Efficient Gröbner Basis Computation in Group Rings and Applications

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1 – Non-Commutative Polynomials

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Idea: Use the potential of **Gröbner basis theory** for computations in group theory.

K field

$K\langle X\rangle$ free associative algebra over the alphabet $X = \{x_1, \ldots, x_n\}$
(This will be called the **non-commutative polynomial ring**.)

X^* monoid of all **words** $x_{i_1} \cdots x_{i_r}$

$I \subseteq K\langle X\rangle$ two-sided ideal generated by $f_1, \ldots, f_s \in K\langle X\rangle$

$R = K\langle X\rangle / I$ finitely presented K-algebra

Main Example:

$G = \langle x_1, \ldots, x_n; \ell_1 = r_1, \ldots, \ell_s = r_s \rangle$ **finitely presented group** (or monoid)

$K\langle G\rangle = \bigoplus_{g\in G} Kg$ **group ring**

$K\langle G\rangle = K\langle X\rangle / I$

$I = \langle \ell_1 - r_1, \ldots, \ell_s - r_s \rangle$ two-sided ideal generated by **binomials**

Definition 1.1 **(a)** A complete ordering σ on X^* is called a **word ordering** if

(1) it is multiplicative, i.e. $w_1 <_\sigma w_2$ implies $w_3w_1w_4 <_\sigma w_3w_2w_4$,

(2) it is a well-ordering. (Equivalently, $1 <_\sigma w$ for all $w \neq 1$.)

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(2) it is a well-ordering. (Equivalently, $1 <_\sigma w$ for all $w \neq 1$.)

(b) For a word $w = x_{i_1} \cdots x_{i_\ell}$, the number $\deg(w) = \ell$ is called the **degree** or the **length** of the word.

Example 1.2 The **length lexicographic word ordering** `llex` is defined by $w_1 <_{\texttt{llex}} w_2$ iff

(1) $\deg(w_1) < \deg(w_2)$ or

(2) $\deg(w_1) = \deg(w_2)$ and the first letter where w_1 and w_2 differ has a larger index in w_1.

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Notice that $x_1 >_{\texttt{llex}} \cdots >_{\texttt{llex}} x_n$.

Remark 1.3 The lexicographic ordering is not a word ordering, because

$$x_1 >_{\texttt{lex}} x_2x_1 >_{\texttt{lex}} x_2x_2x_1 >_{\texttt{lex}} \cdots$$

yields a set of words without minimal element.

Definition 1.4 Let $f \in K\langle X\rangle \setminus \{0\}$. Then we have a unique representation $f = c_1 w_1 + \cdots + c_s w_s$ with $c_i \in K \setminus \{0\}$ and $w_i \in X^*$ satisfying $w_1 >_\sigma w_2 >_\sigma \cdots >_\sigma w_s$.

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Example 1.5 For the non-commutative polynomial $f = x_2x_1x_2 + x_1x_2 + 1$ we have $\mathrm{Lw}_{\mathtt{llex}}(f) = x_2x_1x_2$ and $\mathrm{Supp}(f) = \{x_2x_1x_2,\ x_1x_2,\ 1\}$.

σ word ordering

$f_1, \ldots, f_s \in K\langle X\rangle \setminus \{0\}$ non-commutative polynomials

$I = \langle f_1, \ldots, f_s\rangle$ two-sided ideal generated by $\{f_1, \ldots, f_s\}$

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Definition 1.6 **(a)** The ideal $\mathrm{Lw}_\sigma(I) = \langle \mathrm{Lw}_\sigma(f) \mid f \in I \setminus \{0\} \rangle$ is called the **leading word ideal** of I.

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(b) A set of non-commutative polynomials $G = \{g_1, \ldots, g_s\}$ in $I \setminus \{0\}$ is called a **σ-Gröbner basis** of I if

$$\mathrm{Lw}_\sigma(I) = \langle \mathrm{Lw}_\sigma(g_1), \ldots, \mathrm{Lw}_\sigma(g_s) \rangle$$

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Example 1.7 For the ideal $I = \langle f_1, f_2, f_3, f_4 \rangle$ generated by $f_1 = x^2 - yx$, $f_2 = xy - zy$, $f_3 = xz - zy$, and $f_4 = yz - zy$ in $\mathbb{Q}\langle x, y, z \rangle$, we have the `llex`-Gröbner basis $G = \{f_1,\ f_2,\ f_3,\ f_4,\ zy^2 - z^2y,\ y^2x - zyx\}$.

Example 1.8 The principal ideal $I = \langle x^2 - yx \rangle$ in $\mathbb{Q}\langle x, y \rangle$ has an **infinite** reduced `llex`-Gröbner basis G. We have

$$\begin{aligned} \mathrm{Lw}_{\mathtt{llex}}(I) &= \langle xy^i x \mid i \geq 0 \rangle \\ G &= \{ xy^i x - xy^{i+1} \mid i \geq 0 \} \end{aligned}$$

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Remark 1.9 Non-commutative Gröbner bases have characterizations similar to commutative Gröbner bases:

(a) special generation of the ideal I

(b) convergence of the associated rewriting system

(c) Buchberger criterion

2 – The Buchberger Procedure

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Idea: Construct an efficient **enumerating procedure** to compute non-commutative Gröbner bases!

If the given ideal has a finite Gröbner basis, the procedure shall stop after finitely many steps and return the answer.

If the given ideal has an infinite Gröbner basis, the procedure shall enumerate Gröbner basis elements for a specified amount of time.

The Division Algorithm

Given monic polynomials $f, g_1, \ldots, g_s \in K\langle X \rangle$ and a word ordering σ on X^*, consider the following steps.

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(1) Starting initially with $j = 1$, $p = 0$ and $h = f$, find the smallest $i_j \in \{1, \ldots, s\}$ such that $\mathrm{Lw}_\sigma(h) = w\,\mathrm{Lw}_\sigma(g_{i_j})w'$ with $w, w' \in X^*$.

(2) If such an i_j exists, set $\ell_j = w$, $r_j = w'$, increase j by one, an replace f by $f - \ell_j g_{i_j} r_j$.

(3) If no such i_j exists, replace p by $p + \mathrm{Lw}_\sigma(v)$ and v by $v - \mathrm{Lw}_\sigma(v)$.

(4) Repeat (1) – (3) until $h = 0$. Then return the pairs (ℓ_j, r_j) and p.

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(4) Repeat (1) – (3) until $h = 0$. Then return the pairs (ℓ_j, r_j) and p.

This is an **algorithm** which computes a representation $f = \sum_j \ell_j g_{i_j} r_j + p$ such that no word in the support of the **normal remainder** $\mathrm{NR}_{\sigma,G}(f) = p$ is divisible by some $\mathrm{Lw}_\sigma(g_i)$ and such that $\ell_j\,\mathrm{Lw}_\sigma(g_{i_j})r_j \le_\sigma \mathrm{Lw}_\sigma(f)$ for all j.

σ (fixed) word ordering (usually `llex`) on X^*

$g_1, \ldots, g_s \in K\langle X \rangle \setminus \{0\}$ **monic** polynomials (i.e. $\mathrm{Lc}_\sigma(g_i) = 1$)

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Definition 2.1 A quadruple $(\ell, r, \ell', r') \in X^{*4}$ is called an **obstruction** for (g_i, g_j) if $\quad \ell \,\mathrm{Lw}_\sigma(g_i)\, r = \ell' \,\mathrm{Lw}_\sigma(g_j)\, r'$.

Definition 2.2 Given an obstruction (ℓ, r, ℓ', r') for (g_i, g_j), the polynomial

$$S(g_i, g_j) \;=\; \frac{1}{\mathrm{Lc}_\sigma(g_i)}\,\ell\, g_i\, r - \frac{1}{\mathrm{Lc}_\sigma(g_j)}\,\ell'\; g_j\, r'$$

is called the corresponding **S-polynomial**.

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is called the corresponding **S-polynomial**.

Definition 2.3 A polynomial $f \in K\langle X\rangle$ has a (weak) **Gröbner representation** with respect to G if there exist $c_i \in K$ and $\ell_i, r_i \in X$ and $j_i \in \{1, \ldots, s\}$ such that

$$f \;=\; \sum_{i=1}^{m} c_i\,\ell_i\, g_{j_i}\, r_i \quad \text{and} \quad \ell_i\, \mathrm{Lw}_\sigma(g_{j_i})\, r_i \leq_\sigma \mathrm{Lw}_\sigma(f)$$

for $i = 1, \ldots, m$.

Theorem 2.4 **(Buchberger Criterion)**
The set G is a σ-Gröbner basis of I if and only if every S-polynomial of two elements of G has a Gröbner representation with respect to G.

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It can be shown that it is indeed sufficient to consider the following **finite** set of **non-trivial** obstructions:

(a) **right obstructions**: $\mathrm{Lw}_\sigma(g_i) \cdot r = \ell' \cdot \mathrm{Lw}_\sigma(g_j)$

(b) **left obstructions**: $\ell \cdot \mathrm{Lw}_\sigma(g_i) = \mathrm{Lw}_\sigma(g_j) \cdot r'$

(c) **center obstructions**: $\ell \cdot \mathrm{Lw}_\sigma(g_i) \cdot r = \mathrm{Lw}_\sigma(g_j)$

Therefore one can check in finitely many steps whether G is a σ-Gröbner basis.

Theorem 2.5 (Buchberger Procedure)
Let $I = \langle g_1, \ldots, g_s \rangle$ be a two-sided ideal in $K\langle X \rangle$ generated by a set $G = \{g_1, \ldots, g_s\}$ of monic polynomials. Perform the following steps:

(1) *Let B be the set of all normal remainders $\mathrm{NR}_{\sigma,G}(S(g_i, g_j))$ of S-polynomials $S(g_i, g_j)$ corresponding to non-trivial obstructions.*

(2) *If $B = \emptyset$, return G and stop. Otherwise, choose $f \in B$ using a* **fair strategy**, *remove it from B and append f to G.*

(3) *Compute the non-trivial obstructions for the pairs (g_i, f) and append the non-zero normal remainders of the corresponding $S(g_i, f)$ to the set B.*

(4) **Interreduce** *G and update the set B correspondingly.*

This procedure enumerates a σ-Gröbner basis of I. If I has finite σ-Gröbner bases, the procedure stops and outputs one of them.

Optimizing the Buchberger Procedure

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Remark 2.6 **(Trivial Obstructions)**

(a) If (ℓ, r, ℓ', r') is an obstruction of (g_i, g_j) i.e. if $\ell \cdot \mathrm{Lw}_\sigma(g_i) \cdot r = \ell' \cdot \mathrm{Lw}_\sigma(g_j) \cdot r'$, then all multiples

$$(w\,\ell,\ r\,w',\ w\,\ell',\ r'\,w') \qquad \text{with } w, w' \in X^*$$

are also obstructions. If the S-polynomial of (ℓ, r, ℓ', r') has a Gröbner representation, the S-polynomials of all such obstructions have Gröbner representations.

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are also obstructions. If the S-polynomial of (ℓ, r, ℓ', r') has a Gröbner representation, the S-polynomials of all such obstructions have Gröbner representations.

(b) **(Product Criterion)** If $\mathrm{Lw}_\sigma(g_i)$ and $\mathrm{Lw}_\sigma(g_i)$ have **no overlap**, then the S-polynomial of every obstruction of (g_i, g_j) has a Gröbner representation.

Proposition 2.7 (Non-Commutative Criterion M)
Let $(\ell_i, r_i, \ell'_i, r'_i)$ *be an obstruction of* (g_i, g_s) *and* $(\ell_j, r_j, \ell'_j, r'_j)$ *an obstruction of* (g_j, g_s)*. If there exist words* $w, w' \in X^*$ *such that* $\ell'_i = w\,\ell'_j$ *and* $r'_i = r'_j\,w$ *then we can remove* $(\ell_i, r_i, \ell'_i, r'_i)$ *from* B *in the execution of the Buchberger Procedure provided* $ww' \neq 1$*.*

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Proposition 2.8 (Non-Commutative Criterion F)
In the setting of the preceding proposition, assume that $w = w' = 1$*, i.e. that* $\ell'_i = \ell'_j$ *and* $r'_i = r'_j$*. Then the obstruction* $(\ell_i, r_i, \ell'_i, r'_i)$ *can be removed from* B *in the execution of the Buchberger Procedure if* $i > j$ *or if* $i = j$ *and* $\ell_i >_\sigma \ell_j$*.*

Proposition 2.9 **(Non-Commutative Criterion B)**
A non-trivial obstruction $(\ell_i, r_i, \ell_j, r_j)$ *of* (g_i, g_j) *can be removed from the set* B *during the execution of the Buchberger Procedure if the following conditions hold.*

(1) *There exist words* $\ell_s, r_s \in X^*$ *such that* $(\ell_i, r_i, \ell_s, r_s)$ *is an obstruction of* (g_i, g_s) *where* g_s *is the newly constructed Gröbner basis element.*

(2) *Each of the obstructions* $(\ell_i, r_i, \ell_s, r_s)$ *and* $(\ell_j, r_j, \ell_s, r_s)$ *is without overlap or a multiple of a non-trivial obstruction.*

3 – Application to Group Rings

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Remark 3.1 **(a)** Notice that in general we have to include indeterminates representing the **inverses** $y_i = x_i^{-1}$ and relations $x_i y_i - 1$, $y_i x_1 - 1$ here.

(b) If x_i represents a group element of finite order, i.e. if we have a relation $x_i^k - 1 \in I$, we do not need y_i.

The Word Problem

Proposition 3.2 **(Ideal Membership)**
Given a two-sided ideal $I = \langle g_1, \ldots, g_s \rangle$ *in* $K\langle X \rangle$ *and a polynomial* $f \in K\langle X \rangle$*, there is a* **semi-decision procedure** *for determining whether* $f \in I$*.*

(1) *Perform one iteration of the Buchberger Procedure.*

(2) *Check whether the normal remainder of* f *after division by the intermediate* **partial Gröbner basis** G *is zero. If it is, return* `TRUE`*. Otherwise, continue with (1).*

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Given a two-sided ideal $I = \langle g_1, \ldots, g_s \rangle$ in $K\langle X \rangle$ and a polynomial $f \in K\langle X \rangle$, there is a **semi-decision procedure** *for determining whether $f \in I$.*

(1) *Perform one iteration of the Buchberger Procedure.*

(2) *Check whether the normal remainder of f after division by the intermediate* **partial Gröbner basis** *G is zero. If it is, return* `TRUE`*. Otherwise, continue with (1).*

Remark 3.3 For a word $w \in X^*$, we have a semi-decision procedure for checking whether w represents the neutral element of G by checking $w - 1 \in I$.

Remark 3.4 A more careful computation, keeping track of the division steps, also solves the **Explicit Membership Problem** (also called **Word Search Problem**): if w represents the neutral element in G, write it as a product of the relators.

Elimination

Let $\{y_1, \ldots, y_m\} \subset \{x_1, \ldots, x_n\}$, and let Y^* be the monoid of words in the letters $y_1, \ldots, y_m$. For a two-sided ideal I of $K\langle X\rangle$, the set $I \cap K\langle Y\rangle$ is a two-sided ideal in $K\langle Y\rangle$. It is called the **elimination ideal** of I obtained by eliminating the indeterminates in $X \setminus Y$.

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Definition 3.5 A word ordering σ on X^* is called an **elimination ordering** for $X \setminus Y$ if $\mathrm{Lw}_\sigma(f) \in K\langle Y\rangle$ implies $f \in K\langle Y\rangle$. Equivalently, an elimination ordering σ is characterized by the property that $w_1 >_\sigma w_2$ if $w_1 \notin Y^*$ and $w_2 \in Y^*$.

Example 3.6 The **total lexicographic word ordering** `tlex` is defined as follows. For $t_1, t_2 \in X^*$ we let $t_1 <_{\mathtt{tlex}} t_2$ if the associated commutative terms $\tilde{t}_1, \tilde{t}_2$ satisfy $\tilde{t}_1 <_{\mathtt{lex}} \tilde{t}_2$ or if $\tilde{t}_1 = \tilde{t}_2$ and $t_1 <_{\mathtt{lex}} t_2$. The ordering `tlex` is an elimination ordering for $\{x_1, \ldots, x_k\}$ for every $k \in \{1, \ldots, n-1\}$.

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Theorem 3.7 (Main Theorem on Elimination)
Let $I \subset K\langle X \rangle$ be a two-sided ideal, and let G be a Gröbner basis of I with respect to an elimination ordering σ for $X \setminus Y$.

Then $G \cap K\langle Y \rangle$ is a Gröbner basis of $I \cap K\langle Y \rangle$ with respect to the restriction of σ.

In particular, a Gröbner basis of $I \cap K\langle Y \rangle$ can be enumerated.

Kernels of Algebra Homomorphisms

Let $I \subset K\langle X\rangle$ be a two-sided ideal, let $\{y_1, \ldots, y_m\}$ be a set of further indeterminates and let $\varphi:\ K\langle Y\rangle \longrightarrow K\langle X\rangle/I$ be the K-algebra homomorphism given by $\varphi(y_i) = \bar{h}_i$ for $i = 1, \ldots, m$.

Kernels of Algebra Homomorphisms

Let $I \subset K\langle X\rangle$ be a two-sided ideal, let $\{y_1, \ldots, y_m\}$ be a set of further indeterminates and let $\varphi : K\langle Y\rangle \longrightarrow K\langle X\rangle/I$ be the K-algebra homomorphism given by $\varphi(y_i) = \bar{h}_i$ for $i = 1, \ldots, m$.

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Proposition 3.9 *We have* $\ker(\varphi) = \Delta \cap K\langle Y\rangle$. *In particular, we can enumerate a Gröbner basis of the kernel of* φ.

The Order of a Group Element

$K\langle G\rangle = K\langle X\rangle/I$ group ring of a finitely presented group.

Corollary 3.10 *For word $w \in X^*$ representing a group element $\bar{w} \in G$, we have a semi-decision procedure to check whether $\bar{w}$ has finite order.*

Proof: Compute the kernel of the K-algebra homomorphism $K[t] \longrightarrow K\langle X\rangle/I$ given by $t \mapsto \bar{w}$. The element $\bar{w}$ has infinite order iff this kernel is zero.

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Remark 3.11 To prove that $\bar{w}$ has infinite order, we can try to add polynomials to the diagonal ideal and show that the larger ideal $\widetilde{\Delta}$ satisfies $\widetilde{\Delta} \cap K\langle Y\rangle = \{0\}$. In particular, we can add the binomials defining a normal subgroup.

The Tits Alternative

If a finitely presented group $G = \langle x_1, \ldots, x_n; \ell_1 = r_1, \ldots, \ell_s = r_s \rangle$ contains a free subgroup of rank 2, two **randomly chosen** elements of G should generate such a subgroup.

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Remark 3.12 Let $w_1, w_2 \in X^*$ be words representing two elements of G. Define a K-algebra homomorphism

$$\varphi : \; K\langle y_1, y_2, z_1, z_2 \rangle \longrightarrow K\langle G \rangle \quad \text{by } \varphi(y_i) = \bar{w}_i, \; \varphi(z_i) = \bar{w}_i^{-1}$$

and compute its kernel. The elements $\bar{w}_1, \bar{w}_2$ generate a free subgroup of G iff $\ker(\varphi) = \langle y_i z_i - 1, \; z_i y_i - 1 \rangle$

The Generalized Word Problem

Proposition 3.13 **(Subalgebra Membership)** *Let* $\varphi : K\langle Y\rangle \longrightarrow K\langle X\rangle/I$ *be a* K*-algebra homomorphism. Given* $f \in K\langle X\rangle$*, we have the following semi-decision procedure to check whether* $\bar{f} \in \mathrm{im}(\varphi)$*.*

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(a) *Run one iteration of the Buchberger Procedure to compute a Gröbner basis of Δ.*

(b) *Check whether the partial Gröbner basis G reduces f to an element in $K\langle Y\rangle$. If this is the case, return* `TRUE` *and stop. Otherwise, continue with (a).*

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If we have $h = \mathrm{NR}_G(f) \in K\langle Y\rangle$ then $f = h(\varphi(y_1), \ldots, \varphi(y_m))$ is an explicit representation of f as an element of $\mathrm{im}(\varphi)$.

Application to groups and monoids:

Let $G = \langle x_1, \ldots, x_n; \ell_1 = r_1, \ldots, \ell_s = r_s \rangle$ be a finitely presented group and let $w_1, \ldots, w_m \in X^*$ be words whose residue classes generate a subgroup H.

Given a word $f \in X^*$, we have a semi-decision procedure for the **Generalized Word Problem** which asks whether $\bar{f} \in H$ holds.

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The element $\bar{f}$ is contained in H iff $\bar{f}$ is contained in the image of the K-algebra homomorphism

$$\varphi : \; K\langle y_1, \ldots, y_m, z_1, \ldots, z_m \rangle \longrightarrow K\langle G \rangle$$

defined by $\varphi(y_i) = \bar{w}_i$ and $\varphi(z_i) = \bar{w}_i^{-1}$.

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If we have $\bar{f} \in H$ then the second part of the preceding proposition yields $\bar{f} = h(\bar{w}_i, \bar{w}_i^{-1})$. This representation solves the **Generalized Word Search Problem**.

4 – Hilbert-Dehn Functions

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Let $I \subseteq K\langle X\rangle$ be a two-sided ideal and $R = K\langle X\rangle/I$.

Definition 4.1 For $i \geq 0$, let $\mathcal{F}_i$ be the K-vector subspace of $K\langle X\rangle$ generated by the words of length $\leq i$. Then $\mathcal{F} = (\mathcal{F}_i)_{i\in\mathbb{N}}$ is an increasing filtration of $K\langle X\rangle$. It is called the **degree filtration**.

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The vector space $\mathcal{F}_i \,/\, (\mathcal{F}_i \cap I)$ measures the (lowest degree representatives of) elements of degree $\leq i$ contained in R.

Definition 4.2 **(a)** The function $\mathrm{HF}_R^{\mathrm{tot}} : \mathbb{N} \longrightarrow \mathbb{N}$ given by

$$\mathrm{HF}_R^{\mathrm{tot}}(i) \;=\; \dim_K(\mathcal{F}_i \,/\, (\mathcal{F}_i \cap I))$$

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(c) If G is a finitely presented group, the function $\mathrm{HD}_G(i) = \mathrm{HF}_{K\langle G\rangle}(i)$ is called the **Hilbert-Dehn function** of G.

The value $\mathrm{HD}_G(i)$ measures the number of **normal words** of degree i, i.e. of words which cannot be reduced with respect to a degree compatible word ordering.

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The value $\mathrm{HD}_G(i)$ measures the number of **normal words** of degree i, i.e. of words which cannot be reduced with respect to a degree compatible word ordering.

In general, the Hilbert function of R in degree i cannot be calculated, but only an upper bound.

σ **degree-compatible** word ordering (e.g. $\sigma = \texttt{llex}$)

Proposition 4.3 **(Macaulay's Basis Theorem)**
The residue classes of the elements of the **set of normal words** $\mathcal{O}_\sigma(I) = X \setminus \mathrm{Lw}_\sigma(I)$ *form a K-vector space basis of R.*

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Corollary 4.4 *We have* $\mathrm{HF}_R(i) = \mathrm{HF}_{K\langle X\rangle/\,\mathrm{Lw}_\sigma(I)}(i)$ *for all* $i \geq 0$.

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Definition 4.5 The power series $\mathrm{HS}_R(t) = \sum_{i\geq 0} \mathrm{HF}_R(i)\, t^i$ is called the **Hilbert series** of R.

Goals: Determine whether R is a finite-dimensional K-vector space; if not, find out whether HF_R has polynomial or exponential growth; compute HS_R.

Checking Finite-Dimensionality of R

Let σ be a degree compatible term ordering, and let G be a σ-Gröbner basis of I.

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Definition 4.6 Let S be a finite set of terms and $\ell = \max\{\mathrm{len}(w) \mid w \in S\}$. The **Ufnarovski graph** Γ_S has

(1) a vertex for each normal word $w \in S$ which has length $\ell - 1$, and

(2) a directed edge (v, w) iff there are x_i, x_j such that $vx_i = x_jw$ and this is a normal word.

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Theorem 4.7 **(Ufnarovski's Finiteness Criterion)** *Assume that G is finite. Then we have* $\dim_K(R) < \infty$ *if and only if* $\Gamma_{\mathrm{Lw}_\sigma(G)}$ *contains no cycle.*

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Moreover, in the case of polynomial growth, the maximal number of disjoint cycles visited by a path in the Ufnarovski graph is the degree of the polynomial growth.

Thus, if we can compute a σ-Gröbner basis of I, we can determine the growth rate of R.

Example 4.9 Let $I = \langle x^2 - y^2 \rangle$ and $\sigma = \texttt{llex}$. Then the σ-Gröbner basis of I is $G = \{x^2 - y^2,\ xy^2 - y^2x\}$. This yields

$$\mathrm{Lw}_\sigma(I) = \langle x^2, xy^2 \rangle \quad \text{and} \quad \mathcal{O}_\sigma(I) = \{1, x, y, xy, yx, y^2, xyx, \ldots\}$$

The Ufnarovski graph of $\mathrm{Lw}_\sigma(I)$ is $\quad y^2 \circlearrowleft \;\rightarrow\; xy \;\leftrightarrows\; yx$. Since there are two non-intersectiong cycles, the algebra $R = K\langle x, y\rangle / \langle x^2 - y^2 \rangle$ has polynomial growth of degree 2.

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Example 4.10 For the monomial algebra $R = K\langle x, y \rangle / \langle x^3,\ xy^2 \rangle$, the Ufnarovski graph is $x \leftrightarrows \underset{\circlearrowleft}{y} \leftarrow x^2$. Since there are two intersecting cycles, the algebra has exponential growth.

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Definition 4.11 The **Gelfand-Kirillov dimension** of R is $\mathrm{GKdim}(R) = \overline{\lim_{i\to\infty}} \frac{\ln(\mathrm{HF}_R^{\mathrm{tot}}(i))}{\ln(i)}$.

The Gelfand-Kirillov dimension is finite iff HF_R has polynomial growth. It is a number in $\{0, 1\} \cup [2, \infty]$.

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(e) This Hilbert series agrees with $\mathrm{HS}_R(t)$.

ApCoCoA

All computations mentioned n this talk can be executed using the packages implemented by X. Xiu in **ApCoCoA 1.9**, see

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Thank you for your attention!

In the end, everything is a gag.

(Charlie Chaplin)