Geodesic growth in right-angled Artin and even Coxeter groups

Laura Ciobanu (joint with Yago Antolín)

University of Neuchâtel, Switzerland

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Question (Loeffler, Meier, Worthington, *IJAC* 2002)

Can two non-isomorphic right-angled Artin groups have the same geodesic growth series?

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Answer: Yes.

(Antolin – C., *EJC* 2013)

The goal of this work is to understand geodesic growth from a qualitative perspective for

- right-angled Artin groups (RAAGs)
- right-angled Coxeter groups (RACGs), and for
- even Coxeter groups.

Coxeter systems

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A Coxeter system is

- a finite simplicial graph $\Gamma = (V, E)$, together with
- a labelling map $m\colon E \to \{1, 2, 3, \dots\}$.

Coxeter group $G_{(\Gamma,m)}$ associated to a Coxeter System (Γ, m),

$$G_{(\Gamma,m)} = \langle\, V \mid v^2 = 1 \ v \in V,\ (uv)^{m(\{u,v\})} = 1 \text{ for } \{u, v\} \in E \,\rangle$$

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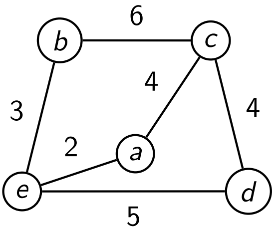

$$\left\langle a, b, c, d, e \;\middle|\; \begin{array}{c} a^2 = b^2 = c^2 = d^2 = e^2 = 1 \\ (bc)^6 = (cd)^4 = (de)^5 = 1 \\ (eb)^3 = (ea)^2 = (ac)^4 = 1 \end{array} \right\rangle$$

RACGs and RAAGs

- The system is even if m only takes even values.
- The system is right-angled if m only takes the value 2.

- The right-angled Coxeter group (RACG) G determined by Γ is

$$\langle s \in S \mid s^2 = 1 \ \forall s \in S, \text{ and } (ss')^2 = 1 \ \forall \{s, s'\} \in E \rangle.$$

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- G - group generated by S

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The geodesic growth function $\gamma : \mathbb{N} \to \mathbb{N}$ is given by

$$\gamma(r) = \text{number of geodesics in G of length } r.$$

Growth series

- Spherical growth series

$$\mathcal{S}_{(G,X)}(z) = \sum_{n=0}^{\infty} \sigma(n) z^n$$

- Geodesic growth series

$$\mathcal{G}_{(G,X)}(z) = \sum_{n=0}^{\infty} \gamma(n) z^n$$

Spherical growth for right-angled Coxeter groups

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Let a_n be the number of *n*-cliques of Γ. The *f*-polynomial of Γ is

$$f_\Gamma(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots a_m x^m$$

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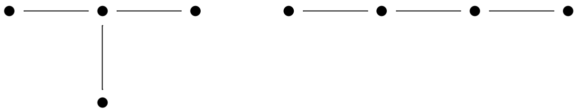
Theorem (Steinberg (1968))

$$\frac{1}{\mathcal{S}(G_\Gamma)(z)} = f_\Gamma\left(\frac{-z}{1+z}\right)$$

The spherical growth of RACGs and RAAGs

- Only depends on the f-polynomial of the simplicial graph.

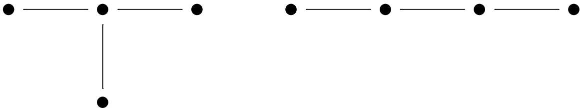
Ex: Two trees with the same number of vertices have the same spherical growth.



$$\boxed{f(x) = 1 + 4x + 3x^2}$$

The geodesic growth of RACGs and RAAGs

- There exist graphs with same f-polynomial but different geodesic growth.



Remarks

- All the groups in this talk have regular languages of geodesics with respect to the standard generating sets $\Longrightarrow$
- All geodesic growth series are rational.
- All Coxeter groups have regular languages of geodesics (Brink-Howlett 1993).
- All Garside groups and lots of Artin groups have regular languages of geodesics (spherical, large etc.)
- Changing the generating sets will modify all statements above.

Main question

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Theorem (A – C)

Let Γ be a link-regular graph. Then the geodesic growth of the right-angled Coxeter (or Artin) group based on Γ is a function of the sizes of the links and the f-polynomial.

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Let Γ be a r-regular, triangle-free graph. Then for RACGs

$$\mathcal{G}(\Gamma) = \frac{1-(r-3)t+2t^2}{1+(-|V|-r+3)t+(-2|V|+2+r|V|)t^2}.$$

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Theorem (A – C)

Let (Γ, m) be an even Coxeter system with Γ triangle-free and star-regular. Then $\mathcal{G}(\Gamma)$ is a function of the star of a vertex and $|V|$.

Corollary. *Let G and G' be two right-angled Artin or Coxeter groups that are link-regular and have the same f-polynomial. Then G and G' have the same geodesic growth.*

The smallest example

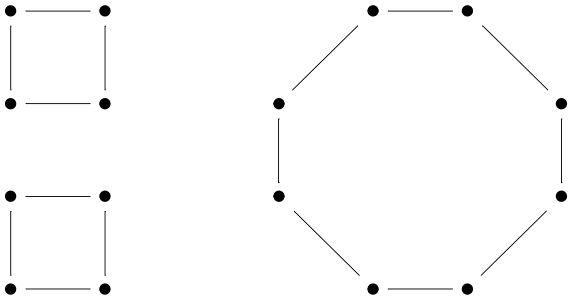

Figure: Two RACGs or RAAGs with the same geodesic growth

The result for RACGs and RAAGs

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Proof:

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For a RAAG: use a result of Droms and Sevatius that connects Cayley graphs of RACSs and RAAGs.

Automaton recognizing geodesics in a RACG

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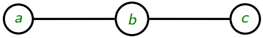
Example ()

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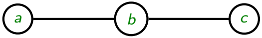
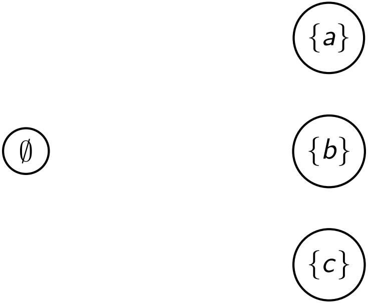
Example ()



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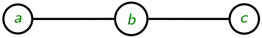
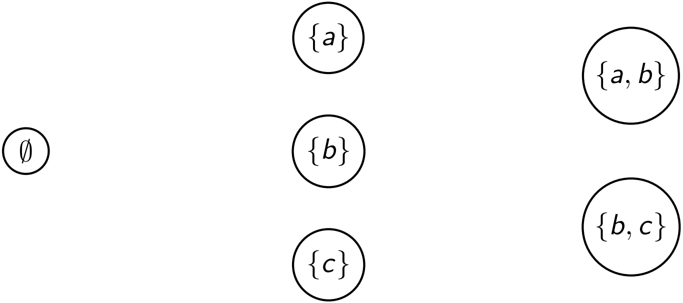
Example ()



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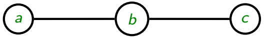
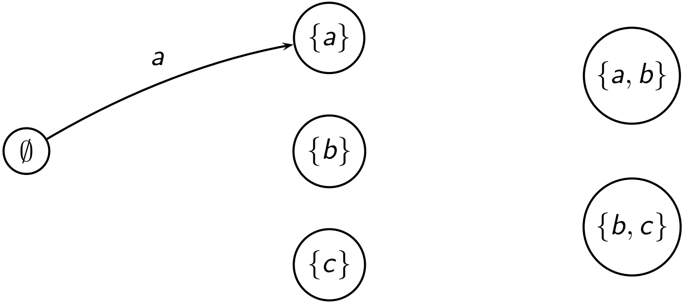
Example ()



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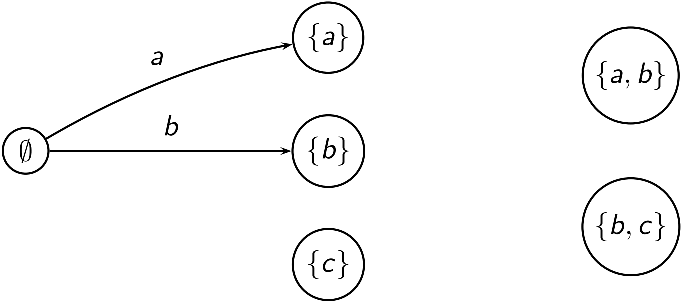

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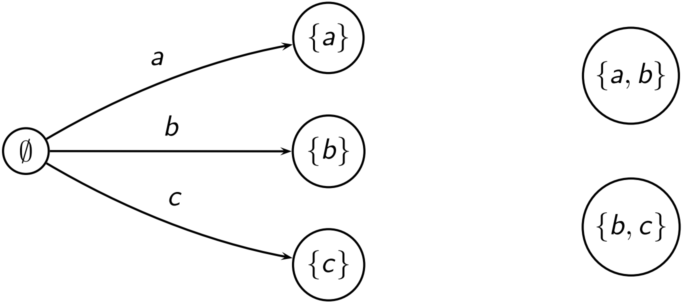

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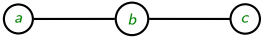

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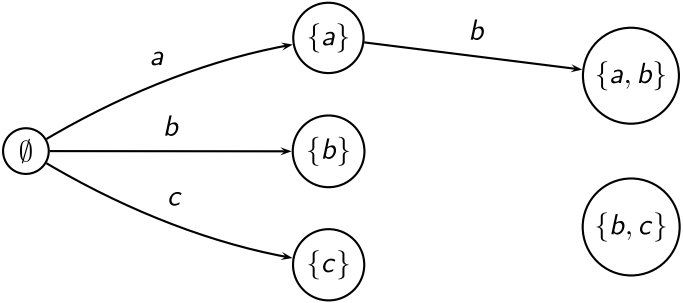

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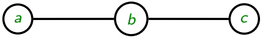
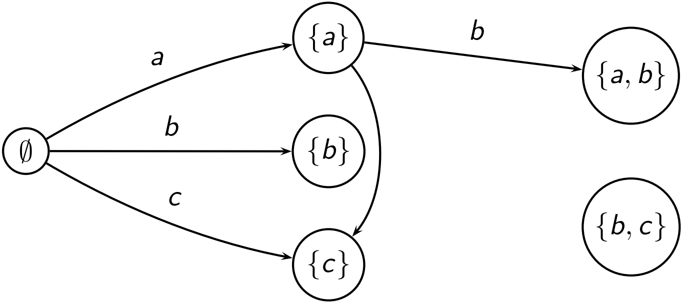
Example ()



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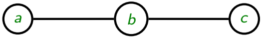
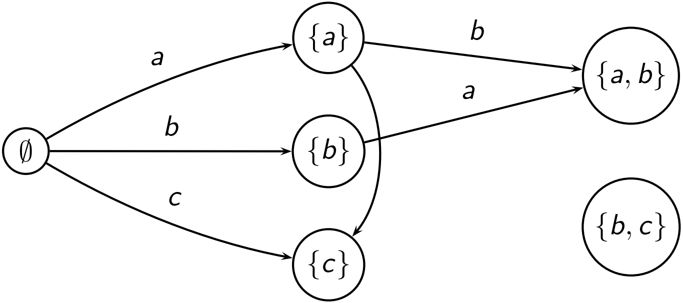
Example ()



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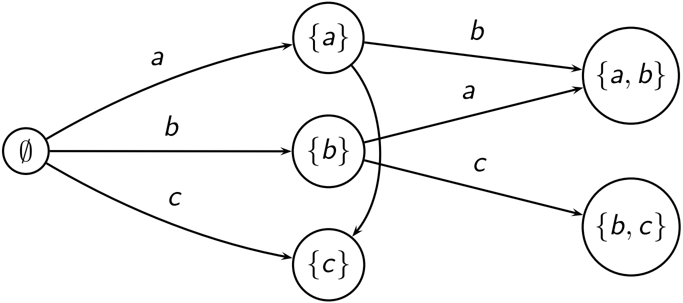

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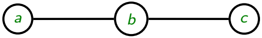
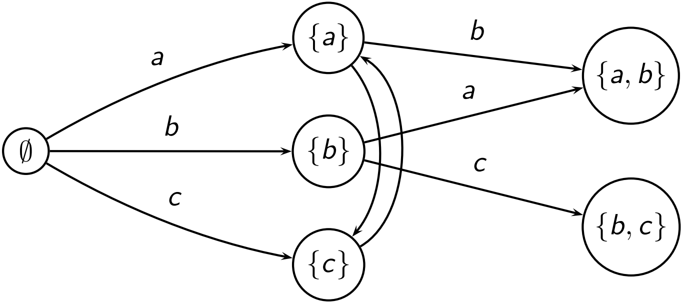
Example ()



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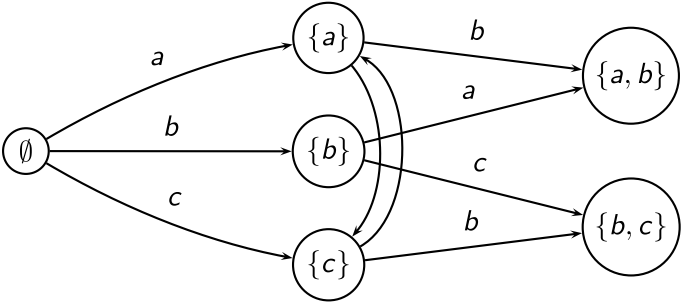

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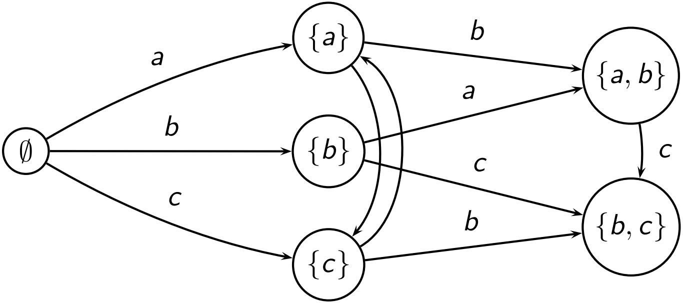

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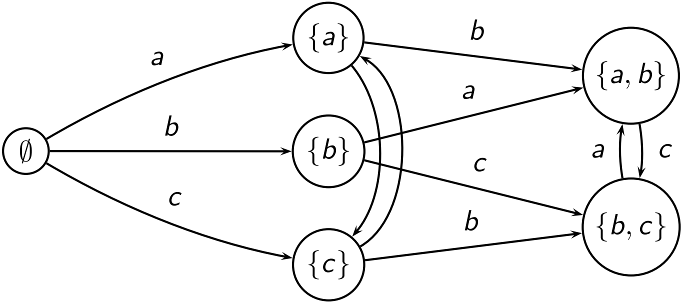

Even Coxeter groups

Theorem 2. [A – C]

Let (W, S) be an even Coxeter system with graph Γ, where Γ is triangle-free and star-regular. The geodesic growth of W depends only on $|S|$ and the isomorphism class of the star of the vertices.

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In particular, if (W_1, S_1) and (W_2, S_2) are triangle-free, star-regular, even Coxeter systems with $|S_1| = |S_2|$ and $St(v) \cong St(u)$, $\forall v \in V\Gamma_1$, $u \in V\Gamma_2$, then W_1 and W_2 have the same geodesic growth.

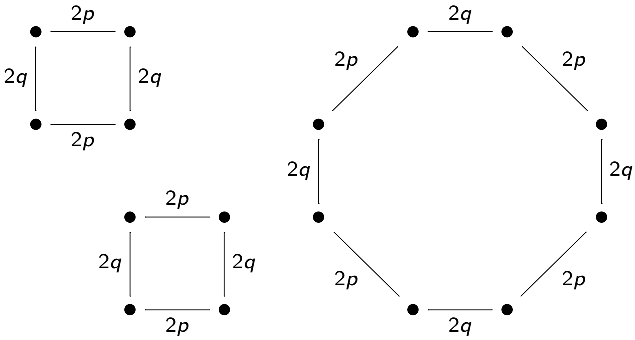

Figure: Two even Coxeter groups with the same geodesic growth

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