Computing zeta functions of groups and rings

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Some counting problems

- Given a finitely generated nilpotent group G, let

$$a_n(G) = \#\Big\{H \leqslant G : |G : H| = n\Big\}.$$

- Given a matrix algebra $\mathcal{A} \leqslant \mathrm{M}_d(\mathbf{Z})$, let

$$a_n(\mathcal{A}) = \#\Big\{\Lambda : \Lambda \text{ is a submodule of } \mathbf{Z}^d \ \&\ |\mathbf{Z}^d : \Lambda| = n\Big\}.$$

- Given an additively finitely generated ring L, let

$$a_n(L) = \#\Big\{\Lambda : \Lambda \text{ is a subring of } L \ \&\ |L : \Lambda| = n\Big\}.$$

- Many variations: normal subgroups of G, ideals of L, ...

Let Γ be one of the above and $a_n = a_n(\Gamma)$.

Goal: compute $(a_1, a_2, \dots)$

Zeta functions

$a_{nm} = a_n a_m$ for $\gcd(n, m) = 1 \rightsquigarrow$ Dirichlet generating functions

Definition

The (subgroup/submodule/subring/...) **zeta function** of Γ is

$$\zeta_\Gamma(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Fact

$\zeta_\Gamma(s)$ *converges for* $\mathrm{Re}(s) > \alpha \iff \sum_{n=1}^{m} a_n = \mathcal{O}(m^\alpha)$.

Example

$\zeta_{\mathbf{Z}}(s) = \sum_{n=1}^{\infty} n^{-s}$, the Riemann zeta function.

Local zeta functions

Definition

The **local zeta function** of Γ at the prime p is

$$\zeta_{\Gamma,p}(s) = \sum_{k=0}^{\infty} a_{p^k} p^{-ks}.$$

Theorem (Grunewald, Segal & Smith 1988)

1. $\zeta_{\Gamma}(s) = \prod_{p \text{ prime}} \zeta_{\Gamma,p}(s)$ (*"Euler product"*)
2. $\zeta_{\Gamma,p}(s) \in \mathbf{Q}(p^{-s})$

Local zeta functions

Theorem (Grunewald & du Sautoy 2000)

Given Γ, there are $\mathbf{Q}$-varieties $V_1, \ldots, V_m$ and $W_1, \ldots, W_m \in \mathbf{Q}(X, Y)$ s.t. for almost all primes p,

$$\zeta_{\Gamma,p}(s) = \sum_{i=1}^{m} \# \overline{V}_i(\mathbf{F}_p) \cdot W_i(p, p^{-s}).$$

Remark

Key steps:

1. Express $\zeta_{\Gamma,p}(s)$ as a p-adic integral.
2. Evaluate the integral using a resolution of singularities.
   Usually infeasible!

Goal

Develop practical methods for computing such V_i and W_i under non-degeneracy assumptions on Γ.

This project

Key ingredients

1. a new concept of non-degeneracy for a class of p-adic integrals
2. an effective method for evaluating non-degenerate integrals
3. a method that modifies the integrals in order to remove degeneracies (WIP)

Inspiration

1. Khovanskii et al. (1970s):
   explicit resolution of singularities under non-degeneracy assumptions w.r.t. certain Newton polyhedra
2. Denef et al. (1980–):
   Igusa's local zeta function enumerating solns of $f(\boldsymbol{x}) \equiv 0 \bmod p^n$
3. Gröbner bases machinery, toric geometry

Cone integrals

Theorem (Grunewald & du Sautoy 2000)

Let Γ have Hirsch length/dimension/additive rank d. Then there are polynomials $f_1 \ldots, f_r$ over $\mathbf{Q}$ s.t. for almost all primes p,

$$\zeta_{\Gamma,p}(s) = (1-p^{-1})^{-d} \int_{V_p} |x_{11}|_p^{s-1} \cdots |x_{dd}|_p^{s-d} \, \mathrm{d}\mu(\boldsymbol{x}),$$

where

$$V_p = \left\{ \boldsymbol{x} = \begin{bmatrix} x_{11} & \cdots & \cdots & x_{1d} \\ & x_{22} & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & x_{dd} \end{bmatrix} \in \mathrm{Tr}_d(\mathbf{Z}_p) \;\middle|\; x_{11} \cdots x_{dd} \mid f_1(\boldsymbol{x}), \ldots, f_r(\boldsymbol{x}) \right\}.$$

Non-degenerate cone integrals

Definition

The **Newton polytope** $\mathrm{New}(f)$ of $f = \sum a_e \boldsymbol{X}^e$: convex hull of $\{e : a_e \neq 0\}$.

Fact

Faces $\tau \subseteq \mathrm{New}(f_1 \cdots f_r)$ define canonical sub-polynomials $f_{i,\tau}$ of the f_i.

Write $\boldsymbol{f} = (f_1, \ldots, f_r)$. For $J \subseteq \{1, \ldots, r\}$, write $\boldsymbol{f}_{J,\tau} = \big(f_{j,\tau}\big)_{j \in J}$.

Definition

$\boldsymbol{f}$ is **non-degenerate** (w.r.t. $\mathrm{New}(f_1 \cdots f_r)$) if

$$\boldsymbol{f}_{J,\tau}(\boldsymbol{x}) = 0 \implies \mathrm{rk}(\boldsymbol{f}'_{J,\tau}(\boldsymbol{x})) = \#J$$

for faces $\tau \subseteq \mathrm{New}(f_1 \cdots f_r)$, subsets $J \subseteq \{1, \ldots, r\}$ and $\boldsymbol{x} \in (\mathbf{C}^\times)^n$.

Evaluating non-degenerate cone integrals

Recall: given Γ, we obtain $\boldsymbol{f} = (f_1, \ldots, f_r)$ s.t. $\zeta_{\Gamma,p}$ is a cone integral involving $\boldsymbol{f}$.

Theorem (R. & Voll)

Suppose $\boldsymbol{f}$ is non-degenerate. Then there are explicit $W_{\tau,J} \in \mathbf{Q}(X, Y)$ indexed by faces $\tau \subseteq \mathrm{New}(f_1 \cdots f_r)$ and subsets $J \subseteq \{1, \ldots, r\}$ s.t.

$$\zeta_{\Gamma,p}(s) = \sum_{\tau,J} c_{\tau,J}(p) W_{\tau,J}(p, p^{-s})$$

for almost all p, where $c_{\tau,J}(p) = \#\left\{u \in (\mathbf{F}_p^{\times})^n \;\middle|\; f_{j,\tau}(u) = 0 \iff j \in J\right\}$.

Heuristic observation

Typical forms of degeneracy can be fixed using a "toric reduction process" (WIP) inspired by Gröbner bases machinery.

Examples

We have

$$\zeta_{\mathbf{Z}[X]/X^3,p}(s) = (1-p^{-1})^{-3}\int_{V_p} |a|^{s-1}|x|^{s-2}|z|^{s-3}\,\mathrm{d}\mu(a,\dots,z),$$

where

$$V_p = \left\{ \begin{bmatrix} a & b & c \\ . & x & y \\ . & . & z \end{bmatrix} \in \mathrm{Tr}_3(\mathbf{Z}_p) \;\middle|\; xz \mid aby - b^2x - acx,\ abz,\ x^3,\ bx^2 \right\}.$$

Newton polytope = $\triangle$, 7 cases. Non-degenerate: ✓
Result:

$$\frac{(1+p^{1-2s})(1+p^{-s}+p^{1-2s}+(p^2-p)p^{-3s}+(p^3-p^2)p^{-4s}-p^{3-5s}-p^{4-6s}-p^{4-7s})}{(1-p^{-s})(1-p^{2-3s})^2(1-p^{4-5s})}.$$

Very similar: $\zeta_{\mathfrak{sl}_2(\mathbf{Z}),p}(s)$
du Sautoy & Taylor (2002): manual resn of singularities; 8 pages

Examples

Submodule ζ-functions for semisimple repns: L. Solomon et al. (1970s)

- $\mathrm{U}_3(\mathbf{Z}) \curvearrowright \mathbf{Z}^3 \equiv \mathfrak{n}_3(\mathbf{Z}) \curvearrowright \mathbf{Z}^3$:

$$\frac{\zeta_p(s)\zeta_p(2s-1)\zeta_p(3s-1)\zeta_p(4s-2)}{\zeta_p(4s-1)}$$

- Nilradical of the Borel subalgebra of $\mathfrak{sp}_4(\mathbf{Z})$ acting on $\mathbf{Z}^4$:

$$\frac{\zeta_p(s)\zeta_p(2s-1)\zeta_p(3s-1)\zeta_p(4s-2)^2\zeta_p(6s-3)}{\zeta_p(4s-1)\zeta_p(6s-2)}$$

- $\mathrm{U}_4(\mathbf{Z}) \curvearrowright \mathbf{Z}^4 \equiv \mathfrak{n}_4(\mathbf{Z}) \curvearrowright \mathbf{Z}^4$:

$$\frac{1-p^{1-4s}+\dots\text{35 terms}\dots-p^{10-30s}}{(1-p^{-s})(1-p^{1-2s})(1-p^{1-3s})(1-p^{1-4s})(1-p^{2-4s})(1-p^{2-5s})(1-p^{2-6s})(1-p^{3-7s})(1-p^{4-8s})}$$

- Can do: $\mathfrak{gl}_2(\mathbf{Z})$ (subrings), $\mathrm{U}_5(\mathbf{Z}) \curvearrowright \mathbf{Z}^5$, $\mathbf{Z}[X]/X^4$, “commutative Heisenberg rings” of rank $\leqslant 5$, … many known examples