Compresed word problem in wreath products

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The word problem for groups

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Novikov 1958, Boone 1959: There exists a finitely presented group with an undecidable word problem.

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one-relator groups $\langle \Sigma, r \rangle$	primitive recursive

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INPUT: Word $w \in \Sigma^*$

OUTPUT: If $w \neq 1$ in G then output "NO", otherwise output words $c_1, \ldots, c_n \in \Sigma^*$ and $r_1, \ldots, r_n \in R \cup R^{-1}$ with

$$w = \prod_{i=1}^{n} c_i r_i c_i^{-1} \text{ in } F(\Sigma).$$

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A straight-line program (SLP) over the alphabet Γ is a sequence of definitions $\mathbb{A} = (A_i := \alpha_i)_{1 \leq i \leq n}$, where either $\alpha_i \in \Gamma$ or $\alpha_i = A_j A_k$ for some $j, k < i$.

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If $|\mathbb{A}|$ is the number of definitions in $\mathbb{A}$, then $|\mathrm{val}(\mathbb{A})| \leq 2^{|\mathbb{A}|}$.

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  - word hyperbolic groups (Saul Schleimer's talk)

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Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of f.g. groups such that the quotient Q is finitely presented.

$\mathrm{WSP}(Q) \in \mathbf{P}$, $\mathrm{CWP}(K) \in \mathbf{P} \Rightarrow \mathrm{WP}(G) \in \mathbf{P}$

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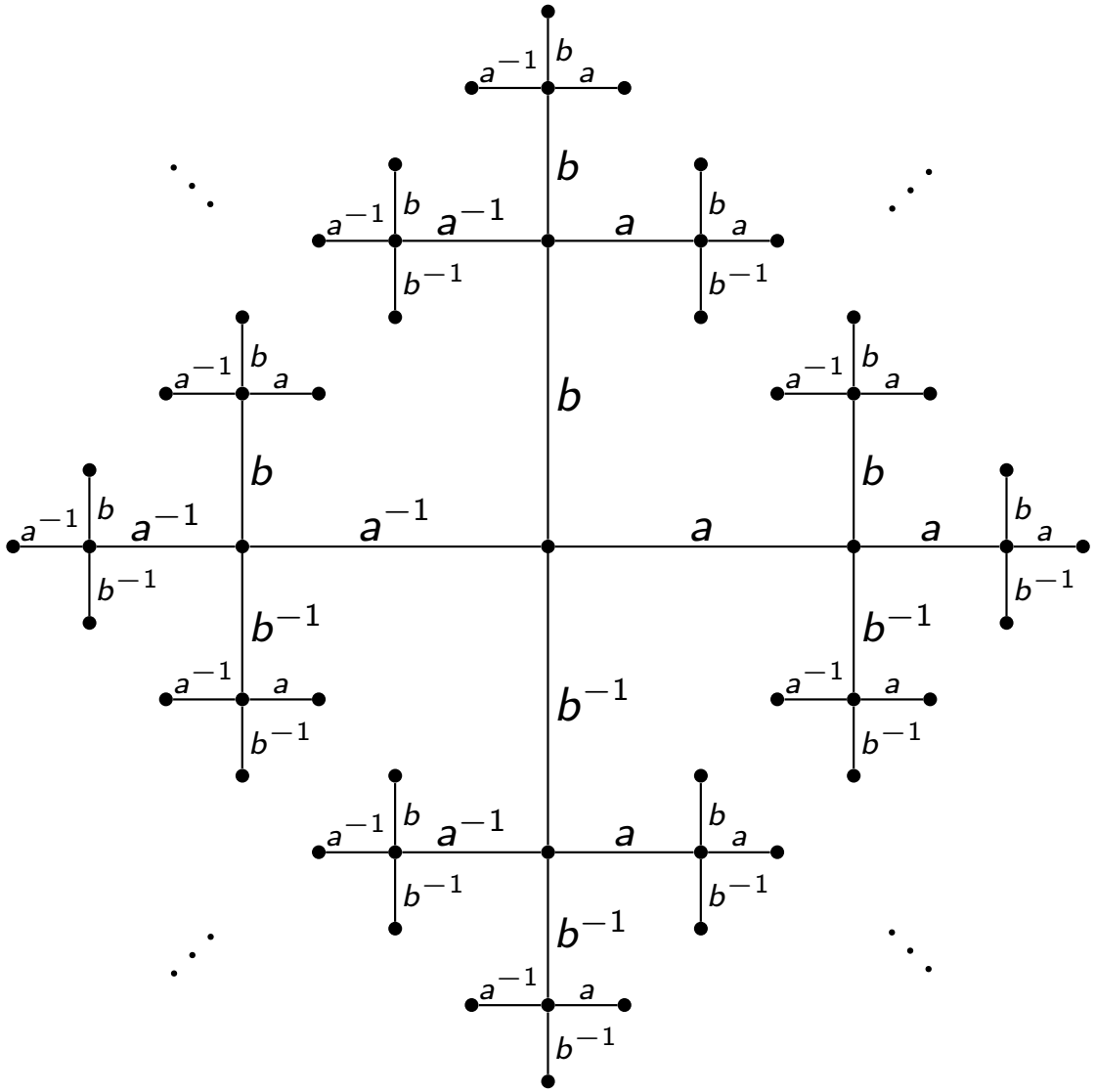
The wreath product $A \wr B$ is the set of all pairs $K \times B$ with the following multiplication, where $(k_1, b_1), (k_2, b_2) \in K \times B$:

$$(k_1, b_1)(k_2, b_2) = (k, b_1 b_2) \text{ with } \forall b \in B : k(b) = k_1(b) k_2(b_1^{-1} b).$$

Wreath product $\mathbb{Z}_2 \wr F(a,b)$ with $\mathbb{Z}_2 = \langle c \mid c^2 = 1 \rangle$

$cbcb^{-1}cabcb^{-1}ca$:



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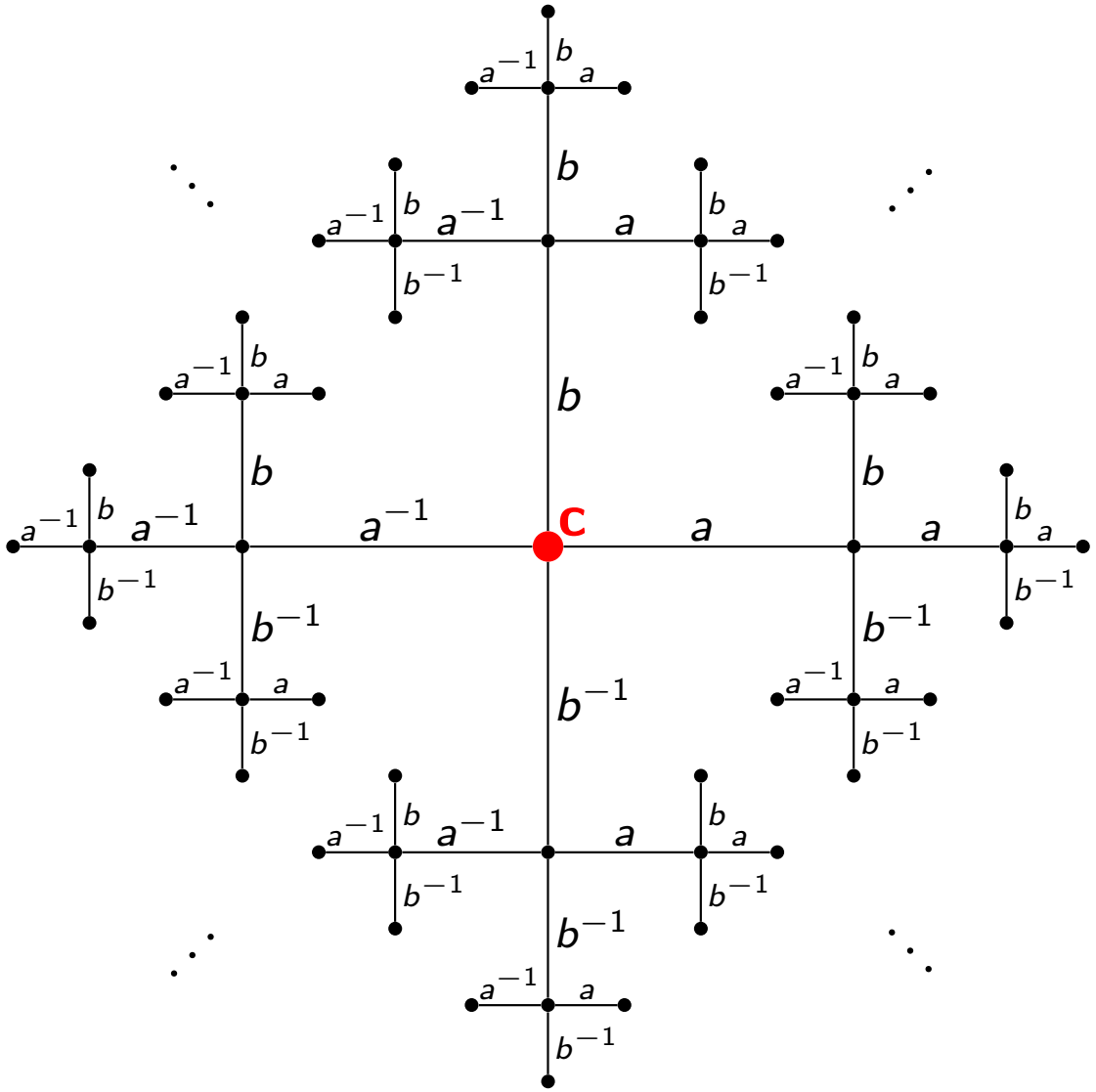

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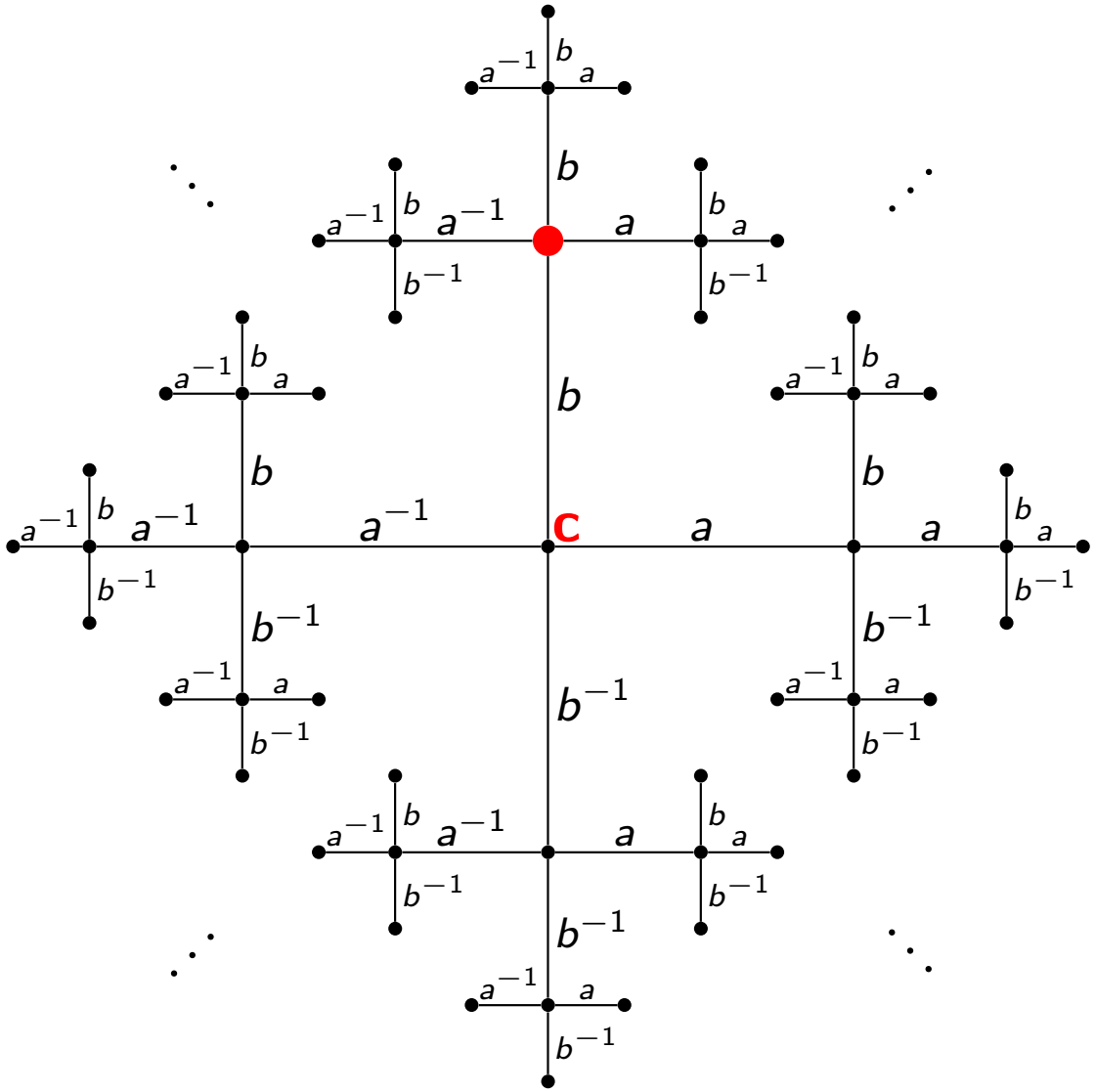

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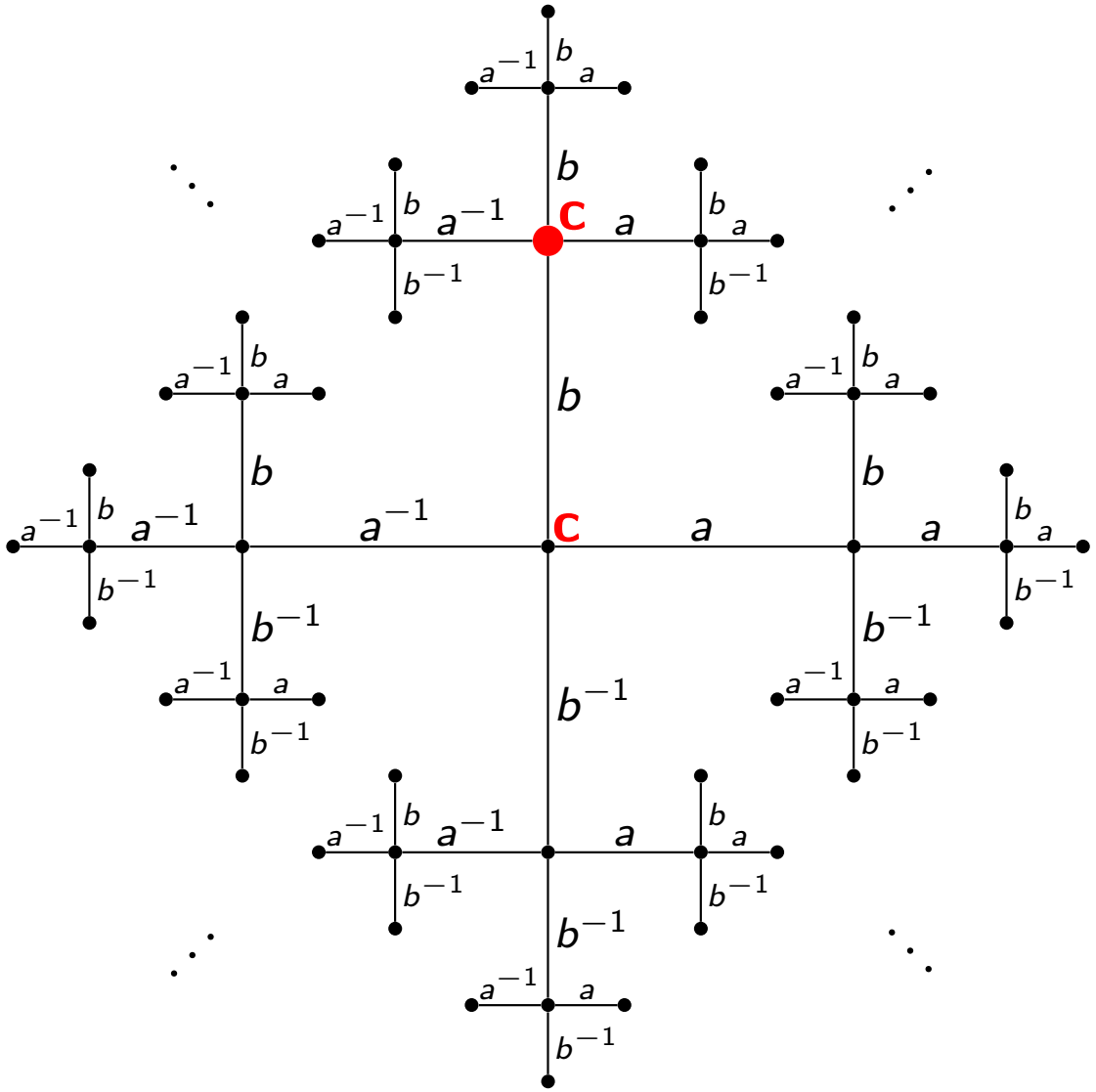

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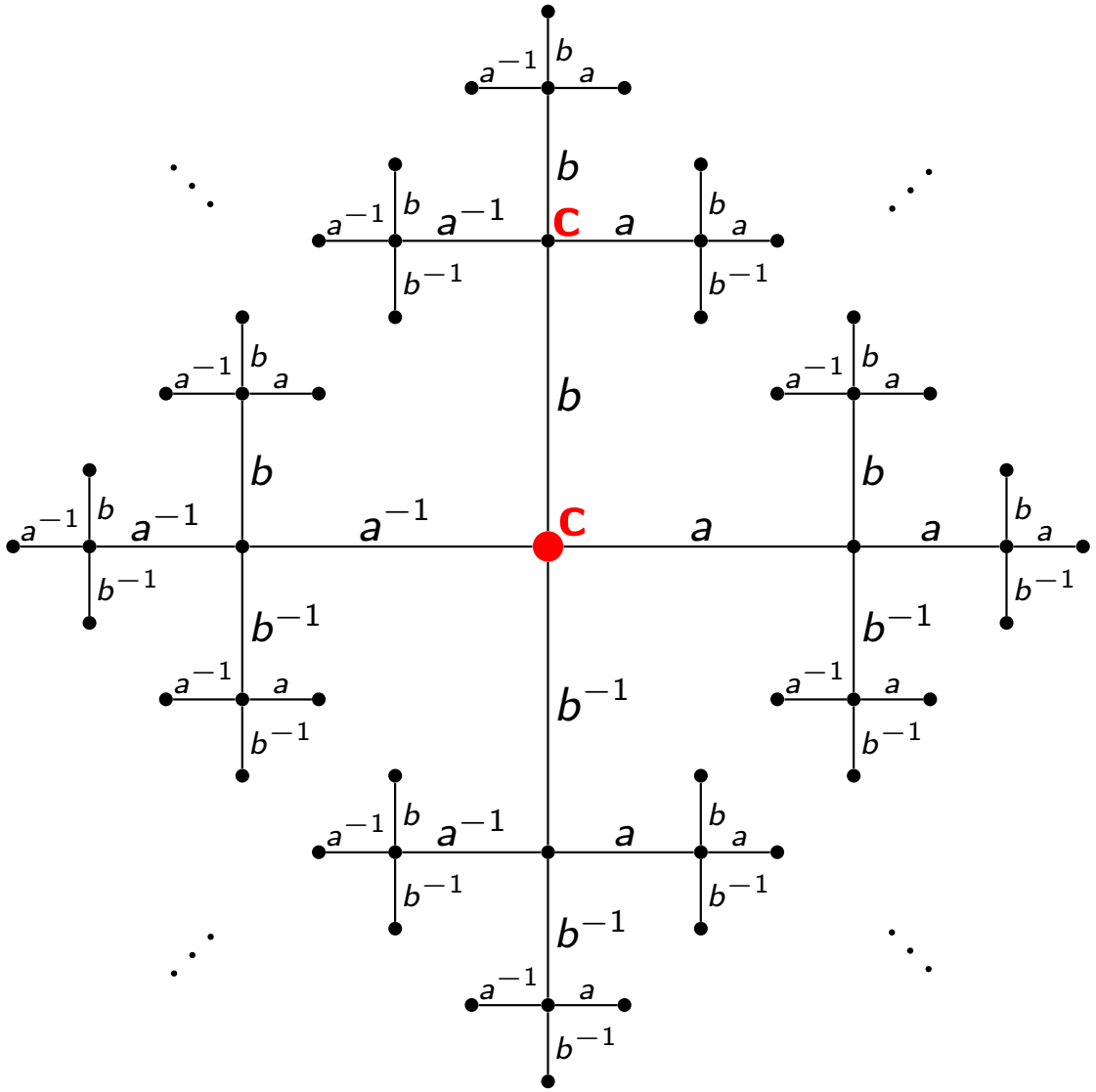

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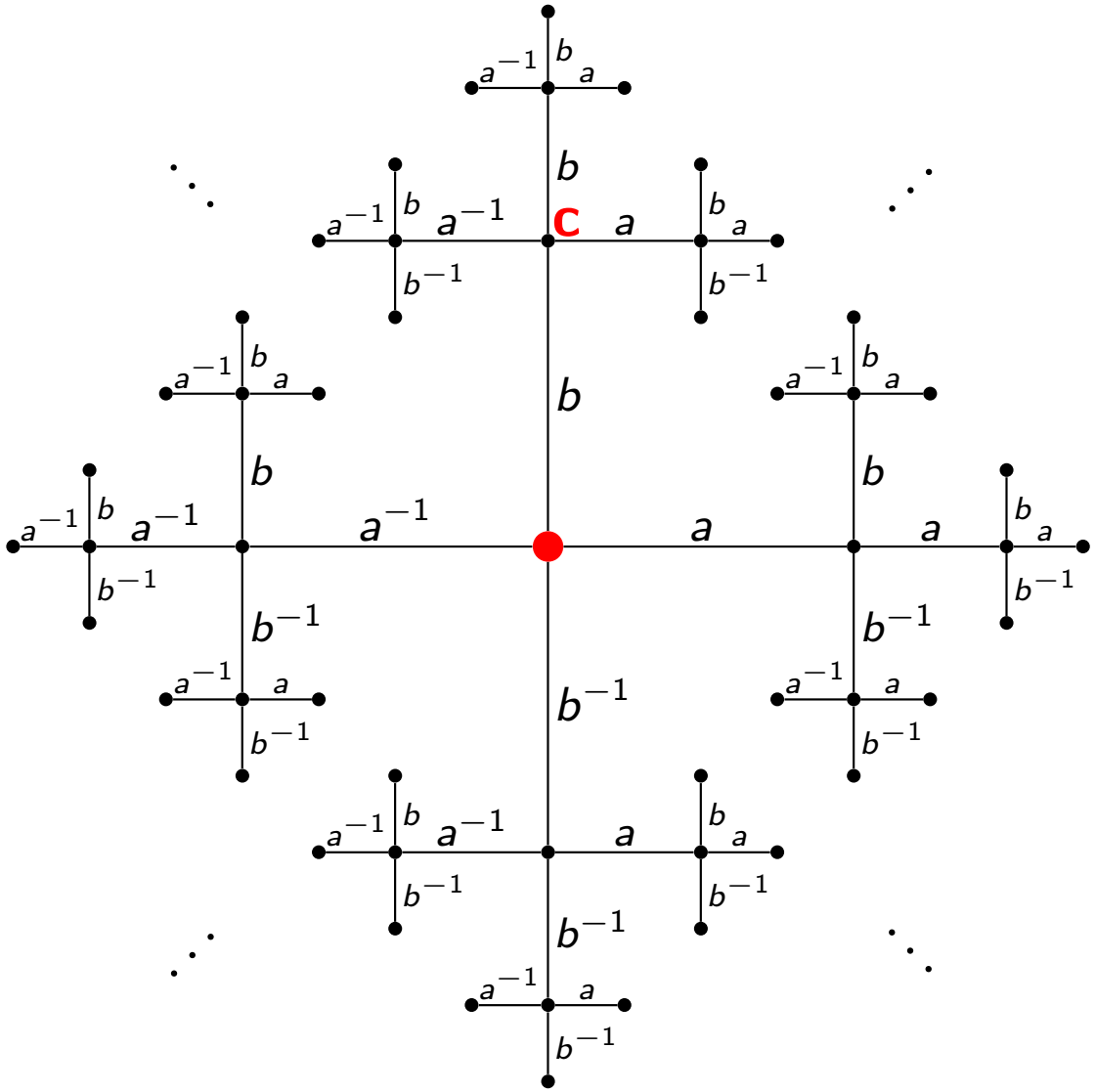

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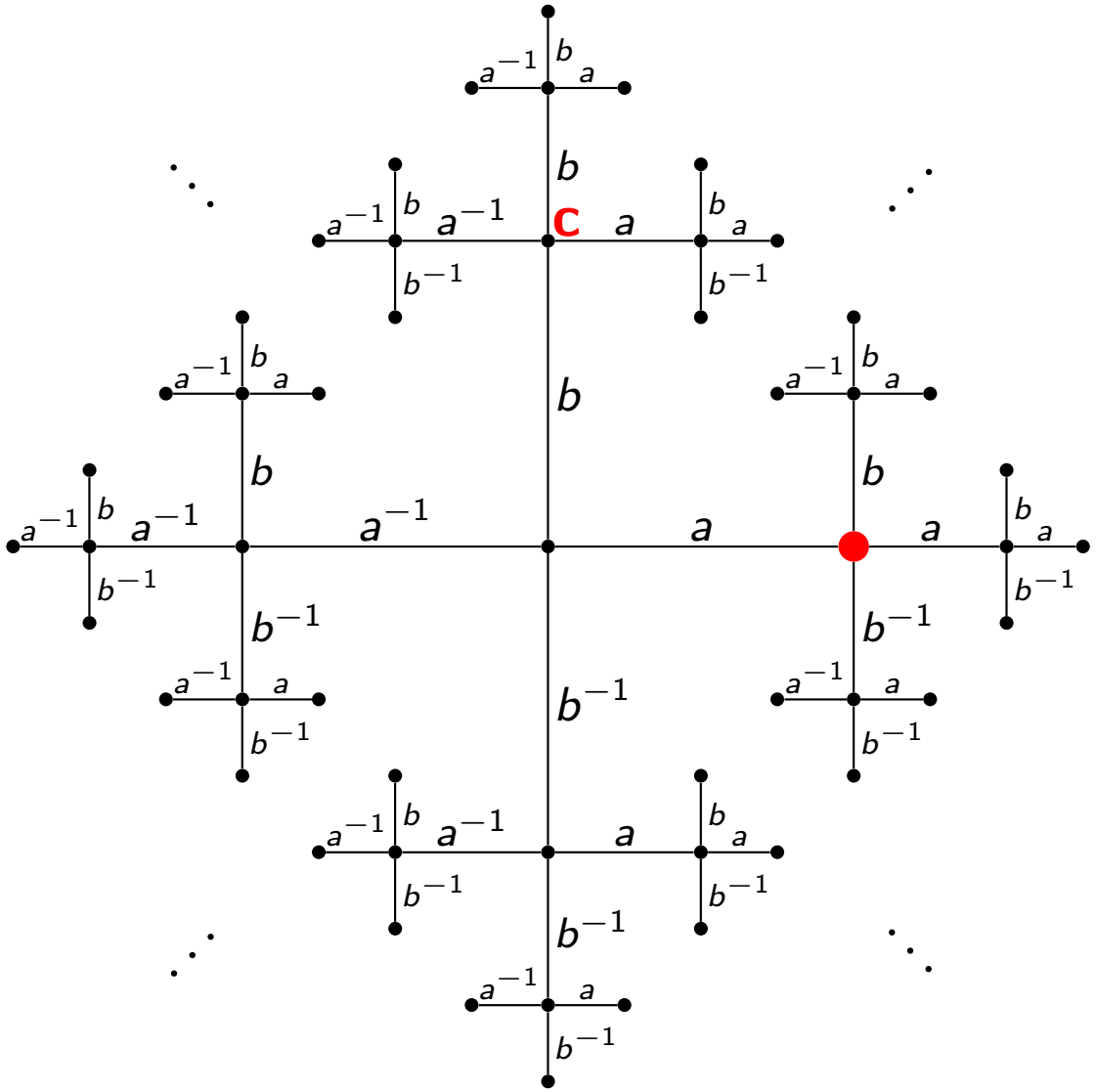

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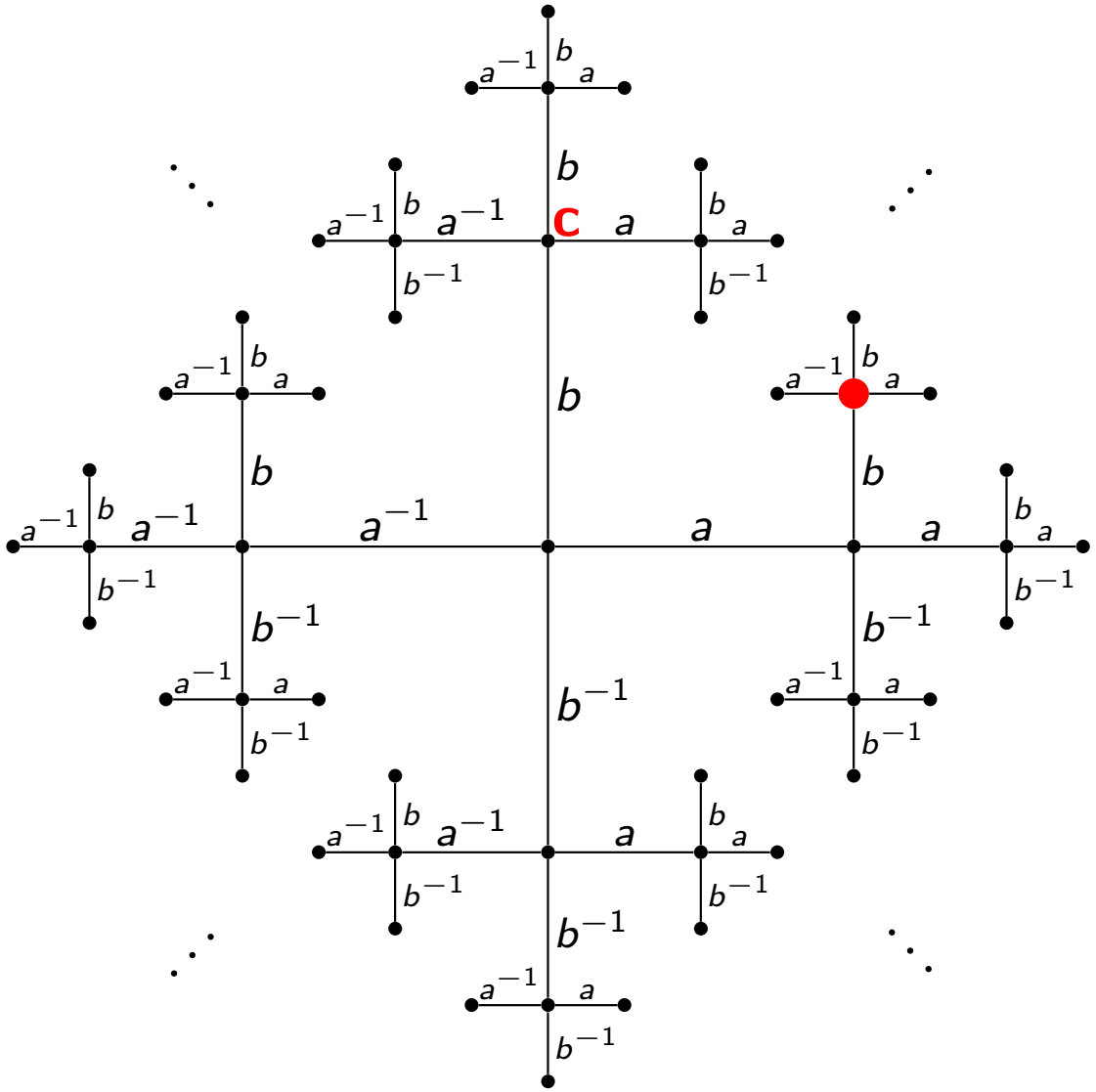

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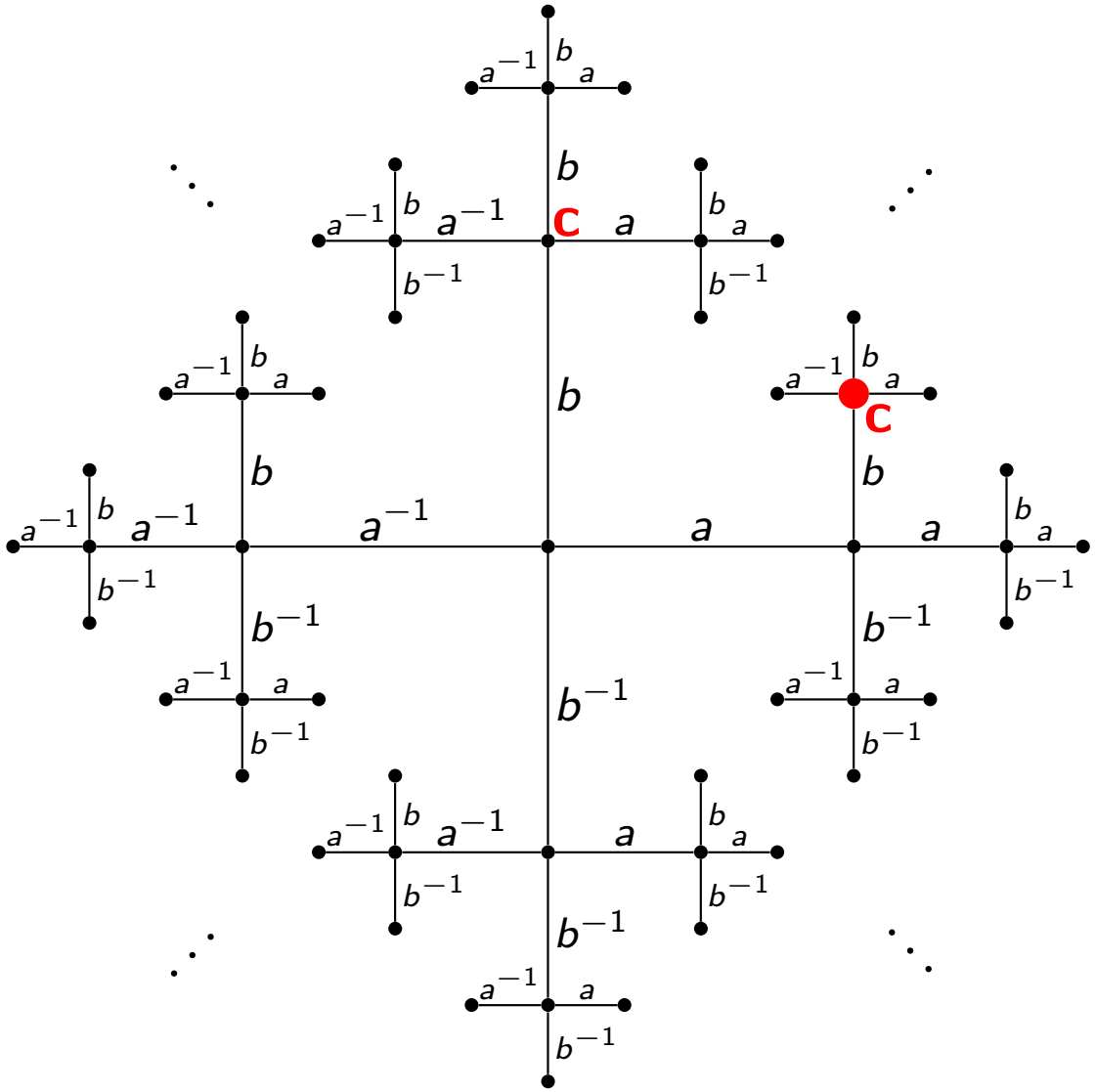

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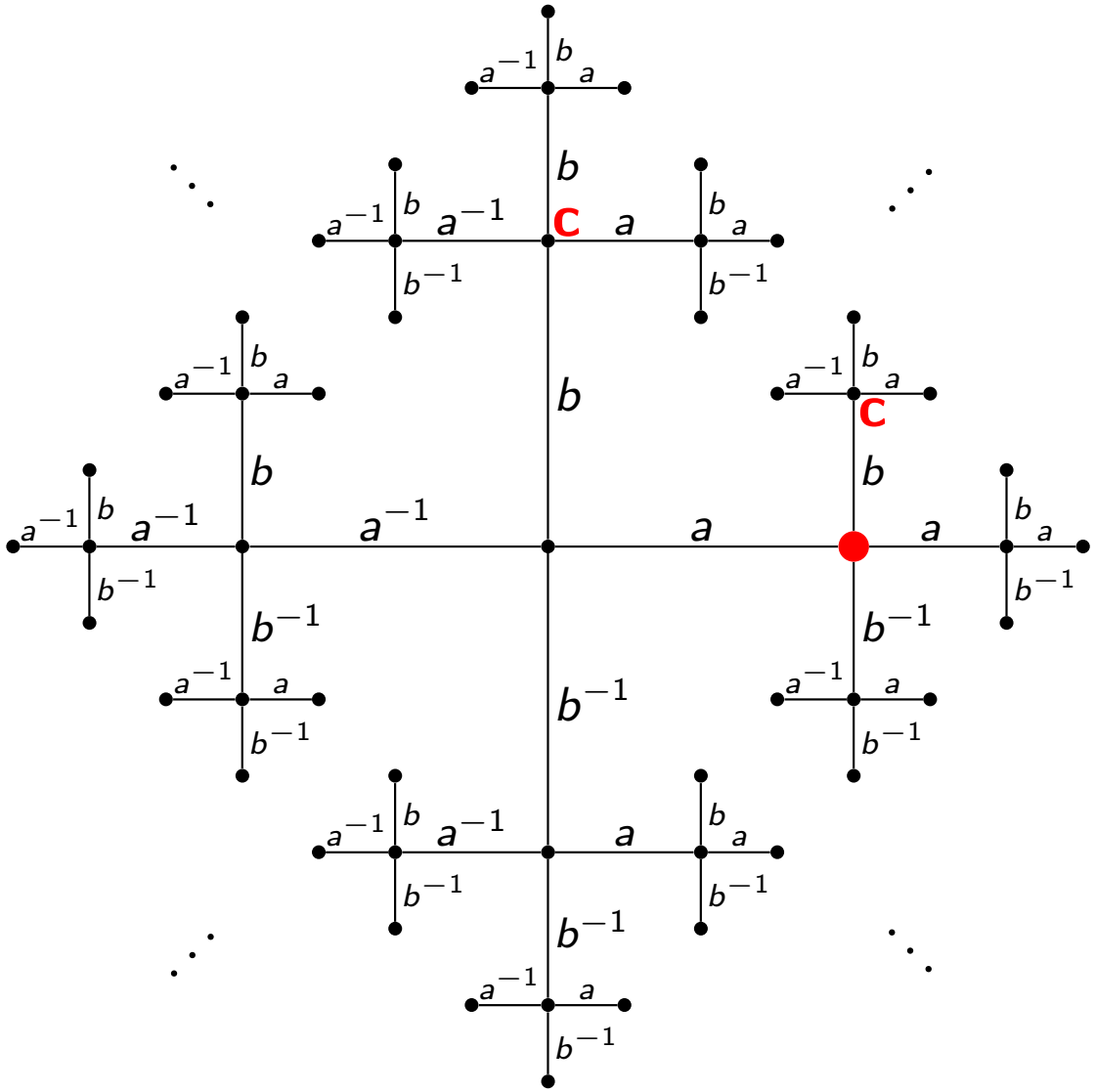

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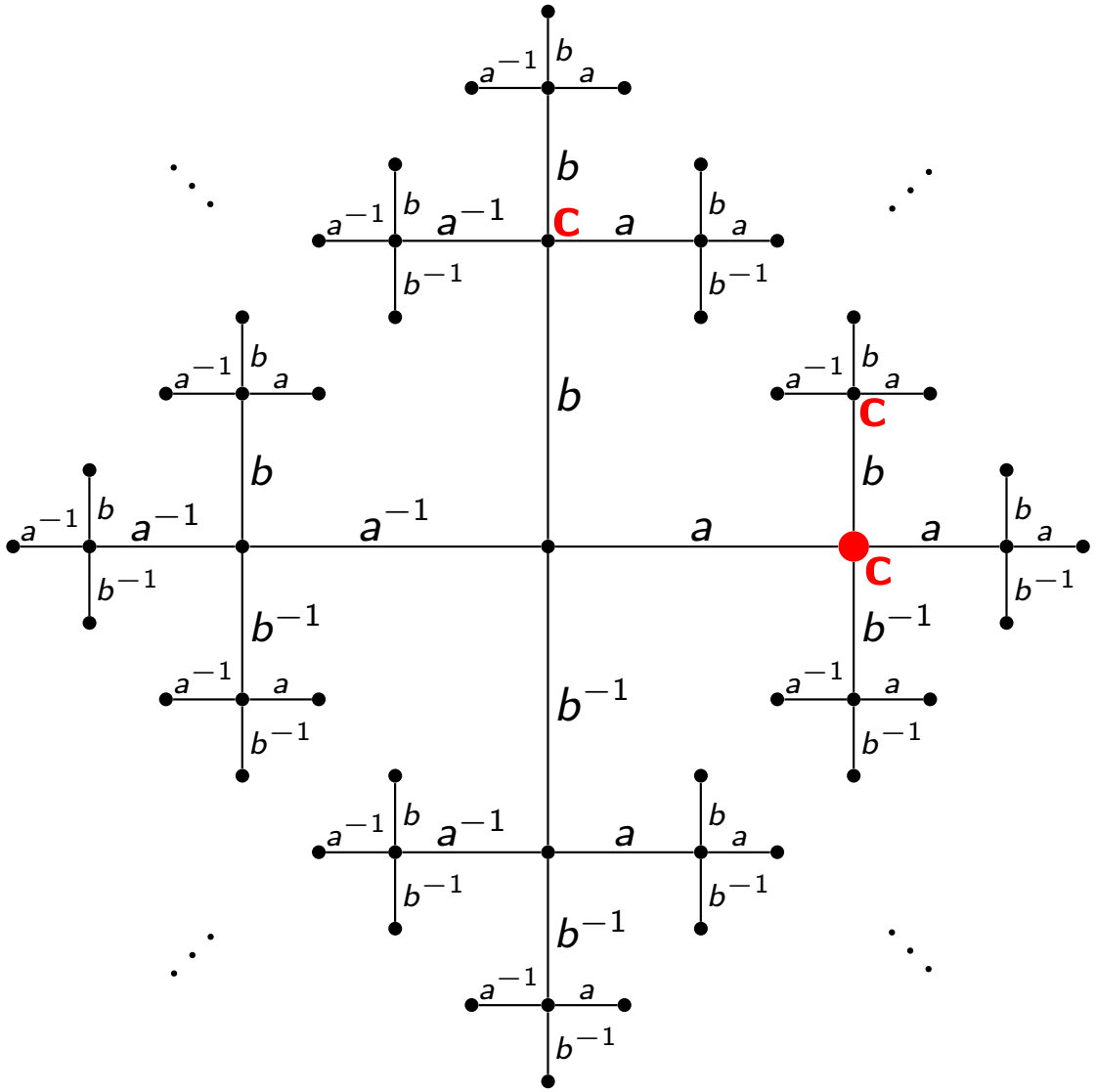

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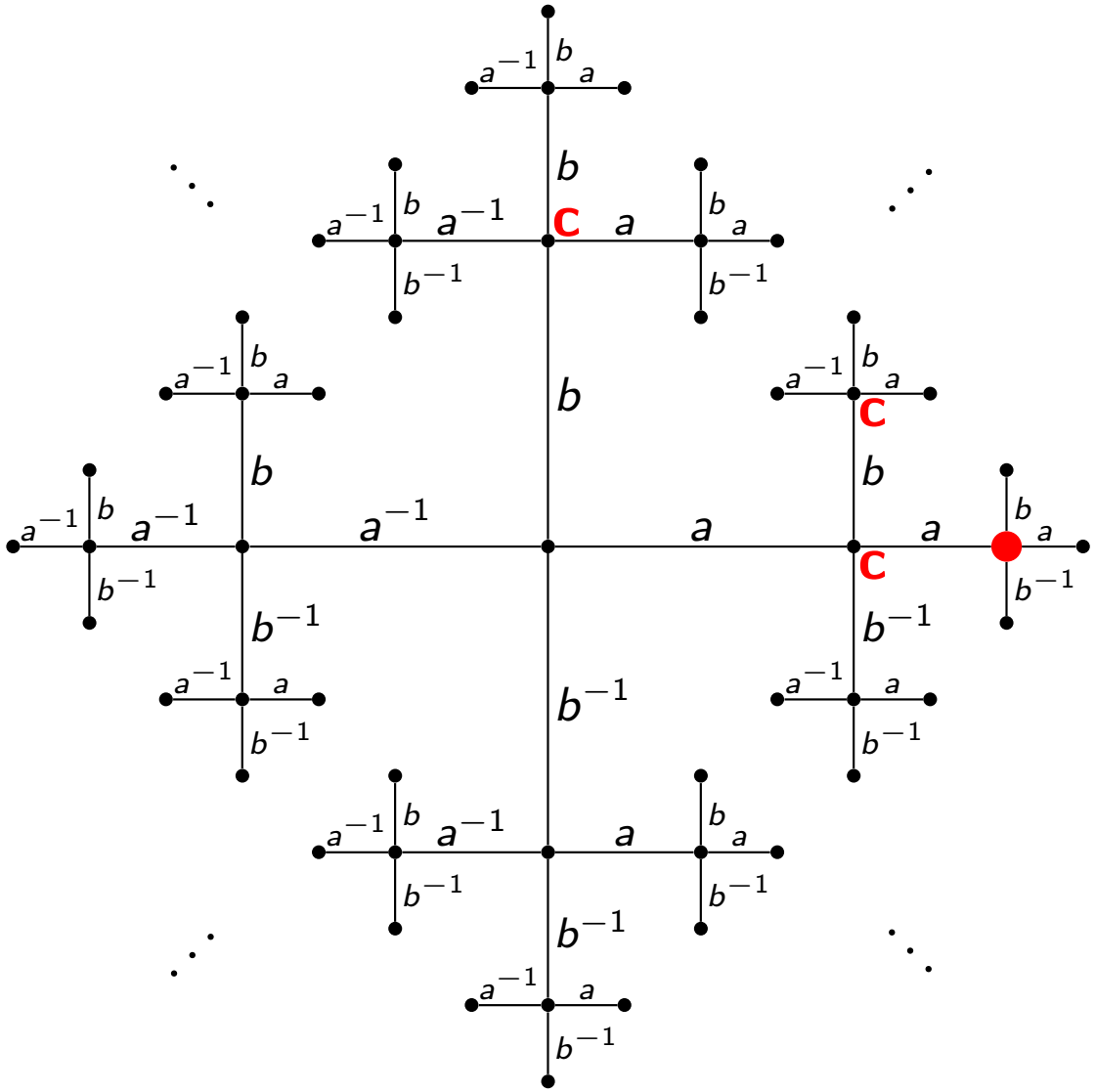

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From $\overline{w}$, z we can construct in poly. time SLPs $\mathbb{A}$, $\mathbb{B}$ such that

$$\mathrm{val}(\mathbb{A}) = \prod_{\overline{x} \in \{0,1\}^n} (t^{\overline{x}\cdot\overline{w}-1}\, c\, t^{s-\overline{x}\cdot\overline{w}}) \quad \text{and} \quad \mathrm{val}(\mathbb{B}) = (t^{z-1}\, c\, t^{s-z})^{2^n}.$$

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$$\rightsquigarrow \quad \exists p \in \mathbb{N} : p\text{-th symbol of } \mathrm{val}(\mathbb{A}) = c = p\text{-th symbol of } \mathrm{val}(\mathbb{B})$$

$$\Leftrightarrow$$

$$\exists \overline{x} \in \{0,1\}^n : \overline{x} \cdot \overline{w} = z$$

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We can construct in poly. time an SLP $\mathbb{C}$ such that

$$\mathrm{val}(\mathbb{C}) = \mathrm{val}(\mathbb{A}_a)t^{-s\cdot 2^n}\mathrm{val}(\mathbb{B}_b)t^{-s\cdot 2^n}\mathrm{val}(\mathbb{A}_{a^{-1}})t^{-s\cdot 2^n}\mathrm{val}(\mathbb{B}_{b^{-1}})t^{-s\cdot 2^n}.$$

Easy word problem but difficult compressed word problem

Let $\mathbb{Z} = \langle t \rangle$.

Choose two elements $a, b \in A$ with $[a, b] \neq 1$.

For $x \in \{a, b, a^{-1}, b^{-1}\}$ let $\mathbb{A}_x$ ($\mathbb{B}_x$) be the SLP that is obtained from $\mathbb{A}$ ($\mathbb{B}$) by replacing every occurrence of the letter c by x.

We can construct in poly. time an SLP $\mathbb{C}$ such that

$$\mathrm{val}(\mathbb{C}) = \mathrm{val}(\mathbb{A}_a)t^{-s\cdot 2^n}\mathrm{val}(\mathbb{B}_b)t^{-s\cdot 2^n}\mathrm{val}(\mathbb{A}_{a^{-1}})t^{-s\cdot 2^n}\mathrm{val}(\mathbb{B}_{b^{-1}})t^{-s\cdot 2^n}.$$

Then we have:

$$\mathrm{val}(\mathbb{C}) \neq 1 \text{ in } A \wr \mathbb{Z}$$

$$\Leftrightarrow$$

$$\exists p \in \mathbb{N} : p\text{-th symbol of } \mathrm{val}(\mathbb{A}) = c = p\text{-th symbol of } \mathrm{val}(\mathbb{B}).$$

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Hence, $\mathrm{CWP}(H \wr G)$ (with G and H finitely generated abelian) reduces to the CWP for finitely generated linear groups.

Randomized complexity classes

A language L belongs to the class **RP** (randomized polynomial time) if there exists a nondeterministic polynomial time bounded Turing machine M such that for every input x:

- If $x \notin L$ then Prob[M accepts x] $= 0$.
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Impagliazzo, Wigderson 1997

If there exists a language in $\mathbf{DTIME}(2^{\mathcal{O}(n)})$ that has circuit complexity $2^{\Omega(n)}$ (seems to be plausible) then $\mathbf{P} = \mathbf{RP} = \mathbf{coRP}$ (actually, $\mathbf{P} = \mathbf{BPP}$).

Polynomial identity testing

An arithmetic circuit is a directed acyclic graph $\mathcal{C}$ such that:

- Every node (gate) is labelled with either 1, -1, a variable $x_1, \ldots, x_n$, or an operator $+$, $\cdot$.
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Polynomial identity testing over the ring $R \in \{\mathbb{Z}\} \cup \{\mathbb{Z}_n \mid n \geq 2\}$

INPUT: An arithmetic circuit $\mathcal{C}$.
QUESTION: Is $p_{\mathcal{C}}$ the zero polynomial in $R[x_1, \ldots, x_n]$?

Complexity of polynomial identity testing

Ibarra, Moran 1983; Agrawal, Biswas 2003

For every ring $R \in \{\mathbb{Z}\} \cup \{\mathbb{Z}_n \mid n \geq 2\}$, polynomial identity testing over R belongs to **coRP**.

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Kabanets, Impagliazzo 2004

If polynomial identity testing over $\mathbb{Z}$ belongs to **P**, then one of the following conclusions holds:

- There is a language in **NEXPTIME** that does not have polynomial size boolean circuits.
- The permanent is not computable by polynomial size arithmetic circuits.

Polynomial identity testing and the compressed word problem

If G is finitely generated linear over field of characteristic 0 (resp. $p \in$ Primes), then $\mathrm{CWP}(G)$ can be reduced to polynomial identity testing over $\mathbb{Z}$ (resp. $\mathbb{Z}_p$).

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In particular, $\mathrm{CWP}(G)$ belongs to **coRP**.

Proof: G can be embedded into $\mathrm{GL}_n(\mathbb{Q}(x_1, \ldots, x_n))$ (resp. $\mathrm{GL}_n(\mathbb{F}_p(x_1, \ldots, x_n))$ for some n (Lipton, Zalcstein 1975).

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Proof: Uses a construction of Ben-Or, Cleve 1992.

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Construct an SLP $\mathbb{A}$ over generators of $\mathrm{SL}_3(\mathbb{Z})$ such that:
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The SLP $\mathbb{A}$ contains for every $\mathcal{C}$-gate A and all $b \in \{-1, 1\}$ and $1 \leq i, j \leq 3$ with $i \neq j$ a variable $A_{i,j,b}$ such that: If $\overline{y} = A_{i,j,b} \cdot \overline{x}$ then $y_i = x_i + b \cdot A \cdot x_j$ and $y_k = x_k$ for $k \in \{1, 2, 3\} \setminus \{j\}$.

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Case 2. $A := B + C$. Set $A_{i,j,b} := B_{i,j,b} + C_{i,j,b}$.

Case 3. $A := B \cdot C$. Let $\{k\} = \{1,2,3\} \setminus \{i,j\}$. Then we set

$$\begin{aligned} A_{i,j,1} &:= B_{k,j,-1} C_{i,k,1} B_{k,j,1} C_{i,k,-1} \\ A_{i,j,-1} &:= B_{k,j,-1} C_{i,k,-1} B_{k,j,1} C_{i,k,1} \end{aligned}$$

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- Compressed word problem for braid groups (they are linear).