A new approach to computation in finitely-presented groups

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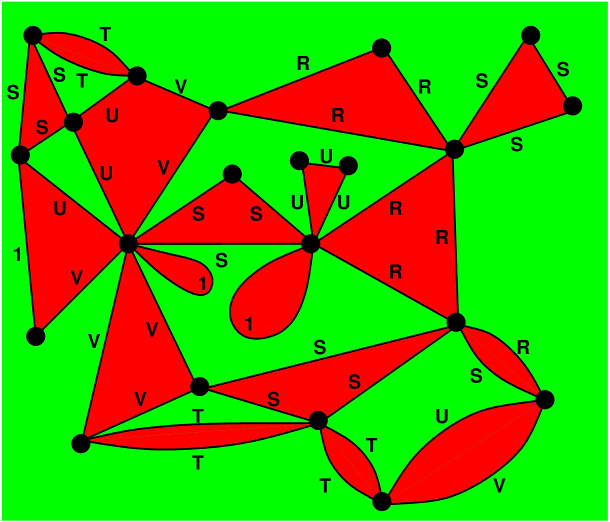
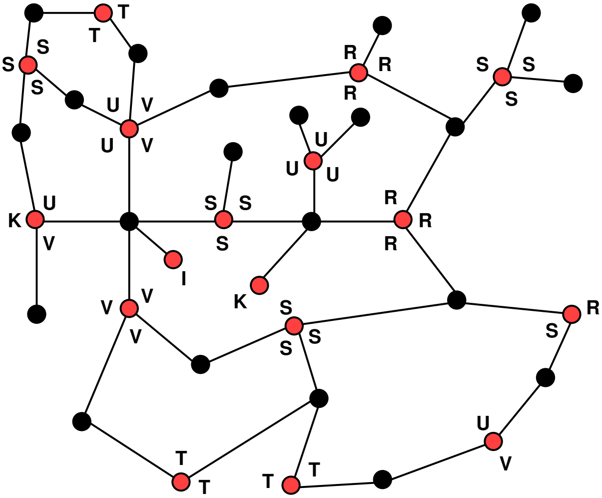
We draw connected finite plane graphs and label the edges:



Faces are oriented clockwise.

We draw connected finite plane bipartite graphs:



Labels are on the red corners. Faces are oriented clockwise.

Introducing infrastructures

Definition

An infrastructure is a semigroup S and two subsets $S_+, S_L \subseteq S$, such that:

$$\text{if } xy \in S_+ \text{ for } x, y \in S, \text{then } yx \in S_+.$$

The elements in S_+ are acceptors. The elements in S_L are labels.

If $0 \in S$ then we usually insist that $0 \notin S_+$, $0 \notin S_L$, and for all $x \in S \setminus \{0\}$ there is a $y \in S$ with $xy \in S_+$.

Lemma (Cyclicity)

Let S be an infrastructure. If $s_1 s_2 \cdots s_k \in S_+$, then all rotations $s_i s_{i+1} \cdots s_k s_1 s_2 \cdots s_{i-1} \in S_+$.

Examples of infrastructures

- Let G be a group. Let $S = G$, $S_L := G \setminus \{1\}$ and $S_+ := \{1\}$.
- Let $S^{(2)} := \{A, 1, 0\}$ with $A \cdot A = 1$ and all other products 0.
  Set $S_+^{(2)} := \{1\}$ and $S_L^{(2)} := \{A\}$.
- Let $S^{(3)} := \{A, A^{-1}, 1, 0\}$ with $A \cdot A^{-1} = A^{-1} \cdot A = 1$ and all other products 0.
  Set $S_+^{(3)} := \{1\}$ and $S_L^{(3)} := \{A, A^{-1}\}$.
- Take any groupoid, adjoin a 0 and set undefined products to 0. Let all identities accept.

Further examples of infrastructures

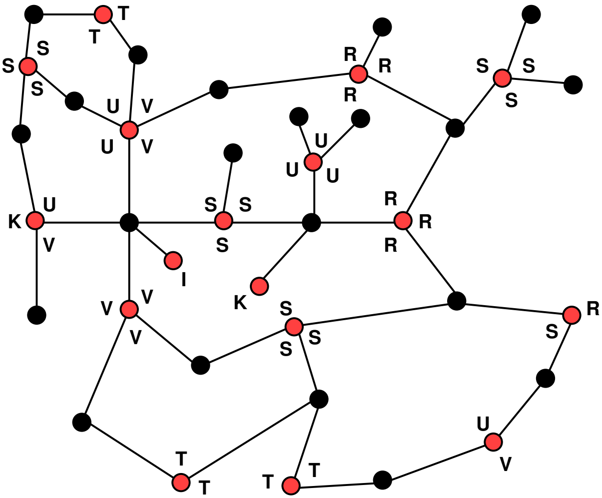
Lemma

The zero direct product of infrastructures (with unions of labels and accepters) is an infrastructure.

e.g. $P := \{R, S, I, T, J, U, V, K, 0\}$, $P_+ := \{I, J, K\}$, $P_L := \{S, R, T, U, V\}$

	R	S	I	T	J	U	V	K
R	S	I	R	·	·	·	·	·
S	I	R	S	·	·	·	·	·
I	R	S	I	·	·	·	·	·
T	·	·	·	J	·	·	·	·
J	·	·	·	·	·	·	·	·
U	·	·	·	·	·	V	K	U
V	·	·	·	·	·	K	U	V
K	·	·	·	·	·	U	V	K

These are two cyclic groups of order 3 and an $S^{(2)}$ for T.


T T
S S S
R R R
S S S
U V U V
U U U
S S S
K U V
R R R
I
K
V V V
S S S
R S
U V
T T
T T

Diagrams

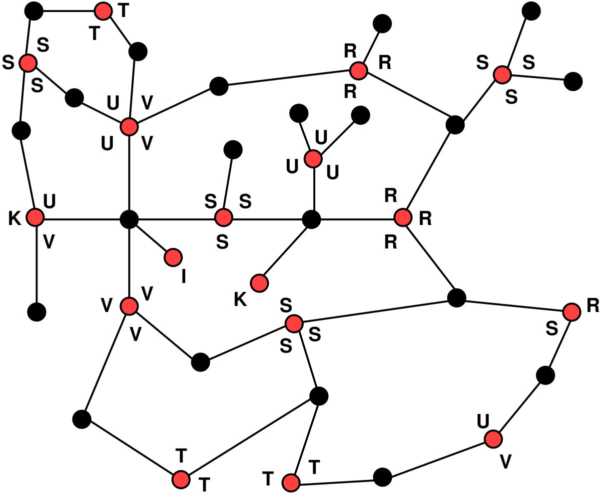
S – infrastructure. Let $\mathcal{R}$ be a set of cyclic words in S_L.

Definition (Valid diagram)

A valid diagram is: a finite set X, permutations R, G, B of X and a function $\ell : X \to S$, such that

- the product $RGB = 1$,
- the group $\langle R, G, B \rangle$ is transitive on X,
- the total number of cycles of R, G and B on X is $|X| + 2$,
- for every R-cycle $x, xR, \ldots, xR^k$ the product $\ell(x) \cdot \ell(xR) \cdot \cdots \cdot \ell(xR^k) \in S_+$, and
- for all but maybe one (the boundary) G-cycle $x, xG, \ldots, xG^k$ the word $(\ell(x), \ell(xG), \ldots, \ell(xG^k))^{\circlearrowleft} \in \mathcal{R}$.

There is a bijection between plane bipartite graphs and such triples R, G, B, up to appropriate equivalence.



We can easily store this on a computer!

Two fundamental problems

A diagram is reduced if $\mathrm{Im}\ell \subseteq S_L$.
S – infrastructure. Let $\mathcal{R}$ be a finite set of cyclic words in S_L.

Problem (Diagram boundary problem)

Algorithmically devise a procedure that decides for any cyclic word $w^{\circlearrowleft}$ in S_L whether or not there is a reduced diagram such that the external face is labelled by w.

Problem (Isoperimetric inequality)

Algorithmically find and prove a function $\mathcal{D} : \mathbb{N} \to \mathbb{N}$, s.t. for every cyclic word w in S_L of length k that is the boundary label of a valid diagram, there is one with at most $\mathcal{D}(k)$ internal faces.

If there is a linear $\mathcal{D}$, we call $\langle S \mid \mathcal{R} \rangle$ hyperbolic.

Applications

These diagrams and their two fundamental problems encode

- the word problem in quotients of the free group,
- the word problem in quotients of free products of groups,
- the word problem for relative presentations
- the rewrite decision problem for rewrite systems,
- the word problem in finite semigroup and monoid presentations,
- jigsaw-puzzles in which you can use arbitrarily many copies of each piece,
- computations of non-deterministic Turing machines,
- etc. ???

You just have to choose the right infrastructure!

Classical Small Cancellation

Consider a group presentation $\mathcal{P} = \langle X | \mathcal{R} \rangle$, relators freely cyclically cancelled, inverse closed.

Suppose that no two relators in $\mathcal{R}$ have common subword of > $1/4$ than either of their lengths.
Suppose also that internal vertices in a reduced van Kampen diagram either have valency 2 or at least 4.

Then $\mathcal{P}$ is a $C'(1/4)$ presentation.

Can show that group presented by $\mathcal{P}$ is hyperbolic.
There exists an efficient algorithm to solve the word problem for $C'(1/4)$ groups.

We want to generalise this idea – and to make it a property of groups rather than of their presentations.

Combinatorial Curvature

Given a plane graph, we endow

- each vertex with $+1$ unit of combinatorial curvature,
- each edge with -1 unit of combinatorial curvature and
- each internal face with $+1$ unit of combinatorial curvature.

Euler's formula

The total sum of our combinatorial curvature is always $+1$.

Given $S, \mathcal{R}$, first find "pieces",

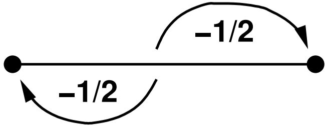
- compute the finite list of all possible edges,
- edges now have different lengths,
- denote the new set of sides of edges in a diagram by $\hat{E}$.

Idea (Officers)

We redistribute the curvature locally in a conservative way.
We call a curvature redistribution scheme an officer.

"Officer Tom":

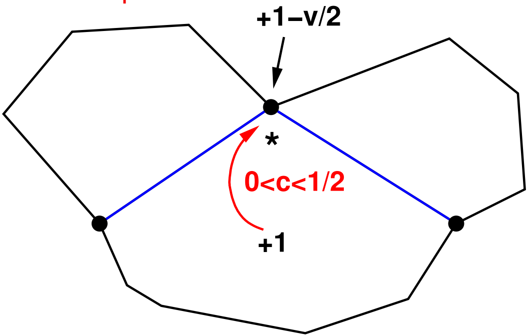
Phase 1: Tom moves the negative curvature to the vertices:



Any vertex in any diagram with valency v (≥ 3) now has $+1 - \frac{v}{2} < 0$.
All internal faces still have $+1$, all edges now have 0.

Phase 2 of Tom

Tom now moves the positive curvature from faces to vertices:



Corner values for Tom

A corner value c of Tom depends on two edges that are adjacent on a face. Tom moves c units of curvature to the vertex v.

Default values for c: $1/6$ if v might have valency 3, and $1/4$ otherwise.

What do officers achieve?

Officers try to redistribute the curvature, such that for all permitted diagrams, after redistribution

- every internal face has $< -\varepsilon$ curvature (for some explicit $\varepsilon > 0$),
- every vertex has ≤ 0 curvature.
- every edge has 0 curvature,
- every face with more than one external edge has ≤ 0 curvature.

Consequence:

All the positive curvature is on faces touching the boundary once.

(Need to show that diagram boundaries have a permitted diagram.)

Using and analysing curvature

- The total positive curvature $\leq n$ (boundary length).
- Let $F := \#$internal faces, then

$$1 < n - F \cdot \varepsilon \quad \Longrightarrow \quad F < \varepsilon^{-1} \cdot n \quad \Longrightarrow \quad \text{hyperbolic}$$

Let $L := \{1, 2, \ldots, \ell\}$ and $a_1, a_2, \ldots, a_\ell \in \mathbb{R}$ and $T := \sum_{m \in L} a_m$.
Define $\pi_L : \mathbb{Z} \to L$ such that $z \equiv \pi_L(z) \pmod{\ell}$.

Lemma (Goes up and stays up)

If $T \geq 0$ then $\exists\, j \in L$ s.t. for all $i \in \mathbb{N}$ the partial sum

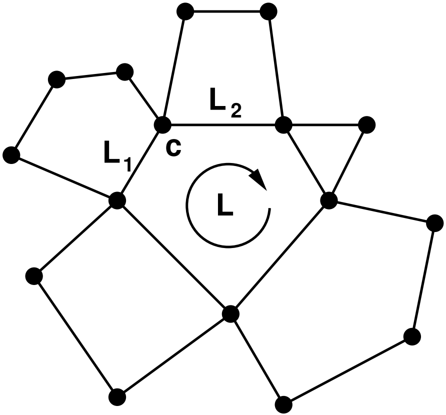
$$t_{j,i} := \sum_{m=0}^{i-1} a_{\pi_L(j+m)} \geq 0.$$

Corollary

Assume that there are $k \in \mathbb{N}$ and $\varepsilon \geq 0$ such that for all $j \in L$ there is an $i \leq k$ with $t_{j,i} < -\varepsilon$, then $T < -\varepsilon \cdot \ell / k$.

Sunflower

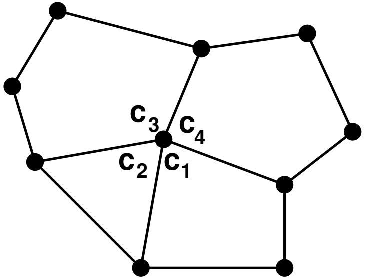
To show that every internal face has curvature $< -\varepsilon$:



Use Goes up and stays up on $\frac{L_1+L_2}{2L} - c$.

Poppy

To show that every internal vertex has curvature ≤ 0:



Use Goes up and stays up on $c + \frac{1-v/2}{v} = c + \frac{2-v}{v}$.

Do valency $v = 3$ first, if nothing found, increase v.

This terminates: higher valencies tend to be negatively curved.

What does Tom achieve?

If Tom found no bad sunflowers or poppies, we have

- determined an explicit ε,
- proved hyperbolicity, and
- can in principle solve the diagram boundary problem.

If we did find bad sunflowers or poppies, we can still

- improve our choices for the corner values (leads to difficult optimisation/linear program problems),
- forbid more diagrams (if possible) (need to show that every boundary is proved by a permitted one),
- or switch to a more powerful officer (with further sight or redistribution), ...

and try again. If $\langle S \mid \mathcal{R} \rangle$ is not hyperbolic, this will not work.