Enumeration of groups whose order factorises in at most 4 primes

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Abstract

Let $\mathcal{N}(n)$ denote the number of isomorphism types of groups of order n. We consider the integers n that are products of at most 4 not necessarily distinct primes and exhibit formulas for $\mathcal{N}(n)$ for such n.

1 Introduction

The construction up to isomorphism of all groups of a given order n is an old and fundamental problem in group theory. It has been initiated by Cayley [6] who determined the groups of order at most 6. Many publications have followed Cayley's work; A history of the problem can be found in [4].
The enumeration of the isomorphism types of groups of order n is a related problem. The number $\mathcal{N}(n)$ of isomorphism types of groups of order n is known for all n at most 2 000, see [4], and for almost all n at most 20 000, see [9]. Asymptotic estimates for $\mathcal{N}(n)$ have been determined in [21]. However, there is no closed formula known for $\mathcal{N}(n)$ as a function in n. Many details on the group enumeration problem can be found in [7] and [5].
Higman [12] considered prime-powers p^m. His PORC conjecture suggests that $\mathcal{N}(p^m)$ as a function in p is PORC (polynomial on residue classes). This has been proved for all $m \leq 7$, see Hölder [13] for $m \leq 4$, Bagnera [1] or Girnat [10] for $m = 5$, Newman, O'Brien & Vaughan-Lee [19] for $m = 6$ and O'Brien & Vaughan-Lee [20] for $m = 7$. To exhibit the flavour of the results, we recall the explicit PORC polynomials for $\mathcal{N}(p^m)$ for $m \leq 5$ as follows.

1 Theorem: (Hölder [13], Bagnera [1])

- $\mathcal{N}(p^1) = 1$ *for all primes* p.
- $\mathcal{N}(p^2) = 2$ *for all primes* p.
- $\mathcal{N}(p^3) = 5$ *for all primes* p.
- $\mathcal{N}(2^4) = 14$ *and* $\mathcal{N}(p^4) = 15$ *for all primes* $p \geq 3$.
- $\mathcal{N}(2^5) = 51$, $\mathcal{N}(3^5) = 71$ *and* $\mathcal{N}(p^5) = 2p + 61 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$ *for all primes* $p \geq 5$.

Hölder [14] determined a formula for $\mathcal{N}(n)$ for all square-free n. For $m \in \mathbb{N}$ let $\pi(m)$ denote the number of different primes dividing m. For $m \in \mathbb{N}$ and a prime p let $c_m(p)$ denote the number of prime divisors q of m with $q \equiv 1 \bmod p$. The following is also proved in [5, Prop. 21.5].

2 Theorem: (Hölder [14])
Let $n \in \mathbb{N}$ be square free. Then

$$\mathcal{N}(n) = \sum_{m|n} \prod_{p \in \pi(\frac{n}{m})} \frac{p^{c_m(p)} - 1}{p - 1}.$$

The aim here is to determine formulas for $\mathcal{N}(n)$ if n is a product of at most 4 primes. If n is a prime-power or is square-free, then such formulas follow from the results cited above. Hence it remains to consider the numbers n that factorise as p^2q, p^3q, p^2q^2 or p^2qr for different primes p, q and r. For each of these cases we determine an explicit formula for $\mathcal{N}(n)$, see Theorems 14, 15, 16 and 17. Each of these formulas is a polynomial on residue classes; that is, there are finitely many sets of number-theoretic conditions on the involved primes so that $\mathcal{N}(n)$ is a polynomial in the involved primes for each of the condition-sets. We summarise this in the following theorem.

3 Theorem: (See Theorems 14, 15, 16, 17 below)
Let p, q and r be different primes and $n \in \{p^2q, p^3q, p^2q^2, p^2qr\}$. Then $\mathcal{N}(n)$ is a polynomial on residue classes.

The enumerations obtained in this paper overlap with various known results. For example, Hölder [13] considered the groups of order p^2q, Western [24] those of order p^3q, Le Vavasseur [16, 17] and Lin [18] those of order p^2q^2 and Glenn [11] those of order p^2qr. Moreover, Laue [15] considered all orders of the form p^aq^b with $a + b \leq 6$ and $a < 5$ and $b < 5$ as well as the orders dividing $p^2q^2r^2$.
So why are these notes written? There are two reasons. First, they provide a uniform and reasonably compact proof for the considered group numbers and they exhibit the resulting group numbers as a closed formula with few case distinctions. Laue's work [15] also contains a unified approach towards the determination of its considered groups and this approach is similar to ours, but it is not easy to read and to extract the results. Our second aim with these notes is to provide a uniform and reliable source for the considered group enumerations. The reliability of our results is based on its proofs as well as on a detailed comparision with the Small Groups library [3].

Acknowledgements

We give more details on the results available in the literature that overlap with the results here in the discussions before the Theorems 14, 15, 16 and 17. Most of these details have been provided by Mike Newman. The author thanks Mike Newman for this and also for various discussions on these notes.

2 Divisibility

For $r, s \in \mathbb{N}$ we define the function $w_r(s)$ via $w_r(s) = 1$ if $s \mid r$ and $w_r(s) = 0$ otherwise. The following remark exhibits the relation of $w_r(s)$ and the underlying gcd's.

4 Remark: *For $r, s \in \mathbb{N}$ it follows that*

$$w_r(s) = \prod_{d|s, d\neq s} \frac{\gcd(r,s) - d}{s - d}.$$

3 Counting subgroups of linear groups

For $r \in \mathbb{N}$ and a group G we denote with $s_r(G)$ the number of conjugacy classes of subgroups of order r in G. We recall the following well-known result.

5 Remark: *Let $n \in \mathbb{N}$ and p prime. Then $\mathrm{GL}(n,p)$ has an irreducible cyclic subgroup of order m if and only if $m \mid (p^n - 1)$ and $m \nmid (p^d - 1)$ for each $d < n$. Further, if there exists an irreducible cyclic subgroup of order m in $\mathrm{GL}(n,p)$, then it is unique up to conjugacy.*

The next theorem counts the conjugacy classes of subgroups of certain orders in $\mathrm{GL}(2,p)$. With C_r^s we denote the s-fold direct product of cyclic groups of order r.

6 Theorem: *Let p, q and r be different primes and let $G = \mathrm{GL}(2,p)$.*

(a) *$s_2(G) = 2$ and for $q > 2$ it follows that*

$$s_q(G) = \frac{q+3}{2} w_{p-1}(q) + w_{p+1}(q).$$

(b) *$s_4(G) = 2 + 3w_{p-1}(4)$ and for $q > 2$ it follows that*

$$s_{q^2}(G) = w_{p-1}(q) + \frac{q^2+q+2}{2} w_{p-1}(q^2) + w_{p+1}(q^2).$$

(c) *$s_{2r}(G) = \frac{3r+7}{2} w_{p-1}(r) + 2w_{p+1}(r)$ and for $r > q > 2$ it follows that*

$$s_{qr}(G) = \frac{qr+q+r+5}{2} w_{p-1}(qr) + w_{p^2-1}(qr)(1 - w_{p-1}(qr)).$$

Proof: Let $m \in \mathbb{N}$ with $p \nmid m$. As a preliminary step in this proof, we investigate the number of conjugacy classes of cyclic subgroups of order m in $\mathrm{GL}(2,p)$. By Remark 5, an irreducible subgroup of this form exists if and only if $m \mid (p^2 - 1)$ and $m \nmid (p-1)$ and its conjugacy class is unique in this case. A reducible cyclic subgroup of order m in $\mathrm{GL}(2,p)$ embeds into the group of diagonal matrices D. Note that $D \cong C_{p-1}^2$ and thus $\mathrm{GL}(2,p)$ has a reducible cyclic subgroup of order m if and only if m divides the exponent $p-1$ of D. If $m \mid (p-1)$, then there exists a unique subgroup $U \cong C_m^2$ in D. This subgroup U contains every cyclic subgroup of order m in D. The group $\mathrm{GL}(2,p)$ acts on D and on U by permutation of the diagonal entries of an element of D.
(a) Each group of prime order q is cyclic. An irreducible subgroup of order q exists if and only if $q \neq 2$ and $q \mid (p+1)$, since $p^2 - 1 = (p-1)(p+1)$ and $\gcd(p-1, p+1) \mid 2$. A reducible subgroup of order q exists if and only $q \mid (p-1)$. In this case, the number of conjugacy classes of reducible cyclic subgroups of order q in $\mathrm{GL}(2,p)$ can be enumerated as 2 if $q = 2$ and $(q+3)/2$ otherwise, as there are $(q+1)$ subgroups of order q in C_q^2 and all but the subgroups with diagonals of the form (a,a) or (a,a^{-1}) for $a \in \mathbb{F}_p^*$ have orbits of length two under the action of $\mathrm{GL}(2,p)$ by permutation of diagonal entries.

(b) We first consider the cyclic subgroups of order q^2 in $\mathrm{GL}(2,p)$. For the irreducible case, note that if $q^2 \mid (p^2-1)$ and $q^2 \nmid (p-1)$, then either $q=2$ and $4 \nmid (p-1)$ or $q>2$ and $q^2 \mid (p+1)$. For the reducible case we note that if $q^2 \mid (p-1)$, then there are $(q^2+q+2)/2$ conjugacy classes of reducible cyclic subgroups of order q^2 in $\mathrm{GL}(2,p)$, as there are (q^2+q) subgroups and all but those with diagonal of the form (a,a) and (a,a^{-1}) for $a \in \mathbb{F}_p^*$ have orbits of length two. Thus the number of conjugacy classes of cyclic subgroups of order q^2 in $\mathrm{GL}(2,p)$ is $1+3w_{p-1}(4)$ if $q=2$ and $(q^2+q+2)/2 \cdot w_{p-1}(q^2)+w_{p+1}(q^2)$ if $q>2$. It remains to consider the subgroups of type C_q^2. Such a subgroup is reducible and exist if $q \mid (p-1)$. In this case there exists a unique conjugacy class of such subgroups.
(c) We first consider the cyclic subgroups of order qr in $\mathrm{GL}(2,p)$. If $q=2$, then $p \neq 2$. Thus $2r \mid (p^2-1)$ and $2r \nmid (p-1)$ if and only if $r \mid (p+1)$. As in the previous cases, this yields that there are $(3r+5)/2 \cdot w_{p-1}(r)+w_{p+1}(r)$ cyclic subgroups of order $2r$ in $\mathrm{GL}(2,p)$. If $q>2$, then $r>2$. The number of cyclic subgroups of order qr in $\mathrm{GL}(2,r)$ in this case is $(qr+r+q+5)/2 \cdot w_{p-1}(qr)+w_{p^2-1}(qr)(1-w_{p-1}(qr)$ using the same arguments as above. It remains to consider the case of non-cyclic subgroups. Such a subgroup H is irreducible and satisfies $q=2$. If H is imprimitive, then C_r is diagonalisable; there is one such possibility if $r \mid (p-1)$. If H is primitive, then C_r is irreducible; there is one such possibility if $r \mid (p+1)$. •

We extend Theorem 6 with the following.

7 Remark: *Let p and q be different primes and let $G = \mathrm{GL}(2,p)$. If H is a subgroup of order q in G, then $[N_G(H) : C_G(H)] \mid 2$. The number of groups H of order q in G satisfying $[N_G(H) : C_G(H)] = 2$ is 0 for $q=2$ and $w_{p-1}(q)+w_{p+1}(q)$ for $q>2$.*

Proof: Consider the groups H of order q in $\mathrm{GL}(2,p)$. If H is irreducible, then H is a subgroup of a Singer cycle and this satisfies $[N_G(H) : C_G(H)] = 2$. If H is reducible, then H is a subgroup of the group of diagonal matrices D. The group D satisfies $[N_G(D) : C_G(D)] = 2$, where the normalizer acts by permutation of the diagonal entries. As in the proof of Theorem 6 (a), only the group H with diagonal of the form (a,a) for $a \in \mathbb{F}_p^*$ satisfies $[N_G(H) : C_G(H)] = 2$. •

8 Theorem: *Let p be a prime, let $G = \mathrm{GL}(3,p)$ and let $q \neq p$. Then $s_2(G) = 3$ and for $q > 2$ it follows that*

$$s_q(G) = \frac{q^2+4q+9+4w_{q-1}(3)}{6} w_{p-1}(q) + w_{(p+1)(p^2+p+1)}(q)(1-w_{p-1}(q)).$$

Proof: We first consider the diagonalisable subgroups of order q in $\mathrm{GL}(3,p)$. These exist if $q \mid (p-1)$. If this is the case, then the group D of diagonal matrices has a subgroup of the form C_q^3 and this contains all subgroups of order q in D. The group D has q^2+q+1 subgroups of order q and these fall under the permutation action of diagonal entries into 3 orbits if $q=2$, into $(q^2+4q+9)/6$ orbits if $q>2$ and $3 \nmid (q-1)$ and into $(q^2+4q+13)/6$ orbits if $q>2$ and $3 \mid (q-1)$. Next, we consider the groups that are not diagonalisable. These can arise from irreducible subgroups in $\mathrm{GL}(2,p)$ or in $\mathrm{GL}(3,p)$. In the first case, there exists one such class if $q>2$ and $q \mid (p+1)$ as in Theorem 6. In the second case, by Remark 5 there exists one such class if $q \mid (p^3-1)$ and $q \nmid (p^2-1)$ and $q \nmid (p-1)$.

Note that the two cases are mutually exclusive. In summary, there exists an irreducible subgroup of order q in $\mathrm{GL}(3,p)$ if $q > 2$ and $q \mid (p+1)(p^2+p+1)$ and $q \nmid (p-1)$. •

We note that $\gcd((p+1)(p^2+p+1), p-1) \mid 6$ for each prime p. Thus for $q \geq 5$ it follows that $w_{(p+1)(p^2+p+1)}(q) = w_{(p+1)(p^2+p+1)}(q)(1 - w_{p-1}(q))$ which simplifies the formula in Theorem 8.

4 Counting split extensions

For two groups N and U let $\Pi(U,N)$ denote the set of all group homomorphisms $\varphi : U \to \mathrm{Aut}(N)$. The direct product $\mathrm{Aut}(U) \times \mathrm{Aut}(N)$ acts on the set $\Pi(U,N)$ via

$$\varphi^{(\alpha,\beta)}(g) = \beta^{-1}(\varphi(\alpha^{-1}(g)))\beta$$

for $(\alpha, \beta) \in \mathrm{Aut}(U) \times \mathrm{Aut}(N)$, $\varphi \in \Pi(U,N)$ and $g \in N$. If $\overline{\beta}$ is the conjugation by β in $\mathrm{Aut}(N)$, then this action can be written in short form as

$$\varphi^{(\alpha,\beta)} = \overline{\beta} \circ \varphi \circ \alpha^{-1}.$$

Given $\varphi \in \Pi(U,N)$, the stabilizer of φ in $\mathrm{Aut}(U) \times \mathrm{Aut}(N)$ is called the group of compatible pairs and is denoted by $Comp(\varphi)$. If N is abelian, then N is an U-module via φ for each $\varphi \in \Pi(U,N)$. In this case $Comp(\varphi)$ acts on $H^2_\varphi(U,N)$ induced by its action on $Z^2_\varphi(U,N)$ via

$$\gamma^{(\alpha,\beta)}(g,h) = \beta^{-1}(\gamma(\alpha^{-1}(g), \alpha^{-1}(h)))$$

for $\gamma \in Z^2_\varphi(U,N)$, $(\alpha,\beta) \in Comp(\varphi)$ and $g,h \in U$. These constructions can be used to solve the isomorphism problem for extensions in two different settings. We recall this in the following.

4.1 Extensions with abelian kernel

Suppose that N is abelian and that N is fully invariant in each extension of N by U; this is, for example, the case if N and U are coprime or if N maps onto the Fitting subgroup in each extension of N by U. The following theorem seems to be folklore.

9 Theorem: *Let N be finite abelian and U be a finite group so that N is fully invariant in each extension of N by U. Let $\mathcal{O}$ be a complete set of representatives of the $\mathrm{Aut}(U) \times \mathrm{Aut}(N)$ orbits in $\Pi(U,N)$ and for each $\varphi \in \mathcal{O}$ let o_φ denote the number of orbits of $Comp(\varphi)$ on $H^2_\varphi(U,N)$. Then the number of isomorphism types of extensions of N by U is*

$$\sum_{\varphi \in \mathcal{O}} o_\varphi.$$

The following theorem proved in [8, Th. 14] exploits the situation further in a special case. Again, C_l denotes the cyclic group of order l.

10 Theorem: (Dietrich & Eick [8])
Let p be a prime, let $N \cong C_p$, and let U be finite with Sylow p-subgroup P so that $P \cong C_p$. Then there are either one or two isomorphism types of extensions of N by U. There are two isomorphism types of extensions if and only if N and P are isomorphic as $N_U(P)$-modules.

4.2 A special type of split extensions

In this section we recall a variation of a theorem by Taunt [23]. As a preliminary step we introduce some notation. Let N and U be finite solvable groups of coprime order. Let $\mathcal{S}$ denote a set of representatives for the conjugacy classes of subgroups in $\mathrm{Aut}(N)$, let $\mathcal{K}$ denote the set of representatives for the $\mathrm{Aut}(U)$-classes of normal subgroups in U and let $\mathcal{O} = \{(S, K) \mid S \in \mathcal{S}, K \in \mathcal{K} \text{ with } S \cong U/K\}$. For $(S, K) \in \mathcal{O}$ let $\iota : U/K \to S$ denote a fixed isomorphism, let A_K denote the subgroup of $\mathrm{Aut}(U/K)$ induced by the action of $Stab_{\mathrm{Aut}(U)}(K)$ on U/K, and denote with A_S the subgroup of $\mathrm{Aut}(U/K)$ induced by the action of $N_{\mathrm{Aut}(N)}(S)$ on S and thus, via ι, on U/K. Then the double cosets of the subgroups A_K and A_S in $\mathrm{Aut}(U/K)$ are denoted by

$$\mathrm{DC}(S, K) := A_K \setminus \mathrm{Aut}(U/K) \; / \; A_S.$$

11 Theorem: *Let N and U be finite solvable groups of coprime order. Then the number of isomorphism types of split extensions $N \rtimes U$ is*

$$\sum_{(S,K)\in\mathcal{O}} |\mathrm{DC}(S, K)|.$$

Proof: Taunt's theorem [23] claims that the number of isomorphism types of split extensions $N \rtimes U$ correspond to the orbits of $\mathrm{Aut}(U) \times \mathrm{Aut}(N)$ on $\Pi(U, N)$. In turn, these orbits correspond to the union of orbits of $A_K \times N_{\mathrm{Aut}(N)}(S)$ on the set of isomorphisms $U/K \to S$. The latter translate to the double cosets $\mathrm{DC}(S, K)$. •

We apply Theorem 11 in two special cases in the following. Again, let C_l denote the cyclic group of order l.

12 Theorem: *Let q^k be a prime-power, let $N = C_{q^k}$ and let U be a finite group of order coprime to q. Denote $\pi = \mathrm{gcd}(|U|, q^{k-1}(q-1))$. For $l \mid \pi$, let $\mathcal{K}_l$ be a set of representatives of the Aut(U)-classes of normal subgroups K in U with $U/K \cong C_l$. For $K \in \mathcal{K}_l$ let $\mathrm{ind}_K = [\mathrm{Aut}(U/K) : A_K]$. If $k \leq 2$ or q is odd, then the number of isomorphism types of split extensions $N \rtimes U$ is*

$$\sum_{l|\pi} \sum_{K\in\mathcal{K}_l} \mathrm{ind}_K.$$

Proof: We apply Theorem 11. The group $\mathrm{Aut}(N)$ is cyclic of order $p^{k-1}(p-1)$. Hence for each $l \mid \pi$ there exists a unique subgroup S of order l in $\mathrm{Aut}(N)$ and this subgroup is cyclic. Next, as $\mathrm{Aut}(N)$ is abelian, it follows that $N_{\mathrm{Aut}(N)}(S) = \mathrm{Aut}(N)$ and $\mathrm{Aut}(N)$ acts trivially on S by conjugation. Hence A_S is the trivial group and $|\mathrm{DC}(S, K)| = \mathrm{ind}_K$ for each K. •

13 Theorem: *Let q^k be a prime-power, let $U = C_{q^k}$ and let N be a finite group of order coprime to q. Let $\mathcal{S}$ be a set of conjugacy class representatives of cyclic subgroups of order dividing q^k in* $\mathrm{Aut}(N)$. *Then the number of isomorphism types of split extensions $N \rtimes U$ equals $|\mathcal{S}|$.*

Proof: Again we use Theorem 11. For each divisor p^l of p^k there exists a unique normal subgroup K in U with $|K| = p^l$ and U/K is cyclic of order p^{k-l}. Note that $A_K = \mathrm{Aut}(U/K)$ for each such K. Hence $|\mathrm{DC}(S, K)| = 1$ in all cases. •

5 Groups of order p^2q

The groups of order p^2q have been considered by Hölder [13], by Lin [18], by Laue [15] and in various other places. The results by Hölder, Lin and Laue agree with ours. (Lin's results have some harmless typos). We also refer to [5, Prop. 21.17] for an alternative description and proof of the following result.

14 Theorem: *Let p and q be different primes.*

(a) *If $q = 2$, then $\mathcal{N}(p^2q) = 5$.*

(b) *If $q > 2$, then*

$$\mathcal{N}(p^2q) = 2 + (q+5)/2 \cdot w_{p-1}(q) + w_{p+1}(q) + 2w_{q-1}(p) + w_{q-1}(p^2).$$

Proof: The classification of groups of order p^2 yields that there are two nilpotent groups of order p^2q: the groups $C_{p^2} \times C_q$ and $C_p^2 \times C_q$. It remains to consider the non-nilpotent groups of the desired order. Note that every group of order p^2q is solvable by Burnside's theorem.

Non-nilpotent groups with normal p-Sylow subgroup. These groups have the form $N \rtimes U$ with $|N| = p^2$ and $U = C_q$. We use Theorem 13 to count the number of such split extensions. Thus we count the number of conjugacy classes of subgroups of order q in $Aut(N)$.

- $N \cong C_p^2$. Then $\mathrm{Aut}(N) \cong GL(2,p)$ and the number of conjugacy classes of subgroups of order q in $\mathrm{Aut}(N)$ is exhibited in Theorem 6 (a).
- $N \cong C_{p^2}$. Then $\mathrm{Aut}(N) \cong C_{p(p-1)}$ is cyclic. Thus there is at most one subgroup of order q in $Aut(N)$ and this exists if and only if $q \mid (p-1)$.

Non-nilpotent groups with normal q-Sylow subgroup. These groups have the form $N \rtimes U$ with $N = C_q$ and $|U| = p^2$. We use Theorem 12 to count the number of such split extensions. For this purpose we have to consider the $\mathrm{Aut}(U)$-classes of proper normal subgroups K in U with U/K cyclic.

- $U \cong C_{p^2}$. Then U has a two proper normal subgroups K with cyclic quotient. The case $K = 1$ arises if and only if $p^2 \mid (q-1)$ and the case $K = C_p$ arises if and only if $p \mid (q-1)$. In both cases $\mathrm{ind}_K = 1$.
- $U \cong C_p^2$. Then there exists one $\mathrm{Aut}(U)$-class of normal subgroups K with cyclic quotient in U and this has the form $K = C_p$ and yields $\mathrm{ind}_K = 1$. This case arises if $p \mid (q-1)$.

Groups without normal Sylow subgroup. Let G be such a group and let F be the Fitting subgroup of G. As G is solvable and non-nilpotent, it follows that $1 < F < G$. As G has no normal Sylow subgroup, we obtain that $pq \mid [G : F]$. Thus $|F| = p$ and $|G/F| = pq$ is the only option. Next, G/F acts on faithfully on F by conjugation, since F is the Fitting subgroup. Hence $pq \mid |Aut(F)| = (p-1)$ and this is a contradiction. Thus this case cannot arise. •

6 Groups of order p^3q

The groups of order p^3q have been determined by Western [24] and Laue [15]. Western's paper is essentially correct, but the final summary table of groups has a group missing in the case that $q \equiv 1 \bmod p$; the missing group appears in Western's analysis in Section 13. There are further minor issues in Section 32 of Western's paper. There are disagreements between our results and the results of Laue [15, p. 224-6] for the case $p = 2$ and the case $q = 3$. We have not tried to track the origin of these in Laue's work.

15 Theorem: *Let p and q be different primes.*

(a) *There are two special cases $\mathcal{N}(3 \cdot 2^3) = 15$ and $\mathcal{N}(7 \cdot 2^3) = 13$.*

(b) *If $q = 2$, then $\mathcal{N}(p^3q) = 15$ for all $p > 2$.*

(c) *If $p = 2$, then $\mathcal{N}(p^3q) = 12 + 2w_{q-1}(4) + w_{q-1}(8)$ for all $q \neq 3, 7$.*

(d) *If p and q are both odd, then*

$$\begin{aligned}\mathcal{N}(p^3q) &= 5 + (q^2 + 13q + 36)/6 \cdot w_{p-1}(q) \\ &\quad +(p+5) \cdot w_{q-1}(p) \\ &\quad +2/3 \cdot w_{q-1}(3)w_{p-1}(q) + w_{(p+1)(p^2+p+1)}(q)(1 - w_{p-1}(q)) \\ &\quad +w_{p+1}(q) + 2w_{q-1}(p^2) + w_{q-1}(p^3).\end{aligned}$$

Before we embark on the proof of Theorem 15, we note that the formula of Theorem 15 d) can be simplified by distinguishing the cases $q = 3$ and $q \neq 3$. For $q = 3$ and p odd it follows that $w_{(p+1)(p^2+p+1)}(q)(1 - w_{p-1}(q)) = w_{p+1}(3)$. Thus Theorem 15 d) for $q = 3$ and p odd reads $\mathcal{N}(3p^3) = 5 + 14w_{p-1}(3) + 2w_{p+1}(3)$. If $q > 3$ and p is odd, then $w_{(p+1)(p^2+p+1)}(q)(1 - w_{p-1}(q)) = w_{(p+1)(p^2+p+1)}(q)$ holds. Again, this can be used to simplify the formula of Theorem 15 d).

Proof: The proof follows the same strategy as the proof of Theorem 14. Burnside's theorem asserts that every group of order p^3q is solvable. It is easy to see that there are five nilpotent groups of order p^3q: the groups $G \times C_q$ with $|G| = p^3$. It remains to consider the non-nilpotent groups of the desired order.

Non-nilpotent groups with normal p-Sylow subgroup. These groups have the form $N \rtimes U$ with $|N| = p^3$ and $U = C_q$. Using Theorem 13, they correspond to the conjugacy classes of subgroups of order q in $\mathrm{Aut}(N)$. There are five isomorphism types of groups N of order p^3. For $p = 2$, the groups $\mathrm{Aut}(Q_8)$ and $\mathrm{Aut}(C_2^3) = GL(3, 2)$ have subgroups of order coprime to 2; that is, $\mathrm{Aut}(Q_8)$ has one conjugacy class of subgroups of order 3 and $GL(3, 2)$ has one conjugacy class of subgroups of order 3 and 7. This leads to the special cases in a) and shows that in all other cases on q this type of group does not exist for $p = 2$. It remains to consider the case $p > 2$.

- $N \cong C_{p^3}$: Then $\mathrm{Aut}(N)$ is cyclic of order $p^2(p - 1)$. Thus $\mathrm{Aut}(N)$ has at most one subgroup of order q and this exists if and only if $q \mid (p - 1)$. This adds

$$c_1 := w_{p-1}(q).$$

- $N \cong C_{p^2} \times C_p$: In this case $\mathrm{Aut}(N)$ is solvable and has a normal Sylow p-subgroup with p-complement of the form C_{p-1}^2. Thus $\mathrm{Aut}(N)$ contains a subgroup of order q if and only if $q \mid (p-1)$. In this case there are $q+1$ such subgroups in C_q^2 and these translate to conjugacy classes of such subgroups in $\mathrm{Aut}(N)$. This adds

$$c_2 := (q+1)w_{p-1}(q).$$

- N is extraspecial of exponent p: Then $Aut(N) \to Aut(N/\phi(N)) \cong GL(2,p)$ is surjective. Thus the conjugacy classes of subgroups of order q in $\mathrm{Aut}(N)$ correspond to the conjugacy classes of subgroups of order q in $GL(2,p)$. These are counted in Theorem 6 (a). Hence this adds

$$\begin{aligned} c_3 &:= \frac{q+3}{2} w_{p-1}(q) + w_{p+1}(q) \text{ if } q > 2, \\ c_3 &:= 2 \text{ if } q = 2. \end{aligned}$$

- N is extraspecial of exponent p^2: Then $\mathrm{Aut}(N)$ is solvable and has a normal Sylow p-subgroup with p-complement of the form C_{p-1}. Thus there is at most one subgroup of order q in $\mathrm{Aut}(N)$ and this exists if and only if $q \mid (p-1)$. This adds

$$c_4 := w_{p-1}(q).$$

- $N \cong C_p^3$: Then $\mathrm{Aut}(N) \cong \mathrm{GL}(3,p)$. The conjugacy classes of subgroups of order q in $\mathrm{GL}(3,p)$ are counted in Theorem 8. Hence this adds

$$\begin{aligned} c_5 &:= \frac{1}{6}(q^2 + 4q + 9 + 4w_{q-1}(3))w_{p-1}(q) + w_{(p+1)(p^2+p+1)}(q)(1 - w_{p-1}(q)) \text{ if } q > 2, \\ c_5 &:= 3 \text{ if } q = 2. \end{aligned}$$

Non-nilpotent groups with normal q-Sylow subgroup. These groups have the form $N \rtimes U$ with $N = C_q$ and $|U| = p^3$. Using Theorem 12, we have to determine the $\mathrm{Aut}(U)$-orbits of proper normal subgroups K in U with U/K cyclic of order dividing $q-1$ and then for each such K determine ind_K.

- $U \cong C_{p^3}$: In this case there are the options $K \in \{1, C_p, C_{p^2}\}$ and all of these have $\mathrm{ind}_K = 1$. Hence this adds

$$c_6 := w_{q-1}(p) + w_{q-1}(p^2) + w_{q-1}(p^3).$$

- $U \cong C_{p^2} \times C_p$: In this case there are two normal subgroups K with $U/K \cong C_p$ and there is one normal subgroup K with $U/K \cong C_{p^2}$ up to $\mathrm{Aut}(U)$-orbits. All of these have $\mathrm{ind}_K = 1$. Thus this adds

$$c_7 := 2w_{q-1}(p) + w_{q-1}(p^2).$$

- U extraspecial of exponent p (or Q_8): In this case there is one $\mathrm{Aut}(U)$-orbit of normal subgroups K with $U/K \cong C_p$ and this has $\mathrm{ind}_K = 1$. Hence this adds

$$c_8 := w_{q-1}(p).$$

- U extraspecial of exponent p^2 (or D_8): Then there are two $\mathrm{Aut}(U)$-orbits of normal subgroups K with $U/K \cong C_p$; one has $\mathrm{ind}_K = 1$ and the other has $\mathrm{ind}_K = p-1$. Thus this case adds

$$c_9 := pw_{q-1}(p).$$

- $U \cong C_p^3$: In this case there is one Aut(U)-orbit of normal subgroups K with $U/K \cong C_p$ and this has $\text{ind}_k = 1$. Thus this adds

$$c_{10} := w_{q-1}(p).$$

Groups without normal Sylow subgroup. Let G be such a group and let F be the Fitting-subgroup of G. Since G is solvable, it follows that F is not trivial. Further, $pq \mid [G : F]$ by construction. Thus $|F| = p$ or $|F| = p^2$ are the only options. Recall that G/F acts faithfully on $F/\phi(F)$. If F is cyclic of order p or p^2, then a group of order pq cannot act faithfully on $F/\phi(F)$. Hence $F \cong C_p^2$ is the only remaining possibility. In this case G/F has order pq and embeds into $Aut(F) \cong GL(2, p)$; thus $q \mid (p + 1)(p - 1)$. As F is the Fitting subgroup of G of order p^2, it follows that G/F cannot have a normal subgroup of order p. Hence $G/F \cong C_q \rtimes C_p$ and $p \mid (q - 1)$. This is only possible for $p = 2$ and $q = 3$ and thus is covered by the special case for groups of order $3 \cdot 2^3$.

Except for the special cases, it now remains to sum up the values $5 + \sum_{i=6}^{10} c_i$ for $p = 2$ and $5 + \sum_{i=1}^{10} c_i$ for $p > 2$. This yields the formulas exhibited in the theorem. •

7 Groups of order p^2q^2

The groups of order p^2q^2 have been determined by Lin [18], Le Vavasseur [17] and Laue [15]. Lin's work is essentially correct (it only has minor mistakes) and it agrees with the work by Laue [15, p. 214-43] and our results. Lin seems unaware of the work by Le Vavasseur [17]. We have not compared our results with those of Le Vavasseur.

16 Theorem: *Let p and q be different primes with $p < q$.*
(a) *There is one special case $\mathcal{N}(2^2 \cdot 3^2) = 14$.*
(b) *If $p = 2$, then $\mathcal{N}(p^2q^2) = 12 + 4w_{q-1}(4)$.*
(c) *If $p > 2$, then*

$$\mathcal{N}(p^2q^2) = 4 + (p^2 + p + 4)/2 \cdot w_{q-1}(p^2) + (p + 6)w_{q-1}(p) + 2w_{q+1}(p) + w_{q+1}(p^2).$$

Proof: First, all groups of this order are solvable by Burnside's theorem and it is obvious that there are 4 isomorphism types of nilpotent groups of this order. Next, we consider Sylow's theorems. Let m_q denote the number of Sylow q-subgroups in a group of order p^2q^2 and recall that $p < q$. Then $m_q \equiv 1 \bmod q$ and $m_q \mid p^2$. Thus $m_q \in \{1, p, p^2\}$. If $m_q = p$, then $q \mid (p-1)$ and this is impossible. If $m_q = p^2$, then $q \mid p^2 - 1 = (p-1)(p+1)$. Thus either $q \mid p - 1$ or $q \mid p + 1$. Again, this is impossible unless $p = 2$ and $q = 3$. Thus if $(p, q) \neq (2, 3)$, then $m_q = 1$ and G has a normal Sylow q-subgroup.

Non-nilpotent groups with normal Sylow q-subgroup. These groups have the form $N \rtimes U$ with $|N| = q^2$ and $|U| = p^2$. We consider the arising cases.

- $N \cong C_{q^2}$ and $U \cong C_{p^2}$: Then $\text{Aut}(N) \cong C_{q(q-1)}$ and thus there are at most one subgroup of order p or p^2 in Aut(N). We use Theorem 13 to determine that this case adds

$$c_1 := w_{q-1}(p^2) + w_{q-1}(p).$$

- $N \cong C_{q^2}$ and $U \cong C_p^2$: This case is similar to the first case and adds

$$c_2 := w_{q-1}(p).$$

- $N \cong C_q^2$: Let $\varphi : U \to \mathrm{Aut}(N) = GL(2,q)$ and denote $K = ker(\varphi)$. If $|K| = p$, then $U/K \cong C_p$. In both cases on the isomorphims type of U there is one $\mathrm{Aut}(U)$-orbit of normal subgroups of order p and this satisfies $\overline{A}_K = \mathrm{Aut}(U/K)$. Hence in this case it remains to count the number c_3 of conjugacy classes of subgroups of order p in $\mathrm{GL}(2,q)$, see Theorem 6 (a). Thus this adds

$$\begin{aligned} c_3 &= 2 \text{ if } p = 2, \\ c_3 &= \frac{p+3}{2} w_{q-1}(p) + w_{q+1}(p) \text{ if } p > 2. \end{aligned}$$

If $|K| = 1$, then $U/K \cong U$. Clearly, there is one $\mathrm{Aut}(U)$-orbit of such normal subgroups and it satisfies $\overline{A}_K = \mathrm{Aut}(U/K)$. Hence in this case it remains to count the number c_4 of conjugacy classes of subgroups of order p^2 in $GL(2,q)$, see Theorem 6 (b). Thus this adds

$$\begin{aligned} c_4 &= 2 + 3w_{q-1}(4) \text{ if } p = 2, \\ c_4 &= w_{q-1}(p) + (\frac{p(p+1)}{2} + 1) w_{q-1}(p^2) + w_{q+1}(p^2) \text{ if } p > 2. \end{aligned}$$

The number of groups of order p^2q^2 can now be read off as $4 + c_1 + c_2 + 2c_3 + c_4$ and this yields the above formulae. •

8 Groups of order p^2qr

The groups of order p^2qr have been considered by Glenn [11] and Laue [15]. Glenn's work has several problems. There are groups missing from the summary tables, there are duplications and some of the invariants are not correct; this affects in particular the summary Table 3. Laue [15, p. 244-62] does not agree with Glenn. We have not compared our results with those of Laue.

17 Theorem: *Let p, q and r be different primes with $q < r$.*

(a) *There is one special case $\mathcal{N}(2^2 \cdot 3 \cdot 5) = 13$.*

(b) *If $q = 2$, then*

$$\mathcal{N}(p^2qr) = 10 + (2r+7)w_{p-1}(r) + 3w_{p+1}(r) + 6w_{r-1}(p) + 2w_{r-1}(p^2).$$

(c) *If $q > 2$, then $\mathcal{N}(p^2qr)$ is equal to*

$$\begin{aligned} &2 + (p^2 - p)w_{q-1}(p^2)w_{r-1}(p^2) \\ &+(p-1)(w_{q-1}(p^2)w_{r-1}(p) + w_{r-1}(p^2)w_{q-1}(p) + 2w_{r-1}(p)w_{q-1}(p)) \\ &+(q-1)(q+4)/2 \cdot w_{p-1}(q)w_{r-1}(q) \\ &+(q-1)/2 \cdot (w_{p+1}(q)w_{r-1}(q) + w_{p-1}(q) + w_{p-1}(qr) + 2w_{r-1}(pq)w_{p-1}(q)) \end{aligned}$$

$+(qr+1)/2 \cdot w_{p-1}(qr)$
$+(r+5)/2 \cdot w_{p-1}(r)(1+w_{p-1}(q))$
$+w_{p^2-1}(qr) + 2w_{r-1}(pq) + w_{r-1}(p)w_{p-1}(q) + w_{r-1}(p^2q)$
$+w_{r-1}(p)w_{q-1}(p) + 2w_{q-1}(p) + 3w_{p-1}(q) + 2w_{r-1}(p)$
$+2w_{r-1}(q) + w_{r-1}(p^2) + w_{q-1}(p^2) + w_{p+1}(r) + w_{p+1}(q)$.

Proof: The exists one non-solvable group of order p^2qr and this is the group A_5 of order 60. Further, there are two nilpotent groups of order p^2qr. In the following we consider the solvable, non-nilpotent groups G of order p^2qr. Let F be the Fitting subgroup of G. Then F and G/F are both non-trivial and G/F acts faithfully on $\overline{F} := F/\phi(F)$ so that no non-trivial normal subgroup of G/F stabilizes a series through $\overline{F}$. This yields the following cases.

- Case $|F| = p$: In this case $\overline{F} = F$ and $Aut(\overline{F}) = C_{p-1}$. Hence G/F is abelian and has a normal subgroup isomorphic to C_p. This is a non-trivial normal subgroup which stabilizes a series through $\overline{F}$. As this cannot occur, this case adds 0.
- Case $|F| = q$: In this case $\overline{F} = F$ and $Aut(\overline{F}) = C_{q-1}$ and $|G/F| = p^2r$. Thus $p^2r \mid (q-1)$ and this is impossible, since $r > q$. Thus this case adds 0.
- Case $|F| = r$: In this case $\overline{F} = F$ and $Aut(\overline{F}) = C_{r-1}$ and $|G/F| = p^2q$. Hence $G = C_r \rtimes C_{p^2q}$. Since $Aut(C_r)$ is cyclic, it has at most one subgroup of order p^2q and this exists if and only if $p^2q \mid (r-1)$. By Theorem 11, this case adds

$$c_1 := w_{r-1}(p^2q).$$

- Case $|F| = p^2$ and $F \cong C_{p^2}$. Then $G = C_{p^2} \rtimes C_{qr}$ and C_{qr} acts faithfully on C_{p^2}. Note that $Aut(C_{p^2})$ is cyclic of order $p(p-1)$. Thus this case arises if and only if $qr \mid (p-1)$ and in this case there is a unique subgroup of $Aut(F)$ of order qr. Again by Theorem 11, this case adds

$$c_2 := w_{p-1}(qr).$$

- Case $|F| = p^2$ and $F \cong C_p^2$. Then $G = C_p^2 \rtimes H$ with H of order qr and H embeds into $Aut(C_p^2) = GL(2,p)$. By Theorem 11, the number of groups G corresponds to the number of conjugacy classes of subgroups of order qr in $\mathrm{GL}(2,p)$. By Theorem 6 (c), this adds

$$\begin{aligned} c_3 &:= \frac{3r+7}{2} w_{p-1}(r) + 2w_{p+1}(r) \text{ if } q = 2, \\ c_3 &:= \frac{qr+q+r+5}{2} w_{p-1}(qr) + w_{p^2-1}(qr)(1-w_{p-1}(qr)) \text{ if } q > 2. \end{aligned}$$

- Case $|F| = pq$: In this case $\phi(F) = 1$ and $Aut(F) = C_{p-1} \times C_{q-1}$. Thus G/F is abelian and hence $G/F \cong C_p \times C_r$. Note that $r > q$ and thus G/F acts on F in such a form that C_p acts as subgroup of C_{q-1} and C_r acts as subgroup of C_{p-1}. This implies that $r \mid (p-1)$ and $p \mid (q-1)$. This is a contradiction to $r > q$ and hence this case adds 0.
- Case $|F| = pr$: In this case $\phi(F) = 1$ and $Aut(F) = C_{p-1} \times C_{r-1}$. Thus G/F is abelian and hence $G/F \cong C_p \times C_q$. It follows that $p \mid (r-1)$ and $q \mid (p-1)(r-1)$. There are two cases to distinguish. First, suppose that the Sylow q-subgroup of G/F acts non-trivially on the Sylow p-subgroup of F. Then $q \mid (p-1)$ and G splits over

F by [8, Th. 14]. Thus the group G has the form $F \rtimes_\varphi G/F$ for a monomomorphism $\varphi : G/F \to \mathrm{Aut}(F)$. As in Theorem 11, the number of such groups G is given by the number of subgroups of order pq in $\mathrm{Aut}(F)$. As the image of the Sylow p-subgroup of G/F under φ is uniquely determined, it remains to evaluate the number of subgroups of order q in $\mathrm{Aut}(F)$ that act non-trivally on the Sylow p-subgroup of F. This number is $1 + (q-1)w_{r-1}(q)$. As second case suppose that C_q acts trivially on C_p. Then $q \mid (r-1)$ and the action of G/F on F is uniquely determined. It remains to determine the number of extensions of G/F by F. By [8, Th. 14] there exist two isomorphism types of extensions in this case. In summary, this case adds

$$c_4 := w_{r-1}(p)(w_{p-1}(q)(1 + (q-1)w_{r-1}(q)) + 2w_{r-1}(q)).$$

- Case $|F| = qr$: In this case G has the form $G \cong N \rtimes_\varphi U$ with $N \cong C_q \times C_r$ and $|U| = p^2$ and $\varphi : U \to \mathrm{Aut}(N)$ a monomorphism. By Theorem 11, we have to count the number of subgroups of order p^2 in $\mathrm{Aut}(N)$. Note that $\mathrm{Aut}(N) \cong C_{q-1} \times C_{r-1}$. Thus the number of subgroups of the form C_p^2 in $\mathrm{Aut}(N)$ is

$$c_{5a} := w_{q-1}(p)w_{r-1}(p).$$

It remains to consider the number of cyclic subgroups of order p^2 in $\mathrm{Aut}(N)$. This number depends on $\gcd(q-1, p^2) = p^a$ and $\gcd(r-1, p^2) = p^b$. If $a, b \leq 1$, then this case does not arise. Thus suppose that $a = 2$. If $b = 0$, then this yields 1 group. If $b = 1$, then this yields p groups. If $b = 2$, then this yields $p(p+1)$ groups. A similar results holds for the dual case $b = 2$. We obtain that this adds

$$\begin{aligned} c_{5b} \;:=\; & w_{q-1}(p^2)(1 - w_{r-1}(p)) \\ + \; & p w_{q-1}(p^2) w_{r-1}(p)(1 - w_{r-1}(p^2)) \\ + \; & p(p+1) w_{q-1}(p^2) w_{r-1}(p^2) \\ + \; & w_{r-1}(p^2)(1 - w_{q-1}(p)) \\ + \; & p w_{r-1}(p^2) w_{q-1}(p)(1 - w_{q-1}(p^2)). \end{aligned}$$

In summary, this case adds

$$\begin{aligned} c_5 &= c_{5a} + c_{5b} \\ &= (p^2 - p) w_{q-1}(p^2) w_{r-1}(p^2) \\ &\quad + (p-1)(w_{q-1}(p^2) w_{r-1}(p) + w_{r-1}(p^2) w_{q-1}(p)) \\ &\quad + w_{q-1}(p^2) + w_{r-1}(p^2) + w_{q-1}(p) w_{r-1}(p). \end{aligned}$$

- Case $|F| = pqr$: In this case G has the form $G \cong N \rtimes_\varphi U$ with $N \cong C_q \times C_r$ and $|U| = p^2$ and φ has a kernel K of order p. Again, we use Theorem 11 to count the number of arising cases. The group U is either cyclic or $U \cong C_p^2$. In both cases, there is one $\mathrm{Aut}(U)$-class of normal subgroups K of order p in U and this satsifies that $\mathrm{Aut}_K(U)$ maps surjectively on $\mathrm{Aut}(U/K)$. Thus it remains to count the number of subgroups of order p in $\mathrm{Aut}(N)$. As $\mathrm{Aut}(N) = C_{q-1} \times C_{r-1}$, this case adds

$$c_6 := 2(w_{q-1}(p) + w_{r-1}(p) + (p-1) w_{q-1}(p) w_{r-1}(p)).$$

- Case $|F| = p^2q$: Then $G \cong F \rtimes_\varphi U$ with $F = N \times C_q$ and $|N| = p^2$ and $U \cong C_r$. By Theorem 13 we have to count the number of conjugacy class representatives of subgroups of order r in $\mathrm{Aut}(F)$. Recall that $r > q$ and $\mathrm{Aut}(F) = \mathrm{Aut}(N) \times C_{q-1}$. Thus it remains to count the number of conjugacy classes of subgroups of order r in $\mathrm{Aut}(N)$. If N is cyclic, then $\mathrm{Aut}(N) \cong C_{p(p-1)}$ and this number is $w_{p-1}(r)$. If $N \cong C_p^2$, then $\mathrm{Aut}(N) \cong \mathrm{GL}(2,p)$ and this number is determined in Theorem 6 (a). In summary, this case adds

$$c_7 := \frac{r+5}{2} w_{p-1}(r) + w_{p+1}(r).$$

- Case $|F| = p^2r$: This case is dual to the previous case with the exception that now the bigger prime r is contained in $|F|$. A group of this type has the form $N \rtimes_\varphi U$ with N nilpotent of order p^2r and $U \cong C_q$. We consider the two cases on N.
If N is cyclic, then $N = C_{p^2} \times C_r$ and $\mathrm{Aut}(N) = C_{p(p-1)} \times C_{r-1}$. In this case we can use Theorem 13 to count the desired subgroups as

$$c_{8a} := w_{p-1}(q) + w_{r-1}(q) + (q-1)w_{p-1}(q)w_{r-1}(q).$$

Now we consider the case that $N = C_p^2 \times C_r$. In this case we use a slightly different approach and note that all groups in this case have the form $M \rtimes_\varphi V$ with $M = C_p^2$ and $|V| = qr$. If V is cyclic, then $\varphi : V \to \mathrm{Aut}(M)$ has a kernel K of order r. The other option is that V is non-abelian and thus of the form $V = C_r \rtimes C_q$. In this case the kernel K of φ has order r or qr. We use Theorem 11 to count the number of arising cases. If V is cyclic, then it is sufficient to count the conjugacy classes of subgroups of order q in $\mathrm{Aut}(M)$. By Theorem 6 this adds

$$c_{8b} := s_q(\mathrm{GL}(2,p)).$$

If V is non-abelian, then $q \mid (r-1)$. We consider Theorem 11 in more detail. First, suppose that $|K| = qr$. Then φ is trivial and uniquely defined. Thus it remains to consider the case that $|K| = r$. Then the subgroup $A_K \leq \mathrm{Aut}(V/K)$ induced by $Stab_{\mathrm{Aut}(V)}(K)$ is the trivial group. Let $\mathcal{S}$ be a set of conjugacy class representatives of subgroups of order q in $\mathrm{GL}(2,p)$ and for $S \in \mathcal{S}$ let A_S denote the subgroup of $\mathrm{Aut}(V/K)$ induced by the normalizer of S in $\mathrm{GL}(2,p)$. Then Theorem 11 yields that the number of groups arising in this case is

$$c_{8c} := w_{r-1}(q)(1 + \sum_{S \in \mathcal{S}} \mathrm{Aut}(V/K)/A_S.$$

Next, note that A_S can be determined via Remark 7. As $\mathrm{Aut}(V/K) = C_{q-1}$, it follows that

$$\begin{aligned} c_{8c} &= 3 \text{ if } q = 2, \\ c_{8c} &= w_{r-1}(q)(1 + \frac{(q-1)(q+2)}{2} w_{p-1}(q) + \frac{q-1}{2} w_{p+1}(q)) \text{ otherwise.} \end{aligned}$$

In summary, this case adds $c_8 = c_{8a} + c_{8b} + c_{8c}$ and this can be evaluated to $c_8 = 8$ if $q = 2$ and if $q > 2$ then

$$c_8 = \frac{(q-1)(q+4)}{2} w_{p-1}(q)w_{r-1}(q)$$

$$+\frac{q-1}{2} w_{p+1}(q) w_{r-1}(q)$$
$$+\frac{q+5}{2} w_{p-1}(q)$$
$$+2w_{r-1}(q) + w_{p+1}(q).$$

It now remains to sum up the values for the different cases to determine the final result. •

9 Final comments

The enumerations of this paper all translate to group constructions. It would be interesting to make this more explicit and thus to obtain a complete and irredundant list of isomorphism types of groups of all orders considered here.

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