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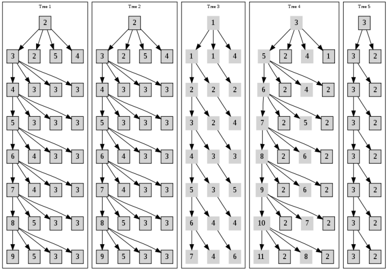

An algorithm for the classification of nil-potent semigroups by coclass

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What is a semigroup?

Definition (groupmonoidsemigroup)

A set *GMS* with a binary operation $\circ$ satisfying

- A1 $(x \circ y) \circ z = x \circ (y \circ z)$
- A2 $x \circ e = e \circ x = x$
- A3 $x \circ x^{-1} = x^{-1} \circ x = e$

Example: $(\mathbb{N}, +)$, $(\mathbb{Z}_n, *)$, matrices over a ring

Applications:

- computer science (automata, formal languages)
- partial differential equations (operators)

Numbers of semigroups

n	# non-equivalent semigroups with n elements	
1	1	
2	4	
3	18	
4	126	[Forsythe '54]
5	1 160	[Motzkin, Selfridge '55]
6	15 973	[Plemmons '66]
7	836 021	[Jürgensen, Wick '76]
8	1 843 120 128	[Satoh, Yama, Tokizawa '94]
9	52 989 400 714 478	[Distler, Kelsey 2009]
10	12 418 001 077 381 302 684	[Distler, Jefferson, Kelsey, Kotthoff 2012]

Outline

- **Nilpotent Semigroups and Semigroup Algebras**
- **Coclass Graph**
- **Infinite Paths in the Coclass Graph**

Nilpotent semigroups

Definition

A semigroup S is *nilpotent* if there is a $c \in \mathbb{N}_0$ such that $|S^{c+1}| = 1$.
The least such c is the *class* of S.
If S is finite then $|S| - 1 - c$ is the *coclass* of S.

Coclass theory for groups

- introduced in 1980 by Leedham-Green und Newmann
- crucial invariant in the classification of nilpotent groups

Can the ideas be transferred to semigroups?

Contracted semigroup algebras

Definition

K - field, S - semigroup with zero, z - zero in S;
$K[S] = \{\sum_{s \in S} a_s s \mid a_s \in K\}$ with addition

$$\sum_{s \in S} a_s s + \sum_{s \in S} b_s s = \sum_{s \in S} (a_s + b_s) s$$

and multiplication

$$\sum_{s \in S} a_s s \cdot \sum_{s \in S} b_s s = \sum_{s,t \in S} a_s b_t st$$

is the *semigroup algebra* of S over K.

$KS = K[S]/\langle z \rangle$ is the *contracted semigroup algebra* of S over K.

Class and coclass of nilpotent algebras

Definition

An algebra A is *nilpotent of class c* if

$$A > A^2 > \cdots > A^c > A^{c+1} = \{0\}.$$

If A is finite-dimensional then $\dim(A) - c$ is the *coclass* of A.

$KS = K[S]/\langle z \rangle$ is a nilpotent algebra if and only if S is nilpotent.

$$\dim(KS) = |S| - 1$$
$$\mathrm{cl}(KS) = \mathrm{cl}(S)$$
$$\mathrm{cc}(KS) = \mathrm{cc}(S)$$

Coclass graph

Given a field K we visualise the isomorphism types of nilpotent semigroups of a coclass r using a graph $\mathcal{G}_{r,K}$.

Vertices: the vertices of $\mathcal{G}_{r,K}$ correspond to the isomorphism types of algebras KS where S is a nilpotent semigroup of coclass r.

Edges: two vertices A and B are adjoined by a directed edge $A \to B$ if $B/B^c \cong A$ where c is the class of B.
(Then A has class $c - 1$, $\dim(A) = \dim(B) - 1$, and $\dim(B^c) = 1$).

Labels: the vertex corresponding to A is labelled by the number of non-isomorphic semigroups S of coclass r with $KS \cong A$.

For a vertex A of $\mathcal{G}_{r,K}$ we denote by $\mathcal{T}(A)$ the subgraph consisting of A and all its descendants.

For a graph $\mathcal{G}$ denote by $\overline{\mathcal{G}}$ the graph without labels.

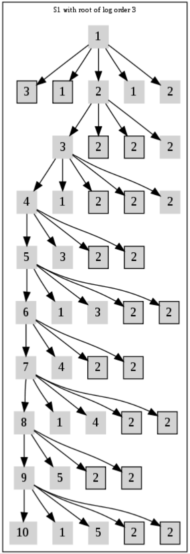
Coclass graph, coclass 1, $K = GF(3)$

class 1, dimension 2 $\rightarrow$

class 2, dimension 3 $\rightarrow$

⋮

class 6, dimension 7 $\rightarrow$



1 semigroup of order 3

9 semigroups of order 4

⋮

14 semigroups of order 8

Conjectures

<u>Clear:</u> Every vertex in $\mathcal{G}_{r,K}$ has at most one parent thus $\mathcal{G}_{r,K}$ is a forest.

We call an infinite path *maximal* if the root of the path has no parent.

Conjecture

For every $r \in \mathbb{N}_0$ and every field K the graph $\mathcal{G}_{r,K}$ has only finitely many maximal infinite paths.

We say that $\mathcal{T}(A)$ is a *coclass tree* if it contains a unique infinite path with root A. It is a *maximal coclass tree* if there is no parent B of A so that $\mathcal{T}(B)$ is a coclass tree.

The conjecture is equivalent to saying that $\mathcal{G}_{r,K}$ consists of finitely many maximal coclass trees and finitely many other vertices.

Conjectures, cont.

Conjecture

Let $\mathcal{T}$ be a maximal coclass tree in $\mathcal{G}_{r,K}$ with maximal infinite path

$$A_1 \to A_2 \to \dots$$

Then there exist positive integers l (defect) and k (period), a graph isomorphism $\mu : \overline{\mathcal{T}}(A_l) \to \overline{\mathcal{T}}(A_{l+k})$, and for each $B \in \mathcal{T}(A_l) \setminus \mathcal{T}(A_{l+k})$ a rational polynomial f_B, so that $f_B(i)$ is the label of $\mu^i(B)$ for all $i \in \mathbb{N}_0$.

For the unlabelled graph $\mathcal{T}(A_l) \setminus \mathcal{T}(A_{l+k})$ is the building block of the periodic part. To obtain the labels for the first block evaluate the polynomials at 0, for the second block at 1, for the third block at 2, ...

If the conjecture holds and if the map μ and the polynomials f_B are given then $\mathcal{T}$ can be constructed from a finite subtree.

Algorithm

Given a coclass r construct a coclass graph in the following way:

1. Choose a field K and classify the maximal infinite paths in $\mathcal{G}_{r,K}$.
2. For each maximal infinite path consider its corresponding maximal coclass tree $\mathcal{T}$ and find:
   - an upper bound l for the defect of $\mathcal{T}$;
   - a multiple k of the period of $\mathcal{T}$;
   - an upper bound d for the degree of the polynomials $f_B(x)$.
3. For each maximal coclass tree $\mathcal{T}$:
   - determine the unlabelled tree $\overline{\mathcal{T}}$ up to depth $l + (d + 1)k$;
   - for each vertex B in the determined part of $\overline{\mathcal{T}}$ compute its label.
4. Determine all parts of $\mathcal{G}_{r,K}$ outside the maximal coclass trees.

Coclass for infinite objects

Let O be a finitely generated infinite semigroup resp. infinite dimensional algebra. Then every quotient O/O^{c+1} is finitely generated, nilpotent of class at most c and hence is finite resp. finite dimensional. Thus O/O^{c+1} has finite coclass.

We say that O is *residually nilpotent* if $\cap_{i\in\mathbb{N}} O^i = 0$ holds. If O is finitely generated and residually nilpotent, then we define its *coclass* $cc(O)$ by

$$cc(O) = lim_{i\to\infty} cc(O/O^i).$$

The coclass of O is finite if and only if there exists $i \in \mathbb{N}$ such that $|O^j \setminus O^{j+1}| = 1$ resp. $dim(O^j/O^{j+1}) = 1$ for all $j \geqslant i$.

Inverse limit

Theorem

For every maximal infinite path in $\mathcal{G}_{r,K}$ there exists a finitely generated infinite dimensional associative K-algebra A of coclass r which describes the path.

Consider a maximal infinite path $A_1 \to A_2 \to \ldots$ in $\mathcal{G}_{r,K}$. For every $j \geqslant k$ let $\nu_{j,k} : A_j \to A_k$ denote the natural homomorphism defined by the path, and let $\widehat{A} = \prod_{i\in\mathbb{N}} A_i$. Define

$$A = \{(a_1, a_2, \ldots) \in \widehat{A} \mid \nu_{j,k}(a_j) = a_k \text{ for every } j \geqslant k\}.$$

Then A is an infinite dimensional associative algebra satisfying $A/A^{c+j} \cong A_j$ for every $j \in \mathbb{N}$, where c is the class of A_1.

Construction of infinite semigroup algebras

Inside the polynomial algebra over K consider the ideal I_K of polynomials with zero constant term. Then $I_K \cong K[\mathbb{N}] \cong K\mathbb{N}_0$.

The algebra I_K is an infinite dimensional contracted semigroup algebra of coclass 0.

S, T – infinite semigroups with $S \cong T/\langle t \rangle$ for some $t \in Ann(T)$. Then the subspace generated by t in KT is a 1-dimensional ideal I satisfying $I \leqslant Ann(KT)$.

If KS is an infinite dimensional contracted semigroup algebra of coclass $r - 1$, then KT is an infinite dimensional contracted semigroup algebra of coclass r.

Computational results for coclass 2

Conjecture

For every field K the graph $\mathcal{G}_{2,K}$ has 6 maximal infinite paths. These are described by the following infinite dimensional algebras:

- $\langle a, b \mid b^2 = ba = a^2 b = 0 \rangle$; *Annihilator* $\langle ab \rangle$.
- $\langle a, b \mid b^2 = ab = ba^2 = 0 \rangle$; *Annihilator* $\langle ba \rangle$.
- $\langle a, b \mid b^3 = ab = ba = 0 \rangle$; *Annihilator* $\langle b^2 \rangle$.
- $\langle a, b \mid b^2 = aba = 0, ab = ba \rangle$; *Annihilator* $\langle ba \rangle$.
- $\langle a, b \mid b^2 = ba, ab = b^2 a = 0 \rangle$; *Annihilator* $\langle ba \rangle$.
- $\langle a, b, c \mid b^2 = c^2 = ab = ba = ac = ca = bc = cb = 0 \rangle \cong I_K \oplus (I_K / I_K^2) \oplus (I_K / I_K^2)$; *Annihilator* $\langle b, c \rangle$.

Thank you for your attention!

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