An L_3-U_3-quotient algorithm for finitely presented groups

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The goal

Let $G = \langle a, b \mid r_1, \ldots, r_k \rangle$ be a finitely presented group.

Compute all quotients of G that are isomorphic to one of the groups $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$, $\mathrm{PGL}(3, q)$, or $\mathrm{PGU}(3, q)$, *simultaneously for every prime power q*.

Studying representations

...using character theory

- We want to find epimorphisms $\delta\colon G \to \mathrm{PSL}(3,q)$.
- As a first step: Study representations $\Delta\colon F_2 \to \mathrm{SL}(3,q)$.
- Main tool: The character $\chi_\Delta\colon F_2 \to \mathbb{F}_q\colon w \mapsto \mathrm{tr}(\Delta(w))$.

Theorem

Let $\Delta_1, \Delta_2\colon \Gamma \to \mathrm{GL}(n,K)$ be absolutely irreducible, where Γ is an arbitrary group and K is arbitrary field.

If $\chi_{\Delta_1} = \chi_{\Delta_2}$, then Δ_1 and Δ_2 are equivalent.

From now on:
"character" = "character of a representation $F_2 \to \mathrm{SL}(3,q)$"

Studying characters

...using commutative algebra

Theorem

For every $w \in F_2$ there exists

$$\tau_w \in \mathbb{Z}[x_1, x_{-1}, x_2, x_{-2}, x_{1,2}, x_{-1,2}, x_{-2,1}, x_{-2,-1}, x_{[1,2]}]$$

such that

$$\chi(w) = \tau_w(\chi(a), \chi(a^{-1}), \chi(b), \ldots, \chi([a,b])).$$

for every character $\chi\colon F_2 \to \mathbb{F}_q$.

We call τ_w the trace polynomial of w and
$t_\chi := (\chi(a), \ldots, \chi([a,b])) \in \mathbb{F}_q^9$ the trace tuple of χ.

Corollary

Every character is uniquely determined by its trace tuple.

Studying characters
... using commutative algebra

Theorem

There exists $r \in \mathbb{Z}[x_1, \ldots, x_{[1,2]}]$ such that $t \in \mathbb{F}_q^9$ is the trace tuple of a character χ if and only if $r(t) = 0$.

Corollary

There is a bijection between the maximal ideals of $R := \mathbb{Z}[x_1, \ldots, x_{[1,2]}]/\langle r \rangle$ and the ($\mathrm{Gal}(\mathbb{F}_q)$-classes of) characters $\chi\colon F_2 \to \mathbb{F}_q$, where q ranges over all prime powers.

For $M \in \mathrm{MaxSpec}(R)$ let χ_M be the corresponding character, and $\Delta_M\colon F_2 \to \mathrm{SL}(3, q)$ a representation with character χ_M.

Let $M \in \mathrm{MaxSpec}(R)$ and $\Delta_M \colon F_2 \to \mathrm{SL}(3, q)$ a corresponding representation.

Theorem

Let G be a finitely presented group. There exists an ideal $I_G \trianglelefteq R$ such that Δ_M factors over G if and only if $I_G \subseteq M$.

Surjectivity of representations

...in ring theoretic terms

Let $M \in \operatorname{MaxSpec}(R)$ and $\Delta_M \colon F_2 \to \mathrm{SL}(3, q)$ a corresponding representation.

Theorem

There exists an ideal $\omega \trianglelefteq R$ such that Δ_M fixes a symmetric form if and only if $\omega \subseteq M$.

Theorem

There exists an ideal $\rho \trianglelefteq R$ such that Δ_M is (absolutely) reducible if and only if $\rho \subseteq M$.

$\vdots$

Examples: Finitely many L_3-U_3-quotients

- $G = \langle a, b | a^2, b^3, (ab^2ab)^4, (ab)^{41} \rangle$ has quotients $L_3(83)$ (twice), $L_3(2543)$ and $U_3(3^4)$.

- $G = \langle a, b | a^2, b^4, (ab)^{11}, [a, bab]^7 \rangle$ has quotients $U_3(769)$, $U_3(9437)$ and $U_3(133078695023)$.

Examples: Infinitely many L_3-U_3-quotients

Classification using algebraic number theory

$G = \langle a, b \mid a^2, b^3, u^4vuvuvuv^4u^2v^2 \rangle$ with $u = ab$ and $v = ab^{-1}$, has infinitely many L_3-quotients, precisely one in every characteristic $\neq 2, 13$.

The isomorphism type of the quotient is

	$p^3 \equiv \pm 1 \mod 13$	$p^3 \not\equiv \pm 1 \mod 13$
$p \equiv 1 \mod 3$	$L_3(p)$ or $PGL(3, p)$	$U_3(p)$
$p \not\equiv 1 \mod 3$	$L_3(p)$	$U_3(p)$ or $PGU(3, p)$

Examples: Infinitely many L_3-U_3-quotients
Classification using combinatorics

- $G = \langle a, b \mid a^2, b^3, [a,b]^5, [a, babab]^3 \rangle$ has infinitely many L_3-quotients, but all are defined in characteristic 2.

  Example: For $\ell > 3$ prime there are

  $(2^{2\ell-1} - 2)/(3\ell)$ quotients isomorphic to PSL$(3, 2^{2\ell})$,

  $(2^{2\ell-1} - 2)/\ell$ quotient isomorphic to PSU$(3, 2^{2\ell})$, and

  $(2^{2\ell} - 2)/(3\ell)$ quotients isomorphic to PGL$(3, 2^{2\ell})$.
- $G = \langle a, b \mid a^3, b^5, aba^{-1}b^2aba^{-1}bab^2a^{-1}b \rangle$ has infinitely many L_3-quotients; finitely many in every characteristic, and infinitely many in characteristic 5.