Passivity Preserving Model Reduction via a Structured Lanczos Method

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***Abstract*— Nowadays, modeling dynamical systems often yields state-space models of very high order (that is, 10,000 or more equations). In order to guarantee a numerical simulation in reasonable time, the dynamical system is reduced to one of the same form which allows simulation of and control-design for the reduced-order state-space model in much less computing time. Usually one would like to obtain a reduced-order system that has the same properties as the original system. In this paper, we will consider stable and passive systems. Antoulas suggests in [1] an approach based on positive real interpolation which is modified by Sorensen [36]. The algorithm is based upon interpolation at selected spectral zeros of the original transfer function to produce a reduced-order transfer function that has the specified roots as its spectral zeros. These interpolation conditions are satisfied through the computation of a basis for a selected invariant subspace of a Hamiltonian matrix which has the spectral zeros as its spectrum. Here we propose to employ a structure-preserving Lanczos algorithm for this part of the computation in order to make use of the underlying structure and physical properties of the problem.**

I. Introduction

This paper is concerned with linear time invariant (LTI) systems

$$\Sigma: \quad \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times p}$. In this setting, u is the input or excitation function, x is the state, and the function $f(x,u) = Ax(t) + Bu(t)$ determines the dynamics of the system Σ. y is the output or set of observations and $h(x,u) = Cx(t) + Du(t)$ describes the way that the observations are deduces from the state and the input. The complexity of Σ is defined as the number of states n. The problem we will address is to approximate Σ with another dynamical system

$$\widehat{\Sigma}: \quad \dot{\hat{x}}(t) = \widehat{A}\hat{x}(t) + \widehat{B}u(t), \quad \hat{y}(t) = \widehat{C}\hat{x}(t) + \widehat{D}u(t), \tag{2}$$

where $\widehat{A} \in \mathbb{R}^{\ell\times\ell}, \widehat{B} \in \mathbb{R}^{\ell\times p}, \widehat{C} \in \mathbb{R}^{p\times\ell}, \widehat{D} \in \mathbb{R}^{p\times p}$. That is, in the new system the number of states (the number of first order differential equations to be solved) is much less than in the original system: $\ell \ll n$. System properties of the original system such as stability, passivity, controllability or observability should be preserved by a model reduction procedure. Often, the existence of a global error bound is required and a small approximation error in terms of $||y-\widehat{y}||$ for an appropriate norm is desired.

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Most approaches for model reduction of LTI systems are based on the idea of projecting the original system Σ onto a system of lower order. Usually, two real $n \times \ell$ matrices V and W with $W^TV = I$ are computed which define the projector $\Pi = VW^T$. The projection of the states of the original system generates a reduced-order model:

$$\widehat{A} = W^TAV, \quad \widehat{B} = W^TB, \quad \widehat{C} = CV, \quad \widehat{D} = D. \tag{3}$$

For a recent survey of methods for computing reduced-order models see [2], [9], [16].

Here we will consider only stable and passive systems Σ. A system is stable, if the matrix A is stable, that is, if all eigenvalues of A lie in the open left half plane. A system is passive if it does not generate any energy internally, and strictly passive, if it consumes or dissipates input energy. A classical result [37] says that a passive system is positive real. For the LTI systems considered here this implies that the corresponding transfer function $G(s) = D + C(sI - A)^{-1}B$ is positive real, that is, $G(s)$ is analytic and $G(s) + (G(s))^H \geq 0$ for $\mathrm{Re}(s) > 0$ (see, e.g., [5] and the references therein). When reducing a stable and passive system, it is important to produce a reduced-order model that preserves the important system properties and response characteristics such as stability and passivity while at the same time using a computationally efficient method.

Passive systems arise in a variety of control problems for mechanical, mechatronic and micro-electro-mechanical systems, see [27]. Circuit simulation is another important source of model reduction problems. In some of these applications, e.g., in the simulation of RLC circuits, the system Σ is stable and passive. A number of methods previously proposed for solving the problem considered here is based on the moment matching property of Krylov subspace methods. For a recent survey of such methods see [16].

For simplicity, let us assume for a moment that $p = 1$; i.e., the system (1) is a single-input single-output system and the transfer functions of the systems (1) and (2) are scalar rational functions. In particular, the transfer function can be expanded around any s_0 for which $(s_0I - A)^{-1}$ exists,

$$G(s) = \sum_{k=0}^{\infty} m_k(s - s_0)^k. \tag{4}$$

Here, the m_k are called moments of the transfer function. A Padé approximation of G is a function F, whose expansion around s_0 matches as many moments of the expansion (4) of G as possible

$$F(s) = \sum_{k=0}^{r} m_k(s - s_0)^k + \text{higher order terms in } (s - s_0).$$

As the transfer functions of (1) and (2) are rational functions, the transfer function of the reduced system (2) can be written as

$$\widehat{G}(s) = \widehat{D} + \widehat{C}(sI - \widehat{A})^{-1}\widehat{B} = \frac{\phi_{\ell-1}(s)}{\psi_\ell(s)},$$

where $\phi_{\ell-1}$ and ψ_ℓ are polynomials of degree at most $\ell-1$, and ℓ, resp.. The two polynomials are uniquely defined by its 2ℓ coefficients. Padé methods try to determine these coefficients such that as many leading moments of both transfer functions are identical as possible. Numerically stable methods for the computation of such approximations rely on the Lanczos algorithm [21] and achieve the matching of the first 2ℓ leading moments of both transfer functions. The most prominent method for achieving this in the general case is the Padé-via-Lanczos method (PVL) [14]. Closely related is the Multipoint-Rational-Interpolation method (MRI) [23], both these methods do not automatically preserve stability and passivity.

Large linear RLC models allow a symmetric formulation of the system with a symmetric transfer function. Based on the symmetric Lanczos algorithm, the method SyPVL is proposed in [18] for such special systems. This method guarantees to preserve stability and passivity for special models, like RC-, RL- or LC-networks. The rational transfer function is approximated by a rational function such that the first 2ℓ moments are matched. At the same time, the dominant poles of the transfer function are captured. A variant for multi-input multi-output systems, called SyMPVL [19], was also proposed. If the nonsymmetric versions of these methods are used to reduce stable and passive models, there is no guarantee that the reduced systems will be stable or passive. An iterative process as described in [4] is necessary in order to ensure stability and passivity of the reduced system.

PRIMA [33] approximates the rational transfer function of the original system (1) by a rational transfer function such that only half as many moments of both transfer functions match compared to the previously described methods. The unused degrees of freedom in the approximation are used to generate a passive reduced-order model for a passive RLC network. This method is based on the Arnoldi algorithm [21]. In (3), $V = W$ is chosen and the projection matrix V is computed as a special basis of a certain block Krylov-subspace. The reduced-order model depends on the network formulation. The idea of not matching the maximal number of possible moments of the transfer functions is also used in [3] in order to obtain a passivity preserving model reduction method for single-input single-output systems. SPRIM [17] is a modification of PRIMA which makes use of the special structure of the matrices induced through the RLC structure.

The mentioned schemes are computationally efficient, but suffer from the fact that they can be used only for special cases and from the lack of global error bounds.

Another class of model reduction methods, which we will not describe in detail as they are quite different from the approach considered below, stems from control theory. Examples of these are Hankel-Norm-Approximation (HNA) [20], Balanced-Truncation-Approximation (BTA) [31] and Singular-Perturbation-Approximation (SPA) [30]. A merit of these control-theoretic approaches is the availability of approximation errors and the superior global accuracy. These schemes are, however, expensive to employ due to the need of solving large-size matrix equations and factorizations. A recent review of such methods can be found in [24]. Passivity preserving balancing methods are described in, e.g, [22], [32], [11], [34]. Other approaches for passivity preserving model reduction are the LR-ADI method [29] and the Laguerre-SVD ansatz [25], [26].

Based on recent work by Antoulas [1] characterizing passivity through interpolation conditions, Sorensen derives in [36] a novel projection method that preserves both stability and passivity. Given 2ℓ distinct points $s_1, \ldots, s_{2\ell}$, let $T_j = s_j I - A$ and

$$\widetilde{V} = [T_1^{-1}B,\ \ldots,\ T_k^{-1}B], \quad (5)$$

$$\widetilde{W} = [T_{k+1}^{-T}C^T,\ \ldots,\ T_{2k}^{-T}C^T]. \quad (6)$$

Assuming that $\det \widetilde{W}^T\widetilde{V} \neq 0$, the projected system defined by (3) where $V = \widetilde{V}$ and $W = \widetilde{W}(\widetilde{V}^T\widetilde{W})^{-1}$, interpolates the transfer function of Σ at the points s_i (see [1, Proposition 4.1]):

$$\widehat{G}(s_i) = G(s_i), \quad i = 1, 2, \ldots, 2\ell.$$

Antoulas proves in [1], that if the interpolation points s_j in (5), (6) are chosen as spectral zeros of the original passive system Σ, the reduced system $\widehat{\Sigma}$ defined by (3) is both stable and passive. The matrices $\widetilde{V}$, $\widetilde{W}$ can be obtained without the explicit computation of spectral zeros. As Sorensen [36] noted, this can be achieved through the computation of certain invariant subspaces of a generalized eigenvalue problem.

In Section II we will briefly review the results by Antoulas [1] and Sorensen [36] and state the passivity preserving model reduction method proposed by Sorensen. In Section III we show how the approach can be rewritten such that a structured eigenvalue problem results and describe an efficient, structure-preserving Krylov subspace method for its solution. Section IV reports a numerical example comparing Sorensen's method with the structure-preserving one proposed here.

II. Sorensen's passivity preserving model reduction method

Let us consider the LTI system (1) where it is assumed that

1) A is stable,
2) Σ is observable and controllable,
3) $D_+ = D + D^T$ is symmetric positive definite,
4) Σ is passive.

Recall, that a real rational function $G(s)$ is positive real if $G(s)$ is analytic and $G(s) + (G(-s))^T \geq 0$ for $Re(s) > 0$. The last property implies the existence of a stable rational matrix function $W(s)$ such that $G(s) + G^T(-s) = W(s)W^T(-s)$. This is the spectral factorization of G, W

is called a spectral factor of G and the zeros of W, i.e., $\lambda_i, i = 1, \ldots, n$, such that $\det W(\lambda_i) = 0$, are called the spectral zeros of G; these definitions and relations can be found in many papers and texts, see e.g. [39], [5]. Denote the set of all spectral zeros by $\mathcal{S}_G$, i.e.,

$$\mathcal{S}_G := \{\lambda | \det\ W(\lambda) = 0\} = \ \{\text{spectral zeros of } G\}.$$

Suppose a reduced-order model $\widehat{\Sigma}$ (2) has been obtained and let $\widehat{G}(s) = \widehat{D} + \widehat{C}(sI - \widehat{A})^{-1}\widehat{B}$ be the associated transfer function. Antoulas notes in [1] that a passive reduced-order model $\widehat{\Sigma}$ will result if certain of the spectral zeros are preserved in the reduced-order model.

Proposition: *If $\mathcal{S}_{\widehat{G}} \subset \mathcal{S}_G$, $\widehat{G}(\lambda) = G(\lambda)$ for all $\lambda \in \mathcal{S}_{\widehat{G}}$, and $\widehat{G}$ is a minimal degree rational interpolant of the values of G on the set $\mathcal{S}_{\widehat{G}}$, then $\widehat{\Sigma}$ is both stable and passive.*

The set $\mathcal{S}_G$ of all spectral zeros is equal to the set of (finite) eigenvalues of

$$\mathcal{A} - \lambda\mathcal{E} = \begin{bmatrix} A & & B \\ & -A^T & -C^T \\ C & B^T & D_+ \end{bmatrix} - \lambda \begin{bmatrix} I & & \\ & I & \\ & & 0 \end{bmatrix}, \quad (7)$$

that is $\mathcal{S}_G = \sigma(\mathcal{A}, \mathcal{E}) \backslash \{\infty\}$ where

$$\sigma(\mathcal{A}, \mathcal{E}) = \{\lambda \in \mathbb{C} | \det(\mathcal{A} - \lambda\mathcal{E}) = 0\}.$$

Making use of this observation, Sorensen suggests in [36] to compute the reduced-order model through the construction of a basis for a selected invariant subspace of the pair $(\mathcal{A}, \mathcal{E})$. His observation is based on the following result where we recall that a quasi-triangular matrix is a triangular matrix which can have additional nonzero entries on the first lower subdiagonal, i.e., $t_{i+1,i} \neq 0$ for some non-neighbouring i.

Proposition 2: *(Generalized Schur decomposition [21]) If M and N are in $\mathbb{C}^{n\times n}$, then there exist unitary matrices U and Q in $\mathbb{C}^{n\times n}$ such that $U^H M Q = T$ and $U^H N Q = S$ are upper triangular. If, for some k, $t_{kk} = s_{kk} = 0$, then $\sigma(M, N) = \mathbb{C}$. Otherwise, $\sigma(M, N) = \{t_{ii}/s_{ii}, s_{ii} \neq 0\} \cup \{\infty \text{ for } s_{ii} = 0\}$. U and Q can be chosen so that the eigenvalues appear in any order along the diagonal of S and T. If M and N are real, then there exist orthogonal matrices U and Q in $\mathbb{R}^{n\times n}$ such that $U^T M Q$ is upper quasi-triangular and $U^T N Q$ is upper triangular. Similar as before, the eigenvalues of (M, N) can be read off.*

Standard software for computing this computation (e.g., MATLAB's `eig`) exists.

Suppose we have computed a partial real generalized Schur decomposition of $(\mathcal{A}, \mathcal{E})$, that is, we have computed matrices U and Q in $\mathbb{R}^{(2n+p)\times\ell}$ with orthonormal columns such that $U^T\mathcal{A}Q = S$ is an upper quasi-triangular matrix in $\mathbb{R}^{\ell\times\ell}$ and $U^T\mathcal{E}Q = T$ is an upper triangular matrix in $\mathbb{R}^{\ell\times\ell}$ such that all eigenvalues of (S, T) have positive real part. Then T is nonsingular and we have from $\mathcal{A}Q = US, \mathcal{E}Q = UT$ that $\mathcal{E}QT^{-1} = U$ and

$$\mathcal{A}Q = \mathcal{E}QT^{-1}S =: \mathcal{E}QR$$

where Q has orthonormal columns ($Q^TQ = I$) and R is real quasi-upper triangular in $\mathbb{R}^{\ell\times\ell}$ such that $\mathrm{Re}(\lambda) > 0$ for all eigenvalues λ of R.

Let $Q^T = [X^T, Y^T, Z^T]$ be partitioned in accordance with the block structure of $\mathcal{A}$; $X \in \mathbb{R}^{n\times\ell}, Y \in \mathbb{R}^{n\times\ell}, Z \in \mathbb{R}^{p\times\ell}$. Then

$$\begin{bmatrix} A & & B \\ & -A^T & -C^T \\ C & B^T & D_+ \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} R. \quad (8)$$

The desired projection matrices V and W can be computed from X and Y in the following way: compute the singular value decomposition [21] of

$$X^TY = Q_x S^2 Q_y^T,$$

where Q_x and Q_y are orthogonal matrices in $\mathbb{R}^{\ell\times\ell}$ and $S^2 \in \mathbb{R}^{\ell\times\ell}$ is a diagonal matrix with nonnegative diagonal entries. Set

$$V = XQ_xS^{-1}, \quad W = YQ_yS^{-1},$$

and

$$\widehat{A} = W^TAV, \quad \widehat{B} = W^TB, \quad \widehat{C} = CV, \quad \widehat{D} = D.$$

The resulting reduced-order system is stable and passive [36].

For small to medium size dense problems $\mathcal{A}$ and $\mathcal{E}$ can be actually formed and the desired generalized Schur decomposition can be obtained from the full one. For large sparse systems this would be impractical and inefficient. An iterative method computing a desired set of eigenvalues and associated eigenvectors (or an associated invariant subspace) is more appropriate. As the best currently available method for this purpose, the implicitly restarted Arnoldi (IRA) algorithm [35] as implemented in MATLAB's `eigs` or ARPACK in Fortran [28] can not deal with the problem under consideration here (as $\mathcal{E}$ is not positive definite), Sorensen suggests to apply a Cayley transformation $\mathcal{C}_\mu = (\mu\mathcal{E} - \mathcal{A})^{-1}(\mu\mathcal{E} + \mathcal{A})$ where $\mu \geq 0$ is a real shift. With a proper choice of μ this will provide for rapid convergence to an invariant subspace corresponding to the ℓ transformed eigenvalues of largest magnitude of

$$(\mu\mathcal{E} - \mathcal{A})^{-1}(\mu\mathcal{E} + \mathcal{A})Q = Q\widehat{R}, \quad (9)$$

so that

$$\mathcal{A}Q = \mathcal{E}QR \quad \text{where} \quad R := \mu(\widehat{R} - I)(\widehat{R} + I)^{-1}. \quad (10)$$

An implementation will require two sparse direct factorizations of $A - \mu I$ and $A + \mu I$. The Cayley transformation may then be applied to an arbitrary vector using a blocked matrix-vector product followed by a Gaussian block elimination. Moreover, it should be noted that as the partial Schur decomposition will automatically keep complex conjugate pairs of eigenvalues together, the parameter ℓ that specifies the order of the reduced system will perhaps need to be adjusted by 1 to accommodate this.

III. STRUCTURED KRYLOV SUBSPACE APPROACH

It is easily seen that $\lambda \in \mathcal{S}_G$ implies $-\overline{\lambda} \in \mathcal{S}_G$ since from $\mathcal{A}u = \lambda\mathcal{E}u$ we have $\widetilde{u}^T\mathcal{A} = -\overline{\lambda}\widetilde{u}^T\mathcal{E}$, where $u^T = [x^T, y^T, z^T]$ and $\widetilde{u}^T = [y^T, -x^T, z^T]$. As Hamiltonian matrices display such an eigenvalue pairing, in this section

we will show how to transform the generalized eigenvalue problem $\mathcal{A} - \lambda\mathcal{E}$ associated with the generalized Schur decomposition (8) into a standard eigenvalue problem for a Hamiltonian matrix. Further it is discussed how the special structure of Hamiltonian matrices can be used here to solve the model reduction problem more efficiently.

From (8) we obtain [2]

$$\begin{aligned} AX + BZ &= XR, && (11)\\ -A^TY - C^TZ &= YR, && (12)\\ CX + B^TY + D_+Z &= 0. && (13) \end{aligned}$$

From (13) it follows that $Z = -D_+^{-1}\{CX + B^TY\}$, as $D_+ = D + D^T$ is symmetric positive definite (assumption 3 in Section II). Substituting this expression for Z into (11), (12) yields

$$\underbrace{\begin{bmatrix} A - BD_+^{-1}C & -BD_+^{-1}B^T \\ C^TD_+^{-1}C & -(A - CD_+^{-1}B)^T \end{bmatrix}}_{\mathcal{H}} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R.$$

$\mathcal{H}$ can be represented as

$$\mathcal{H} = \begin{bmatrix} \widetilde{A} & \widetilde{G} \\ \widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \tag{14}$$

with $\widetilde{G} = \widetilde{G}^T$ and $\widetilde{Q} = \widetilde{Q}^T$. Obviously, all eigenvalues of $\mathcal{H}$ are eigenvalues of the pair $(\mathcal{A}, \mathcal{E})$. Hence, instead of considering the partial generalized Schur decomposition

$$\mathcal{A}Q = \mathcal{E}QR$$

we can consider the partial Schur decomposition of $\mathcal{H}$

$$\mathcal{H}S = S\Lambda, \tag{15}$$

where $S \in \mathbb{R}^{2n\times\ell}$ and $\Lambda \in \mathbb{R}^{\ell\times\ell}$.

Each matrix with a block structure like $\mathcal{H}$ in (14) with $\widetilde{G} = \widetilde{G}^T$ and $\widetilde{Q} = \widetilde{Q}^T$ is called a Hamiltonian matrix. Hamiltonian matrices have a number of interesting properties which allow for efficient, numerically robust methods for the computation of their eigenvalues. If λ is a real or a purely imaginary eigenvalue of a real Hamiltonian matrix, then $-\lambda$ is also an eigenvalue. Complex eigenvalues even appear in quadruples $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$. (This property of the eigenvalues of $\mathcal{H}$ holds also for the finite eigenvalues of $(\mathcal{A}, \mathcal{E})$ and hence for the spectral zeros $\mathcal{S}_G$.)

Standard methods for solving eigenvalue problems do not preserve the Hamiltonian form of the matrix in the course of the computation (usually already the first computational steps in such an algorithm will destroy the Hamiltonian structure). Due to unavoidable rounding errors these methods will in general not compute exact pairs or quadruples of eigenvalues, in particular eigenvalues on the imaginary axis might move into the open right or left half plane. Moreover, the convergence of some of the standard methods might be slowed down as they often converge to the eigenvalue of largest modulus first, but there might be four of them. These problems can be dealt with if a special structure-preserving method is employed which preserves the Hamiltonian structure in every single computational step. For Hamiltonian eigenvalue problems of small to medium size, special structure-preserving algorithms such as the Hamiltonian SR algorithm [12], [13], or the methods described in [10], [8] can be used. For higher dimensional problems the symplectic Lanczos method for Hamiltonian matrices [6], a structure-preserving Krylov subspace method, is suitable.

For simplicity, let us assume that $p = 1$; the system (1) is a single-input single-output system. The standard Lanczos method for nonsymmetric matrices generates in a suitable way matrices $V = [v_1, v_2, \ldots, v_\ell]$ and $W = [w_1, w_2, \ldots, w_\ell]$ which form the basis of the Krylov subspaces

$$\begin{aligned} \mathcal{K}_\ell(\mathcal{H}, x) &= \text{span}\{x, \mathcal{H}x, \mathcal{H}^2x, \ldots, \mathcal{H}^{\ell-1}x\}\\ &= \text{span}\{v_1, v_2, \ldots, v_\ell\},\\ \mathcal{K}_\ell(\mathcal{H}^T, y\} &= \text{span}\{y, \mathcal{H}^Ty, (\mathcal{H}^T)^2y, \ldots, (\mathcal{H}^T)^{\ell-1}y\}\\ &= \text{span}\{w_1, w_2, \ldots, w_\ell\}, \end{aligned}$$

with $W^TV = I$. Short recurrences for computing V and W columnwise can be given. In particular, it holds that

$$\begin{aligned} \mathcal{H}V &= VT_\ell + r_\ell e_\ell^T,\\ \mathcal{H}^TW &= WT_\ell^T + s_\ell e_\ell^T, \end{aligned}$$

where r_ℓ and s_ℓ are certain vectors of length ℓ, e_ℓ denotes the ℓth unit vector and T_ℓ is a nonsymmetric tridiagonal matrix

$$T_\ell = \begin{bmatrix} \alpha_1 & \gamma_1 & & \\ \beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \gamma_{\ell-1} \\ & & \beta_{\ell-1} & \alpha_\ell \end{bmatrix}$$

for certain scalars $\alpha_j, \beta_j, \gamma_j$. T_ℓ is certainly not a Hamiltonian matrix. The standard Lanczos method is not structure-preserving.

Imposing the additional requirement that the matrix V has to be symplectic, that is, $V^TJV = J$ for

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

and making use of the Hamiltonian structure of $\mathcal{H}$, one can show that there is no need to compute W explicitly (this property is valid for a broader class of matrices, see [15]). Thus work and storage can essentially be halved here. In particular, a symplectic $2n\times 2\ell$ matrix V is computed whose columns are a symplectic basis for $\mathcal{K}_{2\ell}(\mathcal{H}, Ve_1)$. A short recurrence for computing V columnwise can be given:

$$\mathcal{H}V = V\widetilde{\mathcal{H}} + r_{2\ell}e_{2\ell}^T, \tag{16}$$

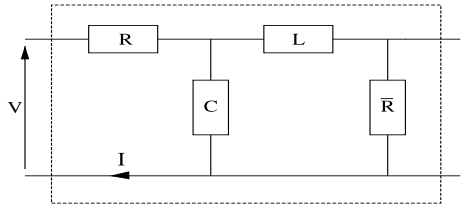

Fig. 1. One section of the circuit as in [24]

where $\widetilde{\mathcal{H}}$ is a Hamiltonian matrix of J-Hessenberg form

$$\widetilde{\mathcal{H}} = \left[\begin{array}{ccc|ccc} \delta_1 & & & \beta_1 & \zeta_2 & \\ & \delta_2 & & \zeta_2 & \beta_2 & \ddots \\ & & \ddots & & \ddots & \ddots \\ & & \delta_\ell & & & \zeta_\ell \\ & & & & & \beta_\ell \\ \hline \nu_1 & & & -\delta_1 & & \\ & \nu_2 & & & -\delta_2 & \\ & & \ddots & & & \ddots \\ & & \nu_\ell & & & -\delta_\ell \end{array}\right]$$

for certain real scalars $\delta_j, \beta_j, \nu_j, \zeta_j$. A detailed description of the symplectic Lanczos method for Hamiltonian matrices can be found in [6]. The partial Schur decomposition (15) can be obtained from (16) by computing the Schur decomposition of $\widetilde{\mathcal{H}}$ using a structure-preserving algorithm like the SR algorithm [13]. For details of the entire process see [7]. Having computed (15), the model reduction can be carried out as explained.

Using this structured approach, we not only save about half of the computational effort compared to the standard approach; this approach offers a fast, efficient and more reliable computation of the reduced-order model. Unfortunately, as for all model reduction methods based on Krylov subspace methods, there does not exist a global error bound.

IV. Numerical Experiment

For the preliminary numerical experiment presented in this section, we consider the RLC ladder network of [24]. This circuit consists of 200 sections interconnected in cascade; each section is as shown in Fig. 1. The input is the voltage V applied to the first section; the output is the current I of the first section. The order of the overall system is $n = 400$. It is assumed that the states are scaled so that all capacitances, inductances and resistances have the value 0.1, except $\overline{R} = 1$. The Hankel singular values of this system decay rapidly so that one can expect a good approximation of the original system by a reduced-order one.

We compare the two approaches described before. For the first approach the matrix pencil $\mathcal{A} - \lambda\mathcal{E}$ (7) is set up. A shift μ is chosen (here $\mu = 5$), the Cayley transformation (9) is computed (actually, not the matrix itself, but an operator that applies this matrix to a vector is set up) and MATLAB's `eigs` is used to compute an $\ell \times \ell$ matrix R and the associated matrix Q as in (10). From this the reduced-order system is computed as described in Section III. This approach will be referred to in the sequel as the Cayley approach.

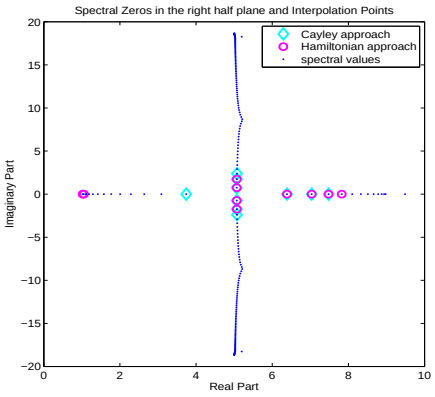

Fig. 2. Choice of spectral zeros

The second approach works with the Hamiltonian matrix $\mathcal{H}$ (14). A desired set of eigenvalues and appropriate invariant subspaces (15) are computed via the structure-preserving symplectic Lanczos algorithm and the reduced system is computed as described in Section III. In order to achieve fast convergence, the use of a shift is recommended, but simply shifting $\mathcal{H}$ by a shift τ yields a non-Hamiltonian matrix $\mathcal{H} - \tau\mathcal{I}$. For $\tau \in \mathbb{R}$ or $\tau \in i\mathbb{R}$, the matrix $\mathcal{H}(\mathcal{H} - \tau I)^{-1}(\mathcal{H} + \tau I)^{-1}$ is Hamiltonian (this shifting strategy reflects the Hamiltonian spectrum). Moreover, if only one shift is used, only a certain part of the spectrum is approximated. But, as the overall approach is based on interpolation it seems reasonable to use interpolation points from different parts of the spectrum in order to achieve better accuracy. In the numerical test based on the Hamiltonian matrix, referred to as the Hamiltonian approach in the sequel, two shifts ($\tau_1 = 1, \tau_2 = 6$) have been used.

The invariant subspaces computed determine the choice of the spectral zeros to be interpolated. Fig. 2 displays the spectral zeros of the transfer function of the original system in the right half plane (please note that a mirror set of spectral zeros can be found in the left half plane) and the spectral zeros chosen by each of the two approaches.

Fig. 3 compares the original system with both reduced ones, Fig. 4 displays the error system. Clearly, the Hamiltonian approach results in a much better approximation.

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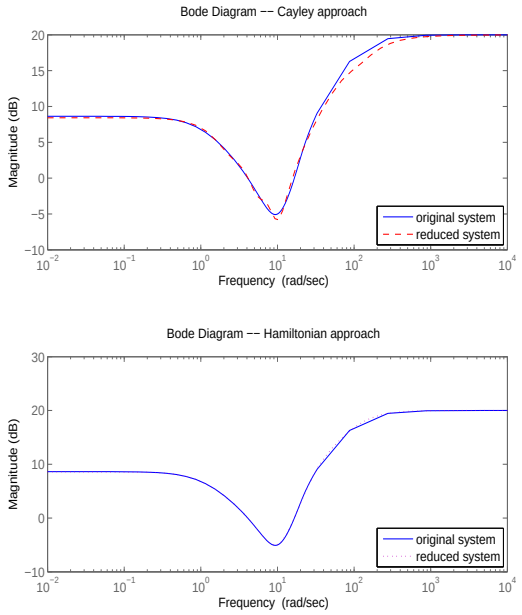

Fig. 3. Bode plot comparing the original system and the reduced-order systems

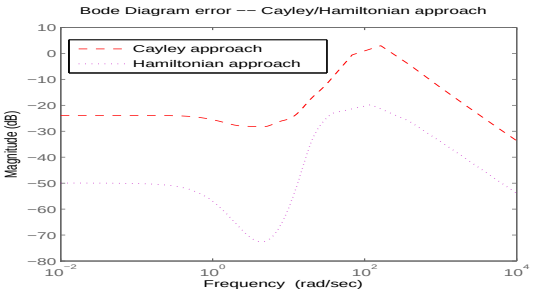

Fig. 4. Bode plot for the error systems

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