\gamma_{l+e+jd+1}(S) \). The subgroup \( \gamma_{l+e+jd+1}(S) \) is a uniserial \( S \)-module and thus generated as normal subgroup in \( S \) by a single element \( t_j \), say. As \( \gamma_{l+e+(j+1)d+1}(S) = \gamma_{l+e+jd+1}(S)^{2} \), we can choose \( t_{j+1} = t_{j}^{2} \) and thus \( t_{j+1} = t_{j}^{2^{j}} \). Finally, \( t = t_1 \) can be written as an element in \( X \) and thus the result follows.

3.2 Parametrised presentations for arbitrary sequences

**Theorem 6** Let \( \mathcal{G} = (G_0, G_1, \ldots) \) be an infinite sequence associated with \( S \). Let \( Y = \{t_1, \ldots, t_d\} \) be a set of \( d \) abstract generators, where \( d \) is the dimension of \( S \). Then

\[
G_j \cong \langle X \cup Y \mid R_j \cup Q_j \cup O \rangle \quad \text{with}
\]

- \( R_j = \{r_i(X)^{y_i}(Y)^{2i}, t(X)t_i^{-1}y_i(Y)^{2i} \mid 1 \leq i \leq m \} \) for some words \( y_1, \ldots, y_m, t, y_t \),
- \( Q_j = \{t_1t_i^{-1}(X)^{c_i(X)}, t_i^{2^{e+j}}, [t_1, t_i] \mid 2 \leq i \leq d \} \) for some words \( c_2, \ldots, c_d \), and
- \( O = \{t_{j}^{n}v_{1,h}(Y) \mid 1 \leq h \leq n \} \) for some words \( v_{1,1}, \ldots, v_{1,n} \) in \( Y \).

**Proof:** The group \( G_j \) is defined as an extension of \( P = S/T \) by \( T/2^{e+j}T \) for \( T = \gamma_{l}(S) \). By Lemma 5, the group \( P \) has a presentation \( P = \langle X \mid R \cup \{t\} \rangle \) for some word \( t \) in \( X \). Further, the group \( T/2^{e+j}T \) is generated by \( d \) elements \( Y = \{t_1, \ldots, t_d\} \), say. The uniserial action of \( P \) on \( T \) implies that we can choose the generators for \( T/2^{e+j}T \) so that \( t_{i+1} = t_{i}^{u_i} \) for some \( u_i \in P \) for \( 1 \leq i \leq d - 1 \). Hence all generators are conjugate under the action of the point group \( P \).
Every extension of $P$ by $T/2^{e+j}T$ has a presentation on $X \cup Y$ with relations of the form

$$
\begin{align*}
r_i(X) &= w_{i,j}(Y) \quad \text{for } 1 \leq i \leq m, \\
t(X) &= w_{t,j}(Y), \\
t_i^{g_i} &= v_{i,h,j}(Y) \quad \text{for } 1 \leq i \leq d, 1 \leq h \leq n \\
t_i^{2^{e+j}} &= 1 \quad \text{for } 1 \leq i \leq d, \\
[t_h, t_i] &= 1 \quad \text{for } 1 \leq h < i \leq d,
\end{align*}
$$

for certain words $w_{i,j}, w_{t,j}$ and $v_{i,h,j}$. We prove that such a presentation can be transformed into the desired presentation.

1. Step. We show that $w_{i,j}(Y) = y_i(Y)^{-2^j}$ for $1 \leq i \leq m$. The extension $G_j$ arises from a cocycle $\epsilon \oplus \mu^j(\beta)$. The cocycle $\epsilon$ defines the main line sequence as extension. For the main line sequence we can choose $w_{i,j} = 1$. Thus the words $w_{i,j}$ for $1 \leq i \leq m$ depend on $\beta$ only. As $\mu$ is multiplication with 2, it follows that the corresponding words $w_{i,j}$ for $G_0$ get powered by 2 when $j$ increases. Hence the result follows.

2. Step. A similar argument as in 1. yields that $w_{t,j}(Y) = t_{1}y_{t}(Y)^{-2^j}$ for some word $y_t$. The words $v_{i,h,j}$ describe the action of the point group $P$ on the module $T/2^{e+j}T$. By [4], this action is induced by the action of $P$ on $T$ and hence we can choose the words $v_{i,h,j}$ independent of $j$.

3. Step. It sufficient to use $i = 1$ in the 3rd, 4th and 5th type of relation, since $t_2, \ldots, t_d$ are all conjugate to $t_1$ under the action of $P$. We add the relations $t_1t_i^{g_i}(X)$ to express this conjugacy relation.

The following theorem yields an improvement on Theorem 6 for sequences which are close to the main line.

**Theorem 7** Let $\mathcal{G} = (G_0, \ldots)$ be an infinite sequence associated with $S$. If $\mathcal{G}/\gamma$ has the parametrised presentation $\langle X \mid R_j \rangle$, then $\mathcal{G}$ has a parametrised presentation $\langle X \mid R_j' \cup \{s^{2^r} \mid s \in R_j \} \rangle$, where $s$ is a word in $X$ and $R_j' = \{rs^{e_r} \mid r \in R_j \}$ for certain $e_r \in \mathbb{Z}$.

**Proof:** Every group $G_j$ is an extension of $\gamma(G_j)$ by $G_j/\gamma(G_j)$. The group $\gamma(G_j)$ is cyclic of order 2 and thus generated by $s_j$, say. Using similar arguments as in the proof of Theorem 6, we can choose $s_j = s_1^{2^j}$. Further, we note that $s_j \in \gamma(G_j) \leq \Phi(G_j)$ and thus $s_j$ and in particular $s = s_1$ can be written as a word in $X$. It remains to observe that the exponents $e_r$ are independent of $j$. This, again, follows from the construction of the groups in the infinite sequences using cocycles $\epsilon \oplus \mu^j(\beta)$ as for the proof of Theorem 6.

**3.3 Bounding the deficiency**

**Corollary 8** Let $\mathcal{G}$ be an infinite sequence associated with $S$. Let $d$ be the dimension of $S$ and $n$ its generator number. Let $k$ be the distance of $\mathcal{G}$ to the main line. Then

a) $\text{def}(\mathcal{G}) \geq \text{def}(S) - (d + n)$.

b) $\text{def}(\mathcal{G}) \geq \text{def}(S) - (k + 1)$.

c) If $\mathcal{G}$ is a main line sequence and $\text{def}(S) = 0$, then $\text{def}(\mathcal{G}) = -1$. 
Proof: a) This follows directly from counting the generators and relations in Theorem 6.
b) We use induction on $k$. If $k = 0$, then $G$ is a main line sequence and the result follows from Lemma 5. Suppose that $G/\gamma$ has distance $k - 1$. Then $\text{def}(G/\gamma) \geq \text{def}(S) - k$ by induction. Theorem 7 implies that $\text{def}(G) \geq \text{def}(G/\gamma) - 1 \geq \text{def}(S) - k - 1$ as desired.
c) By [1], the main line groups have a non-trivial Schur multiplicator and hence they have negative deficiency. This together with b) for $k = 0$ yields the desired result.

Note that the distance $k$ of an infinite sequence of 2-groups of coclass $r$ is bounded by $2^{r-1}(2^{r-1} + r + 3)$ as proved in Theorem 11.3.7 of [4]. This together with Corollary 8 implies Theorem 2.

We believe that a much closer connection between the deficiencies of an infinite sequence and its associated pro-$p$-group holds. We propose the following conjecture.

Conjecture 9 If $G$ is an infinite sequence of deficiency zero, then $S$ is a pro-$2$-group of deficiency zero.

4 Pro-2-groups with trivial Schur multiplicator

The construction of the infinite sequences of 2-groups of deficiency zero requires as a first step the determination of the infinite pro-2-groups with trivial Schur multiplicator together with short presentations for them.

4.1 Coclass at most 3

By [1], there are 5 infinite pro-2-groups of coclass at most 3 and trivial Schur multiplicator. Pro-2-presentations of these groups are given in the following.

- $S_1 := \langle a, t \mid a^2 = 1, t^a = t^{-1} \rangle$.
- $S_2 := \langle a, t \mid a^4 = 1, t^a = t^{-1} \rangle$.
- $S_3 := \langle a, t \mid a^8 = 1, t^a = t^{-1} \rangle$.
- $S_4 := \langle a, b, c, t_1, t_2, d \mid a^2 = b, b^2 = t_2, c^2 = t_1, d^2 = c, e^a = b, t_1^a = t_2, t_2^a = t_1, c^b = c, t_2^a = t_1, t_2^a = t_1 \rangle$.
- $S_5 := \langle a, b, c, t_1, t_2, d \mid a^2 = d, b^2 = t_2, c^2 = t_1, d^2 = c, e^a = b, t_1^a = t_2, t_2^a = t_1, c^b = c, t_2^a = t_1, t_2^a = t_1 \rangle$.

The following table lists some properties of these groups.

<table>
<thead>
<tr>
<th>group</th>
<th>coclass</th>
<th>dimension</th>
<th>abelian invariants</th>
<th>metacyclic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>1</td>
<td>1</td>
<td>(2, 2)</td>
<td>yes</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>1</td>
<td>(2, 4)</td>
<td>yes</td>
</tr>
<tr>
<td>$S_3$</td>
<td>3</td>
<td>1</td>
<td>(2, 8)</td>
<td>yes</td>
</tr>
<tr>
<td>$S_4$</td>
<td>3</td>
<td>1</td>
<td>(2, 4)</td>
<td>no</td>
</tr>
<tr>
<td>$S_5$</td>
<td>3</td>
<td>2</td>
<td>(2, 4)</td>
<td>no</td>
</tr>
</tbody>
</table>

The abelian invariants exhibit that all these groups are 2-generated groups. Note that $S_1, S_2$ and $S_3$ have deficiency zero by their defining presentations. We consider the deficiency of $S_4$ and $S_5$ in the following.

Lemma 10 $S_4 = \langle a, u \mid a^2 = u^4, (u^2)^a = u^{-2} \rangle$ and thus $\text{def}(S_4) = 0$. 


Proof: The defining presentation of $S_4$ exhibits that the subgroup $\langle b \rangle$ of $S_4$ is a cyclic normal subgroup of order 4 and its quotient is isomorphic to $S_1$. Let $u = at^{-1}$ and replace $t$ by $u$ in the defining presentation. Then

$$S_4 = \langle a, b, u \mid a^2 = b^2, b^4 = 1, a^2 = u^2b, b^{u^{-1}a} = b^{-1}, b^a = b^{-1} \rangle,$$

$$= \langle a, b, u \mid a^2 = b^2, b^4 = 1, b = u^2, b^{u^{-1}a} = b^{-1}, b^a = b^{-1} \rangle.$$

Now we eliminate $b$ using that $b = u^2$ and find

$$S_4 = \langle a, u \mid a^2 = u^4, u^8 = 1, (u^2)^a = u^{-2} \rangle.$$

Note that $(u^2)^a = u^{-2}$ implies that $(u^4)^a = ((u^2)^a)^2 = u^{-4}$ and $a^2 = u^4$ implies that $(u^4)^a = u^4$ holds. Hence $u^8 = 1$ follows from the other relations and thus we finally obtain the desired presentation.

Lemma 11 $S_5 = \langle a, b \mid [b, a^2] = 1, a^2 = [b, a]^2, (b^2)^{[b, a]} = b^{-2} \rangle$ and thus $\text{def}(S_5) \in \{0, -1\}$.

Proof: The defining presentation of $S_5$ exhibits that $D = \langle d \rangle$ is a central subgroup of order 2 in $S_5$. Its quotient is a uniserial 2-adic space group of dimension 2 with translation subgroup $T/D = \langle t_1, t_2 \rangle D/D$ and point group $\langle a, b, c \rangle T/T$ of order 8. We eliminate the generators $t_1, t_2$ and $d$ in the defining presentation of $S_5$ using that $d = a^2$, $t_1 = c^2$ and $t_2 = b^2$. This yields

$$S_5 = \langle a, b, c \mid a^4 = 1, b^a = c, c^a = b, c^b = c^{-1}b^2a^2, (c^2)^b = c^{-2}, (b^2)^c = b^{-2}, (a^2)^b = a^2, (a^2)^c = a^2 \rangle.$$

Note that the centrality of $a^2$ follows already from the relations $b^a = c$ and $c^a = b$. Thus the relations $(a^2)^b = a^2$ and $(a^2)^c = a^2$ are redundant. Further, the relation $(c^2)^b = c^{-2}$ is redundant, since $c^b = c^{-1}b^2a^2$ and $(b^2)^c = b^{-2}$. Now we substitute $c$ by $u = b^{-1}c$ and obtain that

$$S_5 = \langle a, b, u \mid a^4 = 1, b^a = bu, u^a = u^{-1}, a^2 = u^2, (b^2)^u = b^{-2} \rangle.$$

The relation $a^4 = 1$ is redundant in this presentation, as $u^a = u^{-1}$ and $a^2 = u^2$ holds. Thus

$$S_5 = \langle a, b, u \mid b^a = bu, u^a = u^{-1}, a^2 = u^2, (b^2)^u = b^{-2} \rangle.$$

Finally, we eliminate $u$ using that $u = [b, a]$ and note that $[b, a]^a = [b, a]^{-1}$ simplifies to $ba^2 = a^2b$. Thus we obtain the desired presentation.

Based on experimental evidence using GAP[15] we propose the following conjecture.

Conjecture 12 $S_5 = \langle a, b \mid a^2 = [b, a]^2, (b^2)^{[b, a]} = b^{-2} \rangle$ and thus $\text{def}(S_5) = 0$. 

4.2 Higher coclasses and 3-generator groups

It is not difficult to construct further infinite pro-2-groups with finite coclass using the functionality of GAP. We iteratedly constructed central extensions with module cyclic of order 2 starting with the group $S_1$. This yields a variety of infinite pro-2-groups with finite coclass and many of them have trivial Schur multiplicator. There are also various 3-generator infinite pro-2-groups with trivial Schur multiplicator arising this way: we found

one such group of coclass 5, five such groups of coclass 6 and twenty-six such groups of coclass 7. A presentation for the obtained 3-generator group of coclass 5 is the following.

$$S_6 := \langle a, t, b, c \mid a^2 = b^2, a^8 = 1, c^2 = 1, t^a = t^{-1} c, b^a = bc, b^t = b^3 c, [c, a] = 1, [c, b] = 1 \rangle.$$

A Smith normal form computation on this presentation of $S_6$ shows readily that $S_6$ has the abelian invariants $(2, 2, 2)$ and hence is a 3-generator group.

**Lemma 13** $S_6 = \langle a, t, b \mid a^2 = b^2, [b, a]^2 = 1, t^a = t^{-1} [b, a], b^t = aba \rangle$ and thus $\text{def}(S_6) \in \{0, -1\}$.

*Proof:* The relation $b^a = bc$ implies that $b^2 = (b^2)^a = bcbc$. As $c^2 = 1$, it follows that $[b, c] = 1$. Hence the relator $[b, c]$ is redundant. Also, $b = b^{a^2} = (bc)^a = b^a c^a = bc^a$ hence $c^a = c^{-1}$ follows. Using $c^2 = 1$, we obtain that $[a, c] = 1$. Hence the relator $[a, c]$ is redundant. Using $t^a = t^{-1} c$, we find that $t^{a^2} = (t^{-1} c)^a = ctc$ and $t^{a^3} = c t^{-1} t$. Hence $t^{a^4} = t$ and $[t, a^4] = [t, b^2] = 1$. Now $b^4 = (b^2)^4 = (b^3 c)^4 = b^{12}$ and thus $b^8 = 1$ follows. This yields that the relator $a^8$ is redundant. In summary, we obtain that

$$S_6 = \langle a, t, b, c \mid a^2 = b^2, c^2 = 1, t^a = t^{-1} c, b^a = bc, b^t = b^3 c \rangle.$$

Now we eliminate $c$ using that $c = [b, a]$ and obtain the desired presentation.

Based on experimental evidence using GAP we suggest the following conjecture.

**Conjecture 14** $S_6 = \langle a, t, b \mid a^2 = b^2, t^a = t^{-1} [b, a], b^t = aba \rangle$ and thus $\text{def}(S_6) = 0$.

5 Infinite sequences – the metacyclic case

Now we investigate the infinite sequences of finite 2-groups arising from the infinite pro-2-groups exhibited in the previous section. We first consider the simplest case: the infinite sequences associated with $S_1, S_2$ and $S_3$. More generally, we consider the infinite pro-2-groups

$$M_n = \langle a, t \mid a^{2^n} = 1, t^a = t^{-1} \rangle.$$

We show in this section that the deficiency of an infinite sequence associated to $M_n$ is either 0 or $-1$ and it is 0 if and only if the groups in the sequence have trivial Schur multiplicator.

**Lemma 15** Let $G = (G_0, \ldots)$ be an infinite sequence associated with $M_n$ for some $n \in \mathbb{N}$. Then each $G_j$ in $G$ is metacyclic and there exist $h, k \in \mathbb{N}_0$ and $l \in \mathbb{Z}_2$ (the 2-adic numbers) so that

$$G_j = \langle a, t \mid a^{2^n} = t^{2^{i+h}}, t^a = t^{-1+l2^j}, t^{2^j+k} = 1 \rangle.$$
Proof: The group \( M_n \) has dimension 1. By Theorem 6, this implies that the sequence \( G \) has a parametrised presentation on generators \( X \cup Y \) with \( X = \{a, t\} \) and \( Y = \{y\} \) and relators \( R_j \cup Q_j \cup O \). We discuss these three sets of relators in more detail.

The set \( R_j \) contains three relators. Since \( Y \) contains one element only, the words \( y_i(Y) \) are all just powers of \( y \). The word \( t(X) \) is a generator of a suitable translation subgroup \( T \) of \( M_n \) and hence can be chosen as \( t^{2m} \) for some \( m \). Thus there exist \( u, v, w \in \mathbb{Z}_2 \) so that the relators in \( R_j \) translate to the relations

\[
    a^{2m} = y^{u2^j}, \quad t^n = t^{-1} y^{v2^j}, \quad \text{and} \quad t^{2m} = y^{1+u2^j}.
\]

Replacing \( y^{1+u2^j} \) by \( y \) does not affect the general form of the relations. Hence we can assume that the third relation in \( R_j \) reads \( t^{2m} = y \). We use this relation to eliminate \( y \) from the generating set and obtain that \( R_j \) has the form \( R_j = \{a^{2m} t^{-u2^j+m}, t^n t^{-v2^j+m}\} \) for some \( u, v \in \mathbb{Z}_2 \). Let \( u = u'2^c \) with \( u' \) invertible in \( \mathbb{Z}_2 \). Then again replacing \( t^{u'} \) by \( t \) does not affect the general form of the relations. Further let \( l = v2^m \) and \( h = m+c \). Then we obtain that \( R_j = \{a^{2m} t^{-u'2^j+h}, t^n t^{-l2^j}\} \).

As \( M_n \) has dimension 1 for every \( n \), it follows that \( Q_j \) contains a single relator only and this has the form \( y^{b+j} \) for some \( b \). The elimination of \( y \) translates this to \( t^{2b+j+m} = t^{k+j} \) for \( k = m + b \).

It remains to show that the relators in \( O \) are redundant. These are two relators of the form \( y^{v_{11}} \) and \( y^{v_{12}} \), where the words \( v_{1j} \) reflect the action of the point group on the translation subgroup. Since \( a \) acts by inverting and \( t \) acts trivially on the translations, we obtain that \( v_{11} = y \) and \( v_{12} = y^{-1} \). The elimination of \( y \) translates this to \( (t^{2m})^a = (t^{2m})^{-1} \) and \( (t^{2m})^t = t^{2m} \). The later relation is obviously redundant. The first relator follows from \( Q_j \) and \( R_j \) if \( m \geq b \). Note that we can choose \( m \) large enough so that this holds and hence both relators in \( O \) are redundant.

This asserts that the infinite sequences \( G \) associated with a group \( M_n \) have deficiency 0 or \(-1 \). The following adaptation of a result by Beyl, see page 66 of [10], shows that they have parametrised presentations of deficiency zero if and only if they have trivial Schur multiplier.

**Theorem 16** Let \( G_j = \langle a, t \mid a^{2m} = t^{2^{j+h}}, t^n = t^{-1+2^j}, t^{2^{j+k}} = 1 \rangle \) as in Lemma 15. Let \( u_j \) be the inverse of \(-1+2^{j-1} \) modulo \( 2^{j+k-1} \) and define \( v_j = u_j + 2^{(j+k)-1} \).

a) Then \( M(G_j) = 1 \) if and only if \( h = k - 1 \).

b) If \( M(G_j) = 1 \), then \( G_j \cong \langle a, t \mid a^{2n} = t^{2^{j+k}}, [a, t^{-v_j}] = t^2 \rangle \).

**Proof:** a) \( |M(G_j)| = 2^{j+h} \gcd(2^{j+k}, -2+2^j) = 2^{j+k-1} - 2^{k-1} = 2^{h-k+1} \) as observed on page 66 in [10].

b) Theorem 2 on page 66 of [10] yields this result for some integer \( v_j \). Following the proof of this Theorem, we find that \( v_j = u_j + 2^{(j+k)-1} \) as desired.

6 Infinite sequences – the non-metacyclic case

Our aim in this section is to investigate some examples of infinite sequences of non-metacyclic groups concerning their deficiencies. We use a combination of theoretical and computational methods.
6.1 Sequences associated with $S_4$

There are five infinite coclass sequences associated with $S_4$, see also [12] or [2], depending on two parameters $(g, f) \in \{(0, 0), (2, 0), (0, 2), (1, 0), (1, 2)\}$. Based on Theorem 6, we find the following parametrised presentations for them:

$$G_j(g, f) = \langle a, u, y \mid a^2u^{-4}y^{g_2}, (u^2)^a a^2y^{f_2}, (u^{-1}a)^{2^7} y^{-1}, y^{2g_2}, y^a, y^uy \rangle.$$

**Lemma 17** $G_j(g, f) = \langle a, u \mid a^2u^{-4}(u^{-1}a)^{g_2+1}, (u^2)^a u^2(u^{-1}a)^{f_2+7}, (u^{-1}a)^{2g_2+9} \rangle$ and thus $G_j(g, f)$ has deficiency at least -1.

**Proof.** The elimination of $y$ is obvious. Further, one of the two relators in $O$ is redundant, since $u^{-1}a$ acts trivially on $y$. Let $j \geq 7$. Then the following relations remain

$$a^2u^{-4} = (u^{-1}a)^{g_2}, (u^2)^a u^2 = (u^{-1}a)^{f_2}, (u^{-1}a)^{2g_2+2}, ((u^{-1}a)^{2^7})^a = (u^{-1}a)^{-2^7}.$$

We replace $a$ by $x = a^{-1}u$ and translate these relations to

$$ux^{-1}ux^{-1}u^{-4} = x^{g_2}, u^2((u^2)^x x^{f_2}, x^{2g_2+2} = 1, (x^{2^7})^u = x^{-2^7}).$$

Now the second relation implies that $(u^2)^x x^{2g_2+2} = x^{-f_2} u^2 x^{f_2}$ and thus

$$(u^2)^x x^{2g_2+2} = x^{-f_2} u^2 x^{f_2+1} = x^{-f_2+1} u^2 x^{f_2+1}.$$  

Hence $(u^2)^x x^{2g_2+2} = x^{-f_2+1} u^2 x^{f_2+1}$ or $[x, u] = 1$ follows. Therefore, the second relation implies that $u^2xu^2 = x^{f_2+1}$. Now the first relation yields $ux^{-1}u = u^4x^{g_2+1}$ or $x^u = x^{-g_2-1}u^{-2}$ and hence $(x^u)^n = x^{-g_2+1}u^{-2}x^{-1}u^{-2} = x^{-g_2+1}x^{2g_2+2}$. We obtain that $(x^u)^n = (u^2)^{2g_2+2} = x^{-8}$ and thus the other relator in $O$ is also redundant.  

As observed in [3], the infinite sequences with $(g, f) = (0, 0)$ and $(g, f) = (2, 0)$ have non-trivial Schur multiplicator. This yields the following.

**Corollary 18** For $(g, f) \in \{(0, 0), (2, 0)\}$ the presentation in Lemma 17 is efficient and thus def$(G_j(g, f)) = -1$ follows in these cases.

The infinite sequences with $(g, f) \in \{(0, 2), (1, 0), (1, 2)\}$ contain groups with trivial Schur multiplicator. For the two sequences with parameters $(1, 0)$ and $(0, 2)$ we could not identify a deficiency zero presentation. But based on experimental evidence using GAP we found the following conjecture.

**Conjecture 19** The sequence defined by the parameters $(1, 2)$ has deficiency zero with the presentation $G_j(1, 2) = \langle a, u \mid a^2u^{-4}t^{2g_2} = 1, (u^2)^a u^2t^{-2g_2+4} = 1 \rangle$ for $t = (u^{-1}a)^{2^7}$.

6.2 Families associated with $S_6$

There are eleven infinite coclass sequences associated with $S_6$. All of them consist of 3-generators groups. We define the families based on 4 parameters $(e, f, g, h)$. We note that all infinite sequences arising in this case have distance at most 3 from the main line. Based on Theorem 7, we exhibit the following parametrised presentations for them:

$$G_j(e, f, g, h) = \langle a, b, t \mid a^2b^{-2}t^{2g_2}, [b, a]^{2}t^{f_2}, t^a(t^{-1}[b, a])^{-1}t^{g_2}, b^t(b^3[b, a])^{-1}b^{h_2}, t^{2g_2+3} \rangle.$$

This implies that the deficiency of the groups $G_j(e, f, g, h)$ is at least -2. Using the algorithm of [3] we determine the Schur multiplicators for the different sequences as follows.
a) The sequences with \((e, f, g, h) \in \{(1, 4, 0, 1), (1, 4, 0, -1), (1, 2, 0, -2), (1, -2, 0, 4), (-1, 2, 0, 4), (1, 2, 0, 2)\}\) are sequences of groups with trivial Schur multiplicators.

b) The sequences with \((e, f, g, h) \in \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (1, 0, 0, 0), (0, 0, 2, 4)\}\) are sequences of groups with Schur multiplicators of order 2.

We investigated the infinite sequences with trivial Schur multiplicators further. Based on experimental evidence, we propose the following conjecture.

**Conjecture 20** All infinite coclass sequences associated with \(S_6\) and having trivial Schur multiplicator have deficiency zero with following presentations

\[
G_j(1, 4, 0, 1) \cong \langle a, b, t \mid a^2b^{-2t^2}, t^a(t^{-1}[b, a])^{-1}, b^j(b^3[b, a])^{-1}t^{2j}\rangle,
\]

\[
G_j(1, 4, 0, -1) \cong \langle a, b, t \mid a^2b^{-2t^2}, t^a(t^{-1}[b, a])^{-1}, b^j(b^3[b, a])^{-1}t^{-2j}\rangle,
\]

\[
G_j(1, 2, 0, -2) \cong \langle a, b, t \mid a^2b^{-2t^2}, t^a(t^{-1}[b, a])^{-1}, b^j(b^3[b, a])^{-1}t^{-2j}\rangle,
\]

\[
G_j(1, 2, 0, 2) \cong \langle a, b, t \mid a^2b^{-2t^2}, t^a(t^{-1}[b, a])^{-1}, b^j(b^3[b, a])^{-1}t^{2j+1}\rangle,
\]

\[
G_j(1, -2, 0, 4) \cong \langle a, b, t \mid a^2b^{-2t^2}, t^a(t^{-1}[b, a])^{-1}, b^j(b^3[b, a])^{-1}t^{2j+1}\rangle,
\]

\[
G_j(-1, 2, 0, 4) \cong \langle a, b, t \mid a^2b^{-2t^2}, t^a(t^{-1}[b, a])^{-1}, b^j(b^3[b, a])^{-1}t^{2j+2}\rangle.
\]

**References**


