The Submonoid Membership Problem for Groups

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[1]Encompasses joint work with Mark Kambites, Markus Lohrey, Pedro Silva and Georg Zetzsche

Integer programming

- INTEGER PROGRAMMING:
  - Given $A \in M_{mn}(\mathbb{Z})$ and $\boldsymbol{b} \in \mathbb{Z}^m$, does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution $\boldsymbol{x} \in \mathbb{N}^n$?

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- The submonoid membership problem for arbitrary groups is a non-commutative analogue of integer programming.

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- Decidability of these problems is independent of Σ.

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- Compare: all finitely generated metabelian groups have decidable generalized word problem (Romanovskii).
- The Rips construction produces hyperbolic groups with undecidable generalized word problem.

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  - The language of geodesic words belonging to a quasiconvex subgroup of a hyperbolic group.

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  - double cosets of finitely generated subgroups.

Examples

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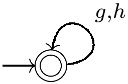

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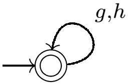

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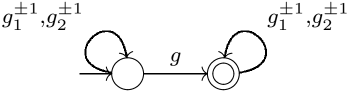

recognizes the double coset $\langle g_1, g_2 \rangle g \langle g_1, g_2 \rangle$.

The theorem of Anissimov and Seifert

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- If $\mathrm{Rat}(G)$ is closed under intersection, then G is a Howson group.

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- The order of g is finite if and only if $g^{-1} \in g^*$, so decidability of submonoid membership gives decidability of order.

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- It reduces to INTEGER PROGRAMMING.

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Theorem (Kambites, Silva, BS (2007))

Decidability of rational subset membership is preserved by free products with amalgamation and HNN-extensions with finite edge groups.

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Theorem (Lohrey, BS (2008))

Every group in the class $\mathscr{C}$ has decidable rational subset membership problem.

Right-angled Artin groups: the generalized word problem

- For Γ a graph, the associated right-angled Artin group is

$$\mathscr{G}(\Gamma) = \langle V(\Gamma) \mid [v,w] : (v,w) \in E(\Gamma) \rangle.$$

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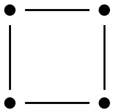
C4 = 

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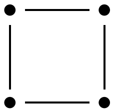
C4 = 

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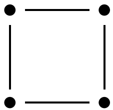

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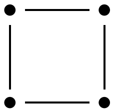

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Theorem (Kapovich, Myasnikov, Weidmann (2005))

The generalized word problem is decidable for chordal right-angled Artin groups.

Right-angled Artin groups: the rational subset problem

Let P4 = $\bullet \text{———} \bullet \text{———} \bullet \text{———} \bullet$ and C4 =

$$\begin{tikzcd} \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[d, dash] \\ \bullet \arrow[r, dash] & \bullet \end{tikzcd}$$

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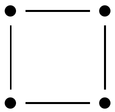
Let P4 =  and C4 = 

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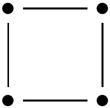
Let P4 = • —— • —— • —— • and C4 = 

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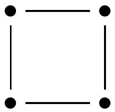
Let P4 = • —— • —— • —— • and C4 = 

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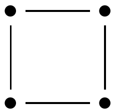
Let P4 =  and C4 = 

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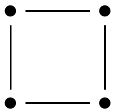
Let P4 =  and C4 = 

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P4 is chordal, yielding our first (but not last!) example of a group with decidable generalized word problem but undecidable submonoid membership problem.

The direct product of two free monoids

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- This is a simple encoding of the Post correspondence problem.

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- Recall: a group has 2 or more ends iff it splits over a finite subgroup.

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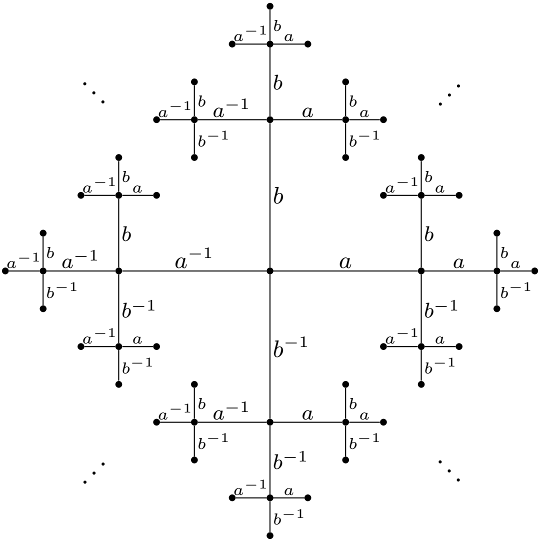
- I.e., $(hf)(h') = f(h^{-1}h')$.

Lamplighter on a tree

The element $cbcb^{-1}cabcb^{-1}ca$ in $\mathbb{Z}_2 \wr F_2$:



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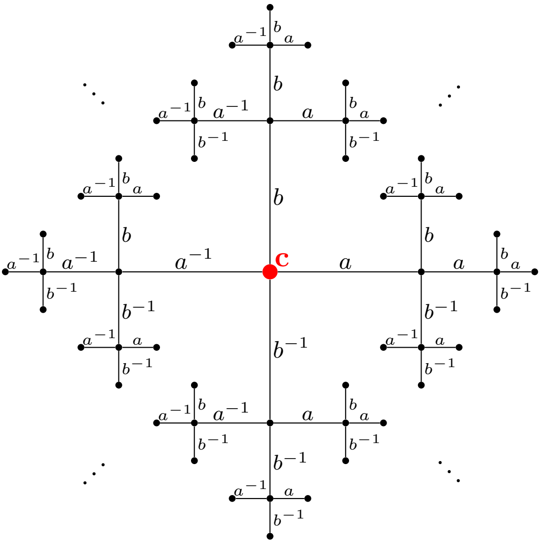

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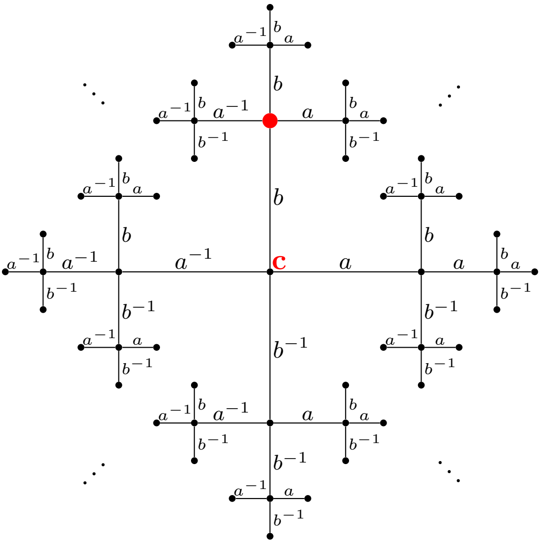

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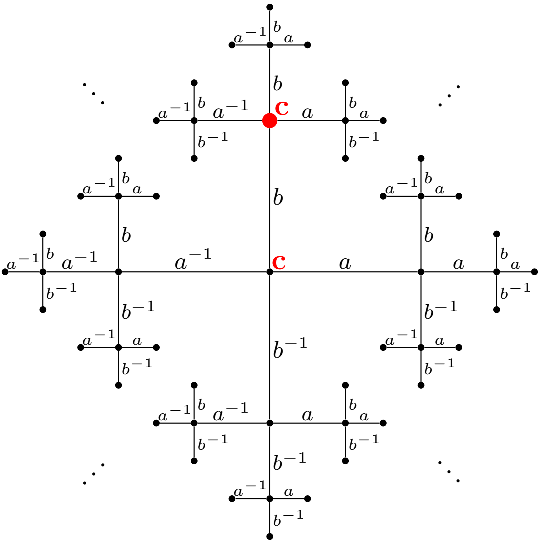

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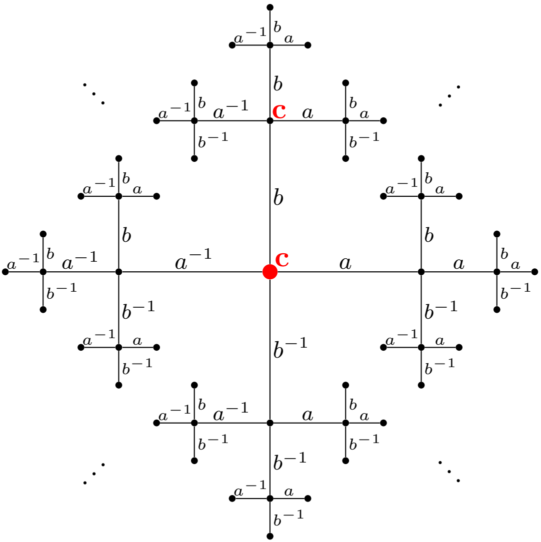

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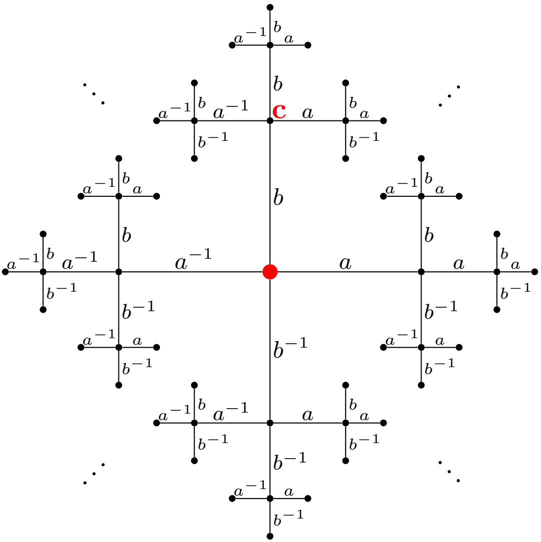

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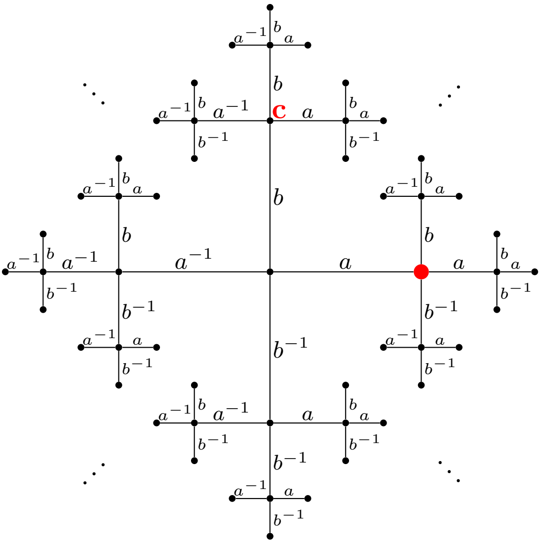

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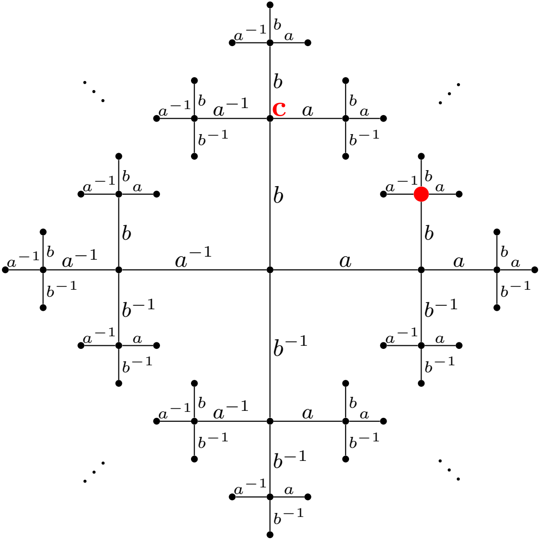

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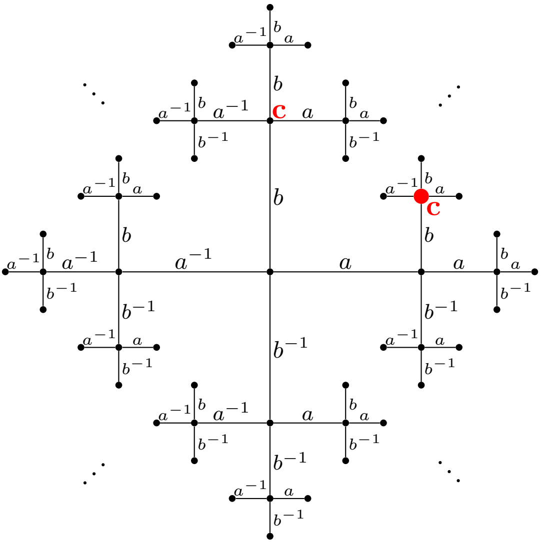

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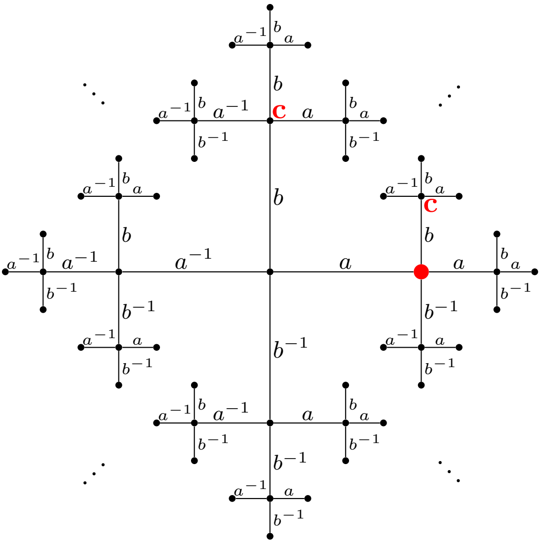

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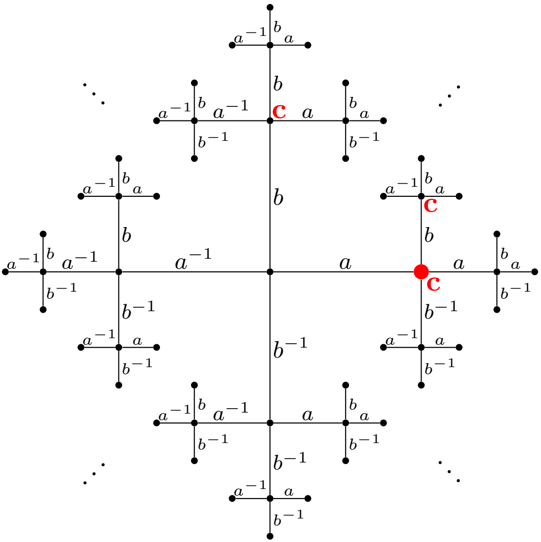

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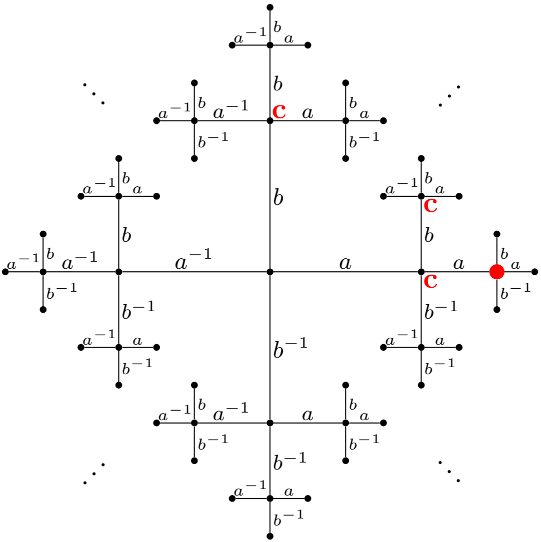

Rational subsets of wreath products with $\mathbb{Z} \times \mathbb{Z}$

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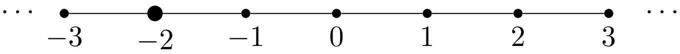

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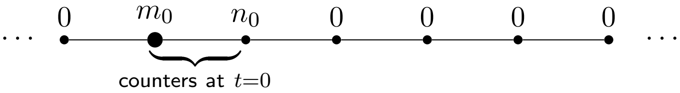

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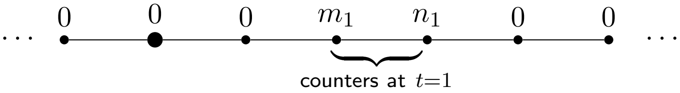

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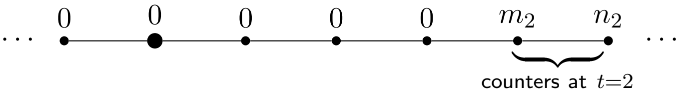

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Corollary

Submonoid membership is undecidable in Thompson's group F.

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- No complexity bounds are obtained.

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Question

Is it true that rational subset membership is undecidable for $G \wr H$ whenever G is non-trivial and H is not virtually free?

The end

THANK YOU FOR YOUR ATTENTION!