

Novel Model Reduction Techniques for Control of Machine Tools

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Summary

Computational methods for reducing the complexity of Finite Element (FE) models in structural dynamics are usually based on modal analysis. Classical approaches such as modal truncation, static condensation (Craig-Bampton, Guyan), and component mode synthesis (CMS) are available in many CAE tools such as ANSYS. In other disciplines, different techniques for Model Order Reduction (MOR) have been developed in the previous 2 decades. Krylov subspace methods are one possible choice and often lead to much smaller models than modal truncation methods given the same prescribed tolerance threshold. They have become available to ANSYS users through the tool *mor4ansys*. A disadvantage is that neither modal truncation nor CMS nor Krylov subspace methods preserve properties important to control design. System-theoretic methods like balanced truncation approximation (BTA), on the other hand, are directed towards reduced-order models for use in closed-loop control. So far, these methods are considered to be too expensive for large-scale structural models. We show that recent algorithmic advantages lead to MOR methods that are applicable to FE models in structural dynamics and that can easily be integrated into CAE software. We will demonstrate the efficiency of the proposed MOR method based on BTA using a control system including as plant the FE model of a machine tool.

Keywords

model order reduction, machine tools, balanced truncation, Krylov subspace method, modal truncation, integrated simulation, control design

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1 Introduction

Nowadays machine tools are complex mechatronic production systems. An efficient method to estimate the attainable capacity and precision of machine tools is the coupled simulation of finite element (FE) models of the mechanical structure and the control system of the electric feed drives. These models are in general sparse but very large in order to achieve high fidelity resolution. This accounts for unacceptable computational and workspace demands in simulation and especially control of these models. In order to design practical controllers meeting the ever growing time constraints of modern industrial applications, it is necessary to transform the high dimensional system of equations from the FE modeling of the mechanical structure into a compact, low dimensional behavioral model. Classically this has been achieved using approaches such as modal truncation, static condensation (Craig-Bampton [5], Guyan [11]), and component mode synthesis (CMS). Modern CAE tools such as ANSYS provide these methods. They are therefore widely used and have proven to yield reliable reduced order models (ROMs).

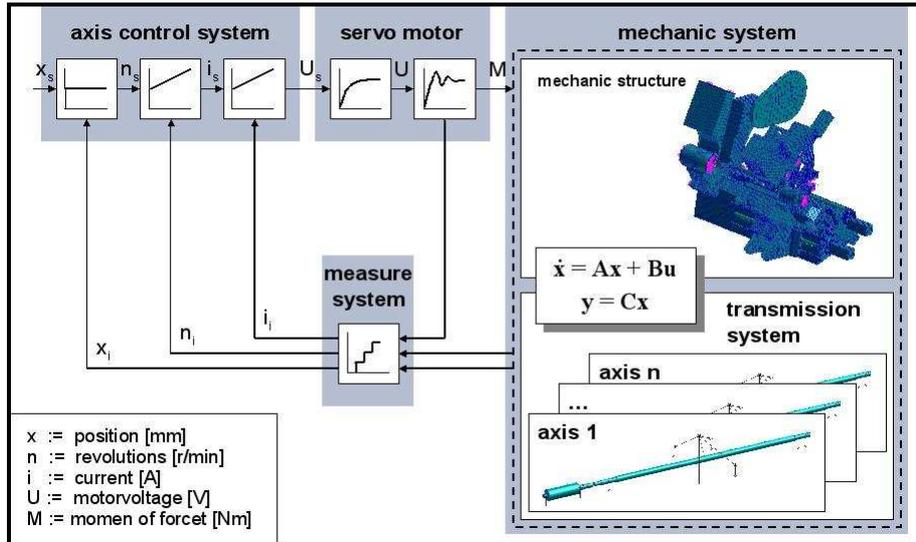


Figure 1: Control loop and embedded structure of a sample system.

Unfortunately, the ROMs generated by the classic approaches are often relatively large and need manual modification by experience of the user, e.g., defining dominant modes. That means they are at least partially heuristic and cannot be generated fully automatic. Here we report on the results achieved within a project that seeks for the extension and application of modern fully automatic model order reduction (MOR) techniques such as Krylov subspace methods and balanced truncation approximation (BTA) to the integrated simulation of machine tools.

1.1 The Simulation Model

The integrated simulation of a machine tool consists of two major parts: the structural model of the machine tool representing its behavior and reaction to certain control inputs, on the one hand, and the control loop generating those inputs, on the other hand. The prior consists of a set of FE semi-discretized partial differential equations resulting in a large scale second order differential-algebraic or ordinary differential equation (DAE or ODE) system of the form:

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = \hat{B}u(t), \quad y(t) = C_v\dot{x}(t) + C_p x(t). \quad (1)$$

Here $M, D, K \in \mathbb{R}^{n \times n}$ are the sparse system matrices reflecting mass, damping and stiffness of the structural model. $\hat{B} \in \mathbb{R}^{n \times p}$ is the input matrix, $C_v, C_p \in \mathbb{R}^{m \times n}$ are the velocity and position output matrices, $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^m$ the measured output and $u(t) \in \mathbb{R}^p$ represents the input signals sent to the system. If M is invertible, (1) is a system of ODEs while a singular mass matrix results in a DAE system (also called descriptor system). In the special case discussed here the damping is considered to follow Rayleigh damping modeling, i.e., the damping matrix is a proportional to mass and stiffness and given as

$$D = \alpha M + \beta K, \quad (2)$$

for certain $\alpha, \beta \in \mathbb{R}$.

1.2 First Order vs. Second Order Form

Model order reduction seeks for a much smaller surrogate model describing the essential system dynamics and preserving the important properties of the original system model. In principle, one has two basic options to generate the ROM. On the one hand it can be desirable to preserve the second order structure of the original system equations in the ROM, i.e., to generate a second order ROM. This approach also has the advantage that when we reduce the system without damping in the first place, we can easily preserve the damping parameters α and β in the ROM and do not have to redo

the reduction every time these parameters are changed. On the other hand it is often easier to include the ROM in existing simulation software if it is available in standard state space or generalized state space representation, i.e., the describing differential equation is of first order. Then the system has to be transformed to first order form either before the ROM is computed, or afterwards. In any case the existing Krylov subspace methods, as well as BTA based reduction have been developed for first order systems. Therefore the remainder of the current section is dedicated to the reformulation of the original second order system in first order form. We will describe one method to achieve this by phase space transformation. Other transformations to first order form can be found, e. g., in [12]. The basic idea for the transformation is to introduce the new state variable $z(t) = [x^T(t), \dot{x}^T(t)]^T$, such that the system takes the form:

$$\underbrace{\begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix}}_{=:E} \dot{z}(t) = \underbrace{\begin{bmatrix} 0 & I_n \\ -K & -D \end{bmatrix}}_{=:A} z(t) + \underbrace{\begin{bmatrix} 0 \\ \hat{B} \end{bmatrix}}_{=:B} u(t), \quad y(t) = \underbrace{\begin{bmatrix} C_p & C_v \end{bmatrix}}_{=:C} z(t), \quad (3)$$

where $E, A \in \mathbb{R}^{2n \times 2n}$, $\hat{B} \in \mathbb{R}^{2n \times p}$ and $I \in \mathbb{R}^{n \times n}$ the identity matrix. This approach obviously doubles the system dimension, but it makes the MOR approaches directly applicable to the transformed system. In [1] we report on a method that can efficiently exploit the sparsity and structure of the original FEM matrices M, D and K . However, the method described there can only be applied when E is invertible. Unfortunately, in the modeling of machine tools often certain rigid body modes cause E to be singular. This issue is addressed in [3]. Due to space limitations, we will not describe the details here, but we note that for the models considered here, a transformation to an ODE system exists which can be obtained by permutations only. The Krylov subspace based approach described in [6], however, remains applicable.

1.3 Structure of the Paper

The remainder of this paper is structured as follows: Section 2 introduces balanced truncation MOR and presents the extension to systems in descriptor representation. In Section 3 we present the basic steps needed for Krylov subspace based reduction of the structural models. Numerical results in both frequency and time domain are given in Section 4. The paper ends with some concluding remarks and the obligatory references.

2 Balanced Truncation

We will now introduce the basic ideas of BTA as given, e. g., in [1]. To this end, in the following section we will consider a system in first order state space representation

$$\Sigma: \begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad (4)$$

where additionally $E = I_n$. Also we assume that all eigenvalues of A (or the matrix pencil (A, E)) lie in the open left complex plane, i.e., A (or the pencil (A, E)) is stable.

Basics

The basic idea of BTA can be summarized as follows. Taking the impulse function δ as the input u of the system, the impulse response turns out to be $h(t) = Ce^{At}B$ for all $t \geq 0$. This can be decomposed into the input-to-state map $x(t) = e^{At}B$ and the state-to-output map $\eta(t) = Ce^{At}$. That means the input $\delta(t)$ generates the state $x(t)$, while the initial state $x(0)$ generates the output $y(t) = \eta(t)x(0)$. The corresponding system Gramians are given as

$$\begin{aligned}
P &= \sum_t x(t)x(t)^* = \int_0^\infty e^{At}BB^*e^{A^*t}dt, \\
Q &= \sum_t \eta(t)^*\eta(t) = \int_0^\infty e^{A^*t}C^*Ce^{At}dt.
\end{aligned} \tag{5}$$

The square roots of the eigenvalues of the product PQ are called the Hankel singular values of the system. They are invariants of the system, since state space transformations $\hat{x}(t) = T x(t)$ give the congruent Gramians

$$\hat{P} = TPT^T, \quad \hat{Q} = T^{-T}QT^{-1},$$

and thus the eigenvalues of $\hat{P}\hat{Q} = TPT^T T^{-T}QT^{-1}$ are the same as those of PQ . It turns out that the Gramians P and Q are the unique self-adjoint positive semi-definite solutions of the two Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0. \tag{6}$$

Further one finds that in order to compute the reduced system not the Gramians themselves, but factors U and V are required, such that $P = UU^T$ and $Q = LL^T$. The Hankel singular values can then be computed as the singular values of the product U^TL . Note that the product U^TL does not need to be computed explicitly [7]. Denoting the singular value decomposition (SVD) of U^TL with

$$U^TL = ZSY^T,$$

where Y, Z are orthogonal matrices and S is a diagonal matrix with non-negative diagonal entries, we can employ the state space transformation $\hat{x} = T_b x$, with

$$T_b = S^{\frac{1}{2}}Z^TU^{-1}. \tag{7}$$

Using this transformation we find that the two Gramians become equal and diagonal, i.e., $P_b = Q_b = S$ since with $A_b = T_bAT_b^{-1}$, $B_b = T_bB$, $C_b = CT_b^{-1}$ we have

$$A_bS + SA_b^T + B_bB_b^T = 0, \quad A_b^TS + SA_b + C_b^TC_b = 0.$$

Note that this presumes invertibility of S which is given if the underlying system is controllable and observable. These prerequisites can be weakened to (A, B) stabilizable and (A, C) detectable [15]

Now assuming that the Hankel singular values are decreasingly ordered, we can perform the model reduction by simple truncation. Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_\ell \geq \sigma_{\ell+1} \geq \dots \geq \sigma_n$, $S_1 = \text{diag}(\sigma_1, \dots, \sigma_\ell)$, $S = \text{diag}(S_1, S_2)$ and block-divide A_b, B_b, C_b accordingly as

$$A_b = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_b = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C_b = [C_1 \quad C_2],$$

such that $\tilde{A} = A_{11} \in \mathbb{R}^{\ell \times \ell}$, $\tilde{B} = B_1 \in \mathbb{R}^{\ell \times p}$ and $\tilde{C} = C_1 \in \mathbb{R}^{m \times \ell}$ then for the reduced order model

$$\begin{aligned}
\tilde{\Sigma}: \quad \tilde{E}\dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \\
\tilde{y}(t) &= \tilde{C}\tilde{x}(t),
\end{aligned} \tag{8}$$

we have the error estimate

$$\sigma_\ell \leq \left\| \Sigma - \tilde{\Sigma} \right\|_{H_\infty} \leq 2(\sigma_{\ell+1} + \dots + \sigma_n). \tag{9}$$

Here $\left\| \Sigma \right\|_{H_\infty}$ denotes the peak frequency response of the system Σ , i.e., the maximum singular value of the transfer function evaluated on the imaginary axis. We point out that neither the full transformation T_b nor the balanced realization (A_b, B_b, C_b) need to be computed explicitly for the ROM to be determined. Instead we perform the truncation directly using the SVD

$$U^T L = [Z_1 \quad Z_2] \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} Y_1^T \\ Y_2^T \end{bmatrix}, \quad (10)$$

define the transformation matrices $W := L^T Y_1^T S_{11}^{-\frac{1}{2}}$, $V := U^T Z_1 S_{11}^{-\frac{1}{2}}$ and find the reduced order system $\tilde{\Sigma}$ as

$$\tilde{A} = W^T A V, \quad \tilde{B} = W^T B, \quad \tilde{C} = C V.$$

If we have $E \neq I_n$, the two Gramians are computed as the solutions of the generalized Lyapunov equations

$$A P E^T + E P A^T + B B^T = 0, \quad A^T Q E + E^T Q A + C^T C = 0. \quad (11)$$

Handling Singular E Matrices

The case of singular E matrices can be treated in different ways, see [1], [8], [12], [14]. For the machine tool models considered here, a simple transformation to an ODE system is described in [3]. In the numerical results reported here, the latter approach is used.

3 Krylov Subspace Methods

Krylov subspace methods are another important class of modern MOR techniques. The basic idea underlying these methods is to compute the columns in the above transformation matrices V, W as adequate bases of certain (block) Krylov subspaces. Given matrices $P \in \mathbb{R}^{s \times s}$ and $Q \in \mathbb{R}^{s \times 1}$ such that P and the columns of Q are linear independent, the block Krylov subspace is defined as

$$K_k(P, Q) = \text{span}\{Q, PQ, P^2Q, \dots, P^{k-1}Q\}. \quad (12)$$

The block Krylov subspace of dimension r for l initial vectors (i.e., columns in Q) can be seen as a union of l Krylov subspaces spanned by $r = l \cdot k$ column vectors.

The stable computation of the basis vectors can be performed by a block-Lanczos- or block-Arnoldi-algorithm where it is important to deflate linear dependent columns. Employing the block-Arnoldi-algorithm provides orthogonal projection matrices, i.e., $V^T V = I$ and $W^T W = I$ whereas using the block-Lanczos-algorithm leads to bi-orthogonal bases, such that $W^T V = I$. The latter bi-orthogonality property is important to interpret these MOR methods as projection techniques. For the Arnoldi-based algorithms, this can be achieved by replacing W by $W(V^T W)^{-1}$. We will restrict ourselves to the basic ideas here. A detailed presentation of the algorithms employed in the numerical tests can be found in [6].

3.1 Reduction of First Order Systems

Applying a Laplace transformation to the system Σ we compute the transfer function

$$H(s) = C^T (sE - A)^{-1} B. \quad (13)$$

The underlying idea of Krylov subspace based methods for computation of reduced order models is to match the leading moments of $H(s)$, i.e., the coefficients $h_j(s_0)$ in a Laurent series expansion

$$H(s_0) = \sum_{j=0}^{\infty} C^T \underbrace{\left[(A - s_0 E)^{-1} E \right]^j (A - s_0 E)^{-1} B}_{=: h_j(s_0)} (s - s_0)^j \quad (14)$$

at the expansion point s_0 . Obviously the quality of the approximation of the original system by the ROM strongly depends on the distance of the current frequency s from the expansion point s_0 . To overcome this problem, rational Krylov subspace methods [9] have been developed employing

multiple expansion points and thus extending the frequency range where good approximations can be expected.

The number of moments to be matched depends on the method taken. For single sided methods, where $W = V$ at least the first $\lfloor r/p \rfloor$ moments are matched [10], the columns of V are computed as a basis of the block Krylov subspace generated by

$$P = (A - s_0 E)^{-1} E \quad \text{and} \quad Q = (A - s_0 E)^{-1} B.$$

For a two sided method the columns of W arise as special basis vectors of the block Krylov subspace with regard to

$$P = (A - s_0 E)^{-T} E^T \quad \text{and} \quad Q = (A - s_0 E)^{-T} C,$$

and at least the first $\lfloor r/p \rfloor + \lfloor r/m \rfloor$ moments of the original transfer function are matched [10]

3.2 Reduction of Second Order Systems

The transfer function of the second order system as in the first order case is given by the Laplace transformation of the system equation:

$$H(s) = (C_p + sC_v)^T (s^2 M + sD + K)^{-1} B. \quad (15)$$

Also in analogy to the first order case we derive the moments $h_j(s_0)$ as the coefficients in a Laurent series expansion at s_0

$$H(s_0) = \sum_{j=0}^{\infty} \underbrace{\left[(-s_0^2 M - s_0 \tilde{D} - \tilde{K})^{-1} M \right]^j}_{=h_j(s_0)} \times (-s_0^2 M - s_0 \tilde{D} - \tilde{K})^{-1} (s - s_0)^j, \quad (16)$$

where $\tilde{D} = 2s_0 M + D$, $\tilde{K} = s_0^2 M + s_0 D + K$ and $\tilde{C}_p = (C_p + sC_v)$. The matrix V in one sided methods here is formed by basis vectors for the block Krylov subspace generated by

$$P = -(s_0^2 M + s_0 D + K)^{-1} M \quad \text{and} \quad Q = -(s_0^2 M + s_0 D + K)^{-1} B.$$

For a two sided method W contains basis vectors of the block Krylov subspace formed by

$$P = -(s_0^2 M + s_0 D + K)^{-T} M^T \quad \text{and} \quad Q = -(s_0^2 M + s_0 D + K)^{-T} \tilde{C}_p.$$

Again multiple expansion points can be used to extend the range of frequencies for which good approximations can be expected (see, e.g. [13])

4 Numerical Tests

In the following we present numerical tests for the two MOR approaches applied to two different test cases. The BTA is applied to a feed drive system model (see Figure 2) with 738 degrees of freedom that allows easy verification of the results. This test system has one input and four outputs.

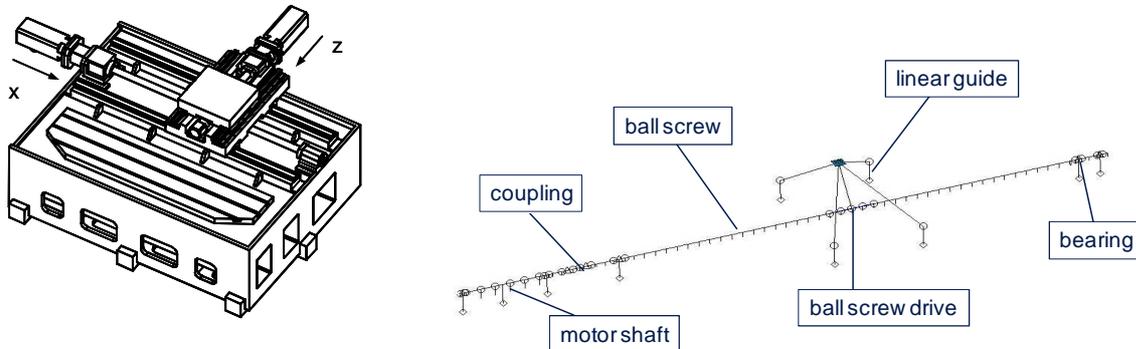


Figure 2: sketch of a drive test stand (left) and FE-structural model of a feed drive system (right)

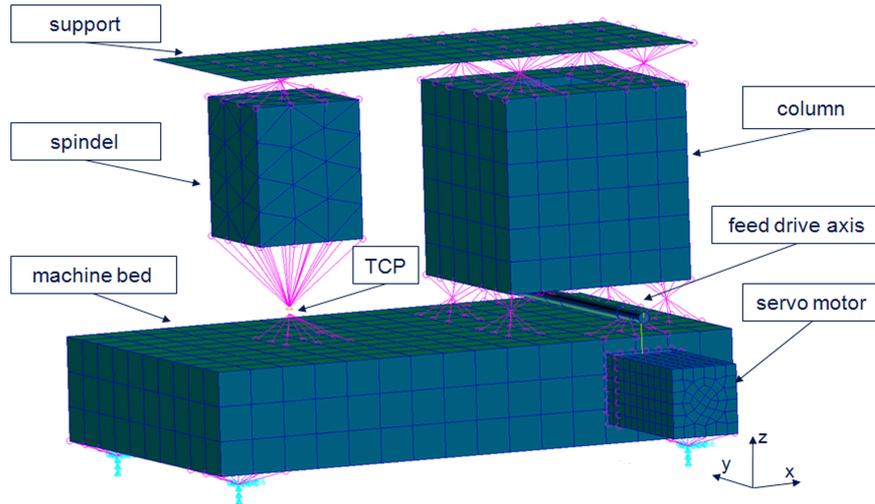


Figure 3: FE model of a simplified test configuration.

Figure 3 shows the system configuration where the feed drive system is inserted in a simple machine tool model. This model already consists of the basic structural components (except from the stock removal process itself) we find in the stock industrial machine tools that define the final interest of the funding project. Its original FE model is composed of a mesh with 4983 degrees of freedom. This model has been used in the tests for the Krylov subspace based reduction methods. Again due to space limitations we will restrict ourselves to the discussion of first order ROMs here. The numerical verification of the reduction to second order ROMs and their application in the closed simulation loop will be reported elsewhere.

4.1 BTA Based Results

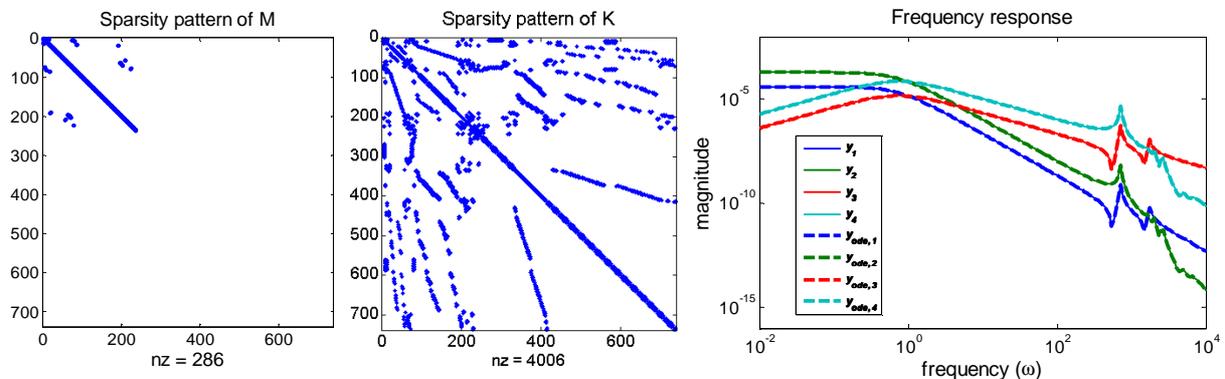


Figure 4 sparsity patterns of mass (left) and stiffness (middle) matrices and frequency responses for DAE (solid) and inherent ODE (dashed)

Figure 4 shows the DAE character of the feed drive model. The procedure to extract the inherent ODE from the original second order DAE described in [3] yields a first order system with $n = 974$, i.e., 502 algebraic constraints are removed. Application of BTA with an error tolerance $\tau = 10^{-10}$ (Note that (9) implies $\|y - y_r\|_2 \leq \tau \|u\|_2$) to the inherent ODE leads to an order 20 first order ROM (corresponding to an order 10 second order model). The absolute and relative errors in the frequency response are plotted in Figure 5. As expected we see that the ROM meets the prescribed error tolerance in all four outputs. The fast ascend of the relative error for high frequencies (especially in the cases of outputs 2 and 4) is due to the fact, that the frequency response is decaying very fast for high frequencies. Thus in the numerical computations of the relative errors we are dividing by very small

numbers such that we have to take into account significant numerical inexactness when interpreting these results.

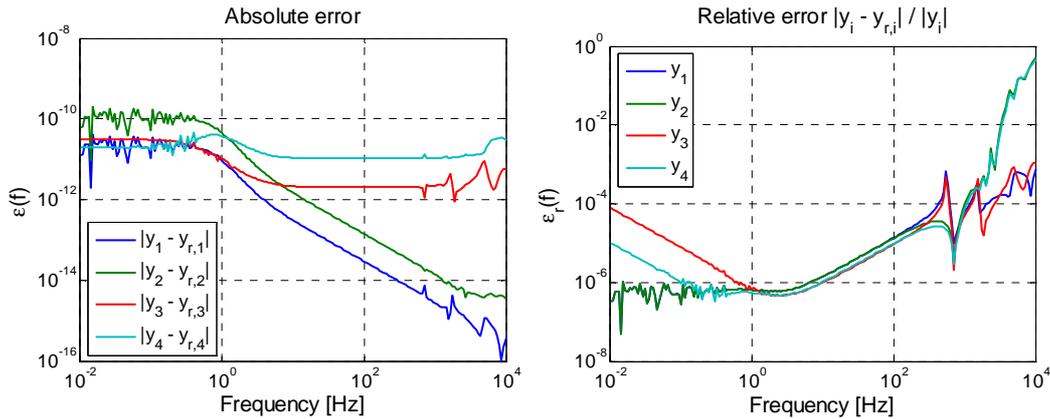


Figure 5 absolute and relative errors in frequency response

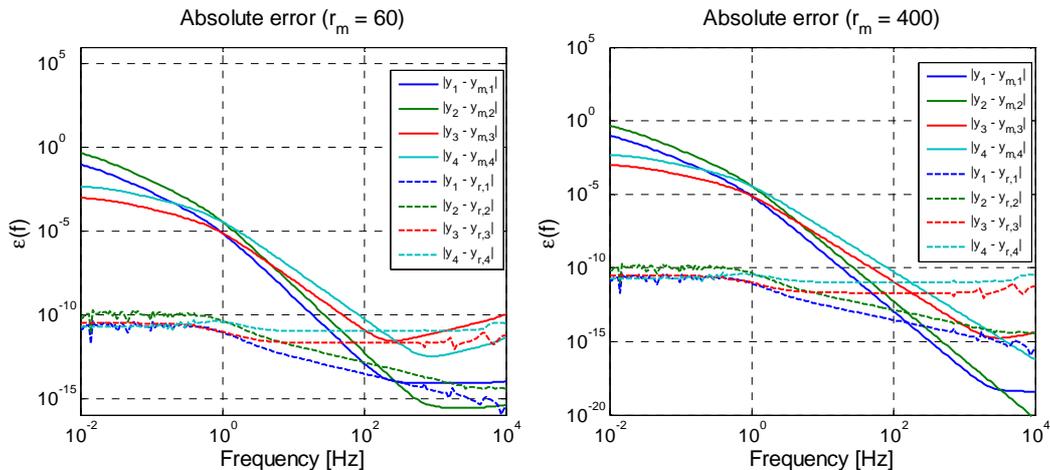


Figure 6 absolute errors of the order 60 (left) and 400 (right) modal reductions (solid) as compared to a BTA ROM (dashed) of order 20

Figure 6 shows the deviation of the absolute errors for modal reduction models (y_m) of order 60 and 400 in comparison to those of an order 20 BTA ROM (y_r). We especially note the larger errors for low frequencies in the case of the modal models. In the desired application of static stress analysis especially these low frequencies play the important role and the handicap of the modal models in this context is obvious. On the other hand for frequencies larger than 1000Hz the modal model of order 400 shows better approximation properties than the BTA based reduction. These high frequencies are not of very much use in practice for the mechanical models in the focus of this research, though.

4.2 Krylov Subspace Based Results for the Simplified Machine Tool Model (Order 4983)

For a detailed description of the different Krylov subspace based reduction algorithms we again refer to [6]. Here we just give the basic ideas of the methods hidden behind the abbreviations used in the upcoming figures. The abbreviations reflect the Krylov algorithm applied in the computation of the orthogonal bases, i.e., Arnoldi stands for the algorithm that generates the orthogonal basis of the Krylov subspace by a block Arnoldi method. In case of Lanczos, the block Lanczos method is applied in the basis computations. For these 2 methods the expansion point $s_0 = 0$ has been chosen. The two algorithms Rational Arnoldi and Rational Lanczos compute the bases for the Krylov subspaces employing rational block Arnoldi/Lanczos methods. As noted above these methods allow for the choice of multiple expansion points extending the frequency range where good transfer function approximations can be expected.

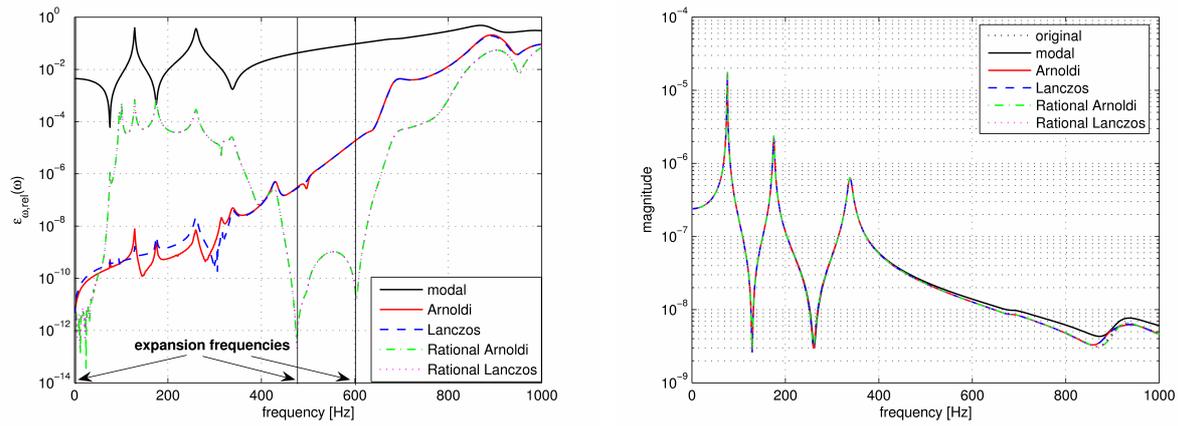


Figure 7 error in the transfer function and expansion points chosen (left) and Bode plots of the transfer function (right)

The three expansion frequencies drawn in Figure 7 are 0.7Hz, 475.5Hz and 603.8Hz. Here we can observe that the rational Krylov methods perform especially well in a vicinity of these frequencies. Another expansion frequency has been chosen at 1075.0Hz outside the drawn frequency range. Having observed very good approximation results in frequency domain in Figure 7, we want to validate the models in time domain simulations. To this end, the different ROMs are applied to compute step response controls in a MATLAB/SIMULINK simulation loop. Since the original system model is much too large to solve the closed loop simulation in reasonable time, we use the modal reduction model of order 400 as the reference in the comparison found in Figure 8.

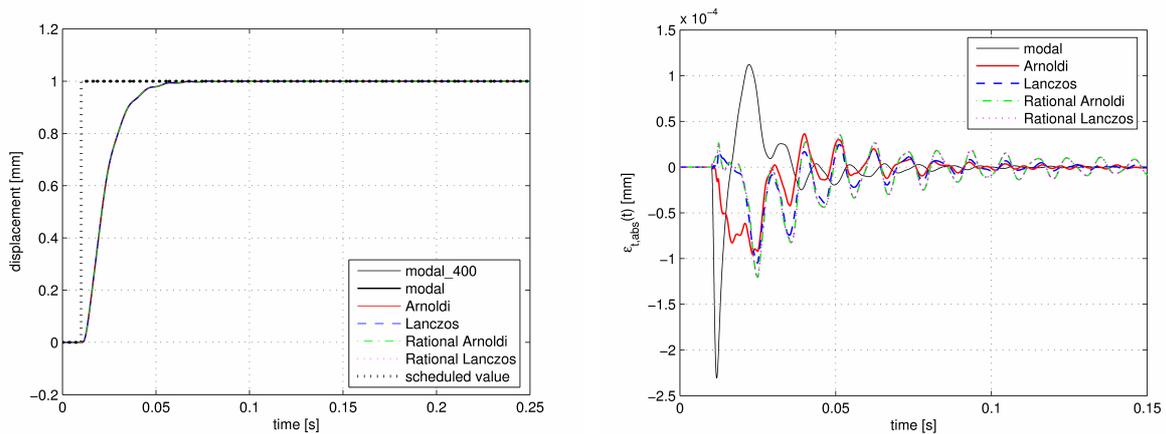


Figure 8: step response (left) and errors in the step response (right) for different order 50 ROMs computed via Krylov and modal reduction (reference: modal ROM of order 400)

We can easily see that all Krylov based ROMs follow the reference model more closely than the modal ROM. We can also observe that the deviation of the rational Krylov method based solutions from the reference decays slower than the others.

5 Conclusions

We have presented novel fully automatic model order reduction techniques for second order systems arising in the context of the integrated numerical simulation of closed-loop control for certain machine tools. We considered balancing based approaches as well as Krylov subspace methods for the generation of the reduced order models. The presented numerical results clearly show the advantages of modern MOR techniques over modal truncation approaches as they are performed by standard CAE software such as ANSYS. It should be noted that these modern methods could be easily integrated into existing CAE tools as the core computations only rely on efficient solvers for linear systems of equations which are available in these software libraries.

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