Context-free Groups and Their Structure Trees

Armin Weiß[1]

Universität Stuttgart

May 22, 2013

[1] Joint work with Volker Diekert

Main Theorem

Theorem

Let Γ be a connected, locally finite graph of finite tree-width and let a group G act on Γ with finitely many orbits and finite vertex stabilizers.
Then there is a tree T such that G acts on T with finitely many orbits and finite vertex stabilizers.

Muller-Schupp-Theorem

Corollary (Muller, Schupp, 1983)

A group is context-free if and only if it is finitely generated virtually free.

Virtually free $\Rightarrow$ context-free: construct a pushdown automaton.
Context-free $\Rightarrow$ virtually free:

1. G context-free $\implies$ Cayley graph finite tree-width.
2. Theorem above implies that G acts on T with finitely many orbits and finite vertex stabilizers.
3. Bass-Serre implies that G is a fundamental group of a finite graph of finite groups.
4. Karrass, Pietrowski, and Solitar (1973) implies that G is virtually free.

Muller-Schupp's proof 1983 / 1985

context-free $\Rightarrow$ virtually free (proof by Muller and Schupp):

1. G context-free $\iff$ Cayley graph is quasi-isometric to a tree.
2. Cayley graphs which are quasi-isometric to a tree have more than one end.
3. Apply Stallings' Structure Theorem (1971).
4. Use the result by Dunwoody (1985) that finitely presented groups are accessible. (This piece was still missing 1983.)
5. Apply the theorem by Karrass, Pietrowski, and Solitar (1973) to see that the group is virtually free.

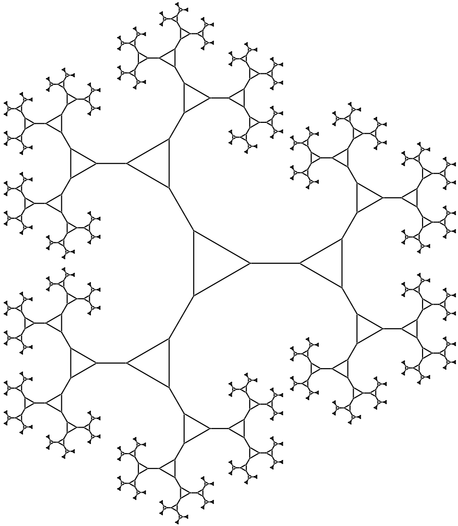
Finite tree-width

Definition

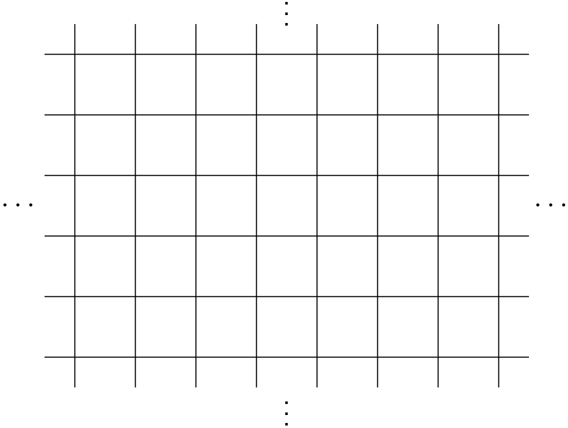
A graph Γ has finite tree-width if there is a tree $T = (V(T), E(T))$ and for every vertex $t \in V(T)$ a *bag* $X_t \subseteq V(\Gamma)$ such that

- Every node $v \in V(\Gamma)$ and every edge $uv \in E(\Gamma)$ is contained in some bag.
- If $v \in X_s \cap X_t$ for two nodes s, t of the tree, then v is contained in every bag of the unique geodesic in the tree from s to t.
- The size of the bags is bounded by some constant.

Modular group



The Cayley graph of $\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ has finite tree-width.

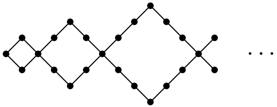


The Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ does not have finite tree-width.

Finite tree-width vs. quasi-isometric to a tree

In general, both classes are incomparable:

- Infinite clique does not have finite tree-width, but is is quasi-isometric to a point.
- The following graph has finite tree-width, but is not quasi-isometric to a tree.



However, for Cayley graphs are equivalent:

- quasi-isometric to a tree,
- finite tree-width.

Cuts

Starting point:
Γ = connected, locally finite graph of finite tree-width
G = group acting on Γ with finitely many orbits and finite vertex stabilizers

For $C \subseteq V(\Gamma)$ let $\delta C = \{ uv \in E(\Gamma) \mid u \in C, v \in \overline{C} \}$ be the *boundary*.

Definition

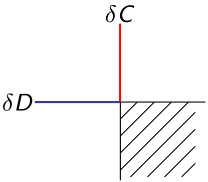
A *cut* is a subset $C \subseteq V(\Gamma)$ such that δC is finite.

Tree sets

Definition

A *tree set* is a set of cuts $\mathcal{C}$ such that

- $C \in \mathcal{C} \Rightarrow \overline{C} \in \mathcal{C}$,
- cuts in $\mathcal{C}$ are pairwise nested, i.e., for $C, D \in \mathcal{C}$ either $C \subseteq D$ or $C \subseteq \overline{D}$ or $\overline{C} \subseteq D$ or $\overline{C} \subseteq \overline{D}$,
- the partial order $(\mathcal{C}, \subseteq)$ is discrete.



The aim is to construct a tree set $\mathcal{C}$.

Why tree sets?

A cut C of tree set defines an undirected edge $\{[C],[\overline{C}]\}$ in a tree for the following equivalence relation.

Definition

For C, $D \in \mathcal{C}$ the relation $C \sim D$ is defined as follows:

Either $C = D$,

or $\overline{C} \subsetneqq D$ and there is no $E \in \mathcal{C}$ with $\overline{C} \subsetneqq E \subsetneqq D$.

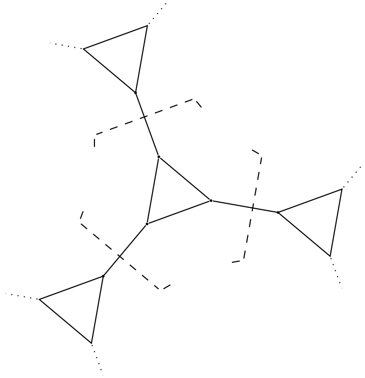
Proposition (Dunwoody, 1979)

The graph $T(\mathcal{C})$ is a tree, where

$$\textit{Vertices: } V(T(\mathcal{C})) = \{\, [C] \mid C \in \mathcal{C} \,\},$$

$$\textit{Edges: } E(T(\mathcal{C})) = \{\, \{[C],[\overline{C}]\} \mid C \in \mathcal{C} \,\}.$$

Vertices in the structure tree



Three cuts in one equivalence class = one vertex in $T(\mathcal{C})$.

Facts about Γ

- There exists some k such that every bi-infinite geodesic can be split into two infinite pieces by some k-cut., i.e., $|\delta(C)| \leq k$.
  - Every bi-infinite geodesic defines two different ends.
  - Every pair of ends can be separated by a k-cut.
- If Γ is infinite, then there exists some bi-infinite geodesic.

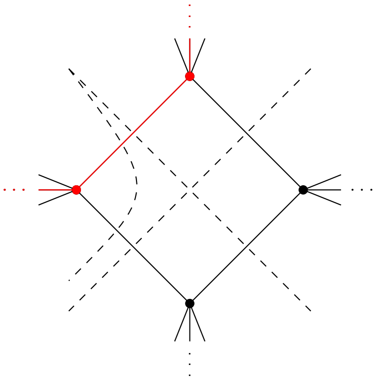
We need $|\mathrm{Aut}(\Gamma)\backslash\Gamma| < \infty$: There are graphs with arbitrarily long geodesics, bi-infinite simple paths, but without any bi-infinite geodesic:

Here: $\mathrm{Aut}(\Gamma) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Aut}(\Gamma)\backslash\Gamma = \mathbb{N}$.

Minimal cuts

Minimal cuts = cuts which are minimal splitting an infinite geodesic.

Minimal cuts still might not be nested:



Optimal cuts

A cut C is optimal, if it cuts a bi-infinite geodesic α with $|\delta C|$ minimal and with a minimal number of not nested cuts.

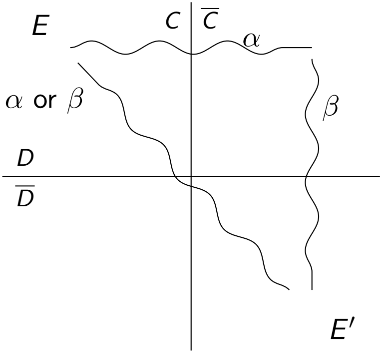
Theorem

- *Every bi-infinite geodesic is split by an optimal cut.*
- *Optimal cuts are pairwise nested.*

Corollary

The optimal cuts form a tree set and the action of G on Γ induces an action of G on $\mathcal{C}_{\text{opt}}$.

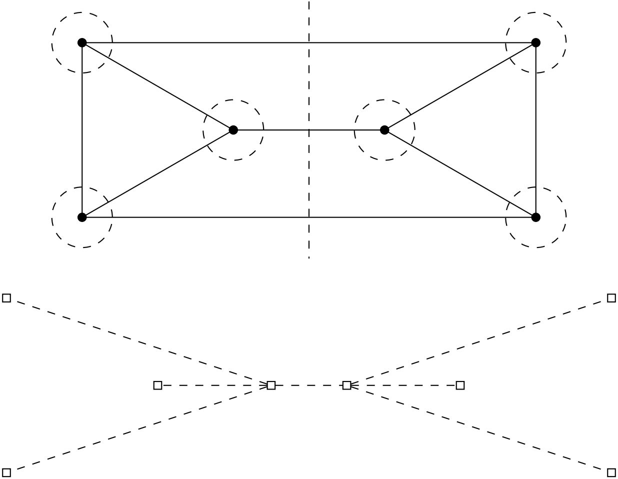
Optimal Cuts



$$\delta E \cup \delta E' \subseteq \delta C \cup \delta D \qquad \delta E \cap \delta E' \subseteq \delta C \cap \delta D$$

$$|\delta E| + \left|\delta E'\right| \leq |\delta C| + |\delta D|$$

Example



Vertex stabilizers

Theorem

The group G acts on the **tree** $T(\mathcal{C}_{\mathrm{opt}})$ *with finitely many orbits and finite vertex stabilizers.*

Proof.

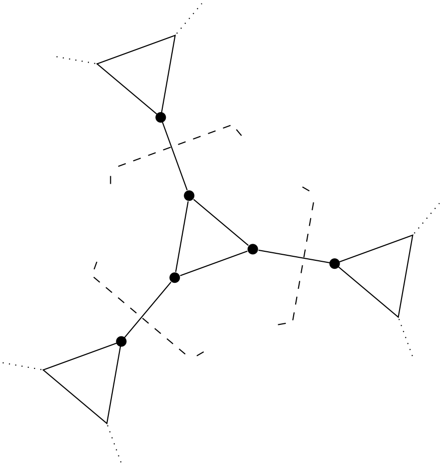
Construct a tree decomposition of Γ assigning to each $[C] \in V(T(\mathcal{C}_{\mathrm{opt}}))$ a block $B[C]$ with

$$B[C] = \bigcap_{D \sim C} N^{\lambda}(D).$$

1. Blocks are connected.
2. The stabilizer $G_{[C]}$ acts with finitely many orbits on $B[C]$.
3. There is no cut in $B[C]$ which splits some bi-infinite geodesic.

□

Vertices in the structure tree and blocks



A block assigned to an equivalence class consisting of three cuts.

Concluding remarks

- Proof based on “Cutting up graphs revisited – a short proof of Stallings’ structure theorem” by Krön (2010).
- Direct, one-step construction of the structure tree.
- Muller-Schupp-Theorem as corollary.
- Solution of the isomorphism problem for context-free groups in elementary time if the minimal cuts can be computed in elementary time. (Known: primitive recursive (Sénizergues, 1993))

Thank you!