Produkte endlicher nilpotenter Gruppen

Diplomarbeit

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Contents

Introduction

A group G is said to be *the product of its subgroups* G_i for $i \in I$ if G is generated by the G_i and the set G_iG_j is a subgroup of G for all $i, j \in I$. In this terminology, a famous result of P. Hall [24] published in 1937 can be formulated as follows: The finite soluble groups G are precisely the finite groups which are the product of certain of their Sylow subgroups. This shows in particular that the finite soluble groups are groups which are the product of certain of their nilpotent subgroups.

For about 20 years, it remained an open question whether the converse of the last statement is also true, namely whether a finite group that is the product of certain nilpotent subgroups is soluble. In 1955, Itô [32] obtained a first result in this direction, namely that a product of two (possibly infinite) abelian subgroups is metabelian.

The original question was answered positively by Wielandt [45] and Kegel [33] in 1958 and 1961 respectively. Their result, stating that a finite group is soluble if and only if it is the product of finitely many nilpotent subgroups, has become known as the Kegel-Wielandt theorem.

This motivates the following general question: If the group G is the product of its subgroups G_i with $i \in I$, and certain group-theoretical properties of the G_i are known, what group-theoretical properties does the group G have?

One problem in answering this question is that in general, the group G is not uniquely determined by the G_i since the concept of a product of subgroups includes e.g. direct and semidirect products. Observe however that these kinds of products are not typical because the factors are not necessarily normal in G (for a finite group that is the product of two nonnormal subgroups see Example 3.6.8 below).

A second question, often encountered when dealing with questions of the first type, is the following: Which subgroups of G inherit the product structure of G, or, more concretely, which subgroups of G are conjugate to (isomorphic to) a *prefactorized* subgroup S of G, i.e. a subgroup that is the product of its subgroups $S \cap G_i$?

In this *Diplomarbeit*, we will be concerned mainly with groups G that are the product of *two* subgroups A and B. Then a subgroup S of G is prefactorized if $S = (S \cap A)(S \cap B)$.

A prefactorized subgroup that also contains $A \cap B$ will be termed *factorized.* Chapter 1 will be dedicated to the study of factorized and in particular prefactorized subgroups in the subgroup lattice of the group $G = AB$ and in quotient groups.

Since products of finite nilpotent subgroups are soluble by the Kegel-Wielandt theorem, we study in Chapter 2 finite groups G which are the product of two subgroups and which satisfy some properties related to solubility. Our results are based on a result of Wielandt [44] who proved that for every prime p, the finite group G has a prefactorized Sylow p-subgroup. We will show in particular that if G is soluble, then it possesses a Hall system that consists entirely of prefactorized Hall subgroups: a Hall system of this type will play an important role in Section 3.3.

Furthermore, we will show in Section 2.2 and Section 2.3 that a relatively large number of subgroups of G is prefactorized or factorized if A and B have coprime indices or orders respectively: if G is finite and soluble, then it is possible to find a prefactorized or factorized subgroup among the conjugates of any subgroup.

In Chapter 3, we specialize to groups G that are the products of two nilpotent subgroups A and B. One of the main tools of this and the following chapter will be a theorem due to Gross which gives important information about the structure of G when G is primitive. This theorem will be proved in Section 3.2 in a somewhat more general form, based on the proof of Gross' result given in [4].

The rest of Chapter 3 is dedicated to finding prefactorized or factorized conjugates for certain subgroups of G: we will show that for every abnormal subgroup of G, there is exactly one factorized conjugate (Proposition 3.3.5). This result will then be used to reduce the question whether certain pronormal subgroups of G are prefactorized or factorized to questions about normal subgroups. As a first application, we will be able to improve a result of Heineken [28], showing that for every Schunck class $\mathfrak{H}$ containing all finite nilpotent groups, G has a unique factorized $\mathfrak{H}$-projector. More generally, we will show in Theorem 3.5.1 that every $\mathfrak{H}$-maximal subgroup has a factorized conjugate. However, these factorized subgroups are not necessarily isomorphic.

A corresponding result about $\mathfrak{F}$-injectors for arbitrary Fitting classes $\mathfrak{F}$ does not hold (see Example 3.6.8). Nevertheless, using our results about pronormal subgroups, we will prove in Section 3.6 that a product of two finite nilpotent subgroups has a prefactorized or factorized $\mathfrak{F}$-injector if the $\mathfrak{F}$-radical of every product of two finite nilpotent subgroups is prefactorized or factorized respectively and that in this case, every finite group that is the product of two nilpotent subgroups has exactly one prefactorized $\mathfrak{F}$-injector. By Proposition 3.6.9, this result applies in particular to all saturated Fitting-formations and thus to all subgroup-closed Fitting classes. This result generalizes a well-known result of Amberg [1] and Pennington [40] stating that the Fitting subgroup of a product of two finite nilpotent subgroups is always factorized.

We do not know wheter our results about injectors and radicals can be extended to certain other classes of subgroups. In this context, it is of interest to ask whether the hypercentre of a product G of two finite nilpotent subgroups is prefactorized, since such a group has a factorized system normalizer (see Corollary 3.3.13). On the other hand, an example due to Gillam [19] shows that $Z(G)$ is not necessarily prefactorized.

Also, a number of other characteristic subgroups of the group $G = AB$ need not be prefactorized (or factorized), among them the derived subgroup G', the nilpotent residual $G^{\mathfrak{N}}$ and the subgroups $O^{\pi}(G)$ (see Example 3.4.4).

If π is an arbitrary set of primes, we show in Chapter 4 how to obtain upper bounds on the π-length of a group that is the product of two finite nilpotent subgroups A and B in terms of certain invariants of A and B from bounds on the p-length of a finite soluble group in terms of a Sylow p-subgroup. We use the bounds found by Hall and Higman [26], supplemented by some additional results of Gross [21] and Berger and Gross [9] when $p = 2$, to show e.g. that for every set π of odd primes,

$$l_{\pi}(G) \leq \max\big\{d(A), d(B)\big\}$$

and thus for any set π of primes,

$$l_{\pi}(G) \leq \max\big\{d(A), d(B)\big\} + 1.$$

Our bounds for the p-lengths are best-possible in the sense of [26] whenever the bounds in terms of the Sylow p-subgroups are best-possible.

As a further result, it will turn out that nilpotent length and the p-lengths of a group that is the product of two finite nilpotent subgroups are not too far removed from each other: We will show in Theorem 4.3.1 that

$$n(G) \leq 2 \max_{p \in \pm\nvdash\mathbb{P}} \big\{l_p(G)\big\},$$

$$n(G) \leq 2 \max_{p \in \pm\nvdash\mathbb{P}} \big\{l_{p'}(G)\big\} + 1,$$

and

$$n(G) \leq \max_{p \in \pm\nvdash\mathbb{P}} \big\{l_p(G) + l_{p'}(G)\big\}.$$

Using essentially the above results, the nilpotent length of a product G of two finite nilpotent subgroups A and B can also be bounded in terms of invariants of A and B. In particular, we show that

$$n(G) \leq 2d(A) + \max\{0,\ 2d_2(A) - 4\} + 1$$

and

$$n(G) \leq d(A) + d(B) + \max\{0,\ d_2(A) - 2,\ d_2(B) - 2\}.$$

These bounds are best-possible at least for groups of odd order.

Although it seems to be an open question whether the derived length of G can be bounded by the derived lengths of A and B, we can still obtain some information about the derived length of G: if π is the set of common prime divisors of the orders of A and B, then we obtain in Section 4.4:

$$\begin{aligned} d\big(G/\Phi(G) \cap O_\pi(G)\big) &\leq \max\big\{c_2(A),\ \tfrac{1}{2}d_{2'}(A)\big(d_{2'}(A)+1\big)\big\} \\ &\quad + \ \max\big\{c_2(B),\ \tfrac{1}{2}d_{2'}(B)\big(d_{2'}(B)+1\big)\big\}, \\ d\big(G/\Phi(G) \cap O_\pi(G)\big) &= \tfrac{1}{2}d(A)\big(d(A)+1\big) + \tfrac{1}{2}d(B)\big(d(B)+1\big) \\ &\quad + \ \max\big\{\tfrac{1}{2}d_2(A)\big(d_2(A)+1\big),\ \tfrac{1}{2}d_2(B)\big(d_2(B)+1\big)\big\} \end{aligned}$$

and also

$$d\big(G/F(G)\big) \leq \big(d(A) + \max\{0, d_2(A) - 1\}\big)\big(d(B) + \max\{0, d_2(B) - 1\}\big).$$

Finally, in Section 4.5, further structural information about the Fitting quotient group $G/F(G)$ will be obtained; this will give rise to relate the class of products of finite nilpotent groups to other classes of finite soluble groups and in particular to the class of groups 'with many Sylow bases' introduced by Huppert in [30].

The notation used is standard and follows Doerk and Hawkes [13], Robinson [43] and Amberg, Franciosi and de Giovanni [4]. For details, the reader is referred to the list of symbols at the end of the text.

Chapter 1
Basic properties of products of groups

1.1 Subgroups of products of groups

Let G be a group. If X and Y are subsets of G, define

$$XY = \{xy \mid x \in X, y \in Y\}.$$

We first recall the following elementary lemma which allows to calculate the cardinality of the set AB; for the proof, see e.g. Robinson [43], 1.3.11 or Doerk and Hawkes [13], A.1.5.

1.1.1 Lemma. *Let A and B be subgroups of the group G, then $|AB|\cdot|A\cap B| = |A|\cdot|B|$. In particular if A and B are finite, then*

$$|AB| = \frac{|A| \cdot |B|}{|A \cap B|}.$$

If X and Y are subgroups of G, one is particularly interested in whether XY is again a subgroup of G. In this case, X and Y are said to be *permutable* because of the well-known equivalence of (i) and (ii) in the following

1.1.2 Lemma. *Let G be a group and let A and B be subgroups of G. Then the following statements about the set AB are equivalent:*

(i) *AB is a subgroup of G;*

(ii) *$AB = BA$ (i.e. A and B are permutable);*

(iii) *$[a, b] \in AB$ for all $a \in A$ and $b \in B$;*

(iv) *$a^b \in AB$ for all $a \in A$ and $b \in B$.*

Proof. (i) $\Rightarrow$ (ii). Let $g \in AB$. Since AB is a group, also $g^{-1} \in AB$: write $g^{-1} = ab$ with $a \in A$ and $b \in B$. Then $g = b^{-1}a^{-1} \in BA$ and therefore $AB \subseteq BA$. Similarly, $BA \subseteq AB$ so that $AB = BA$.

(ii) $\Rightarrow$ (iii). Let $a \in A$ and $b \in B$, then we have $[a, b] = a^{-1}b^{-1}ab \in ABAB = A(AB)B = AB$ as required.

(iii) $\Rightarrow$ (iv). Since $a^b = a[a, b]$ for all $a \in A$ and $b \in B$, we have $a^b \in A(AB) = AB$.

(iv) $\Rightarrow$ (i). Let a_1, $a_2 \in A$ and b_1, $b_2 \in B$. We have to show that $(a_1b_1)^{-1}(a_2b_2) \in AB$. But this is true since

$$(a_1b_1)^{-1}(a_1b_1) = b_1^{-1}a_1^{-1}a_2b_2 = (a_1^{-1}a_2)^{b_1}b_1^{-1}b_2 \in (AB)BB = AB.$$

□

The following lemma will be needed later.

1.1.3 Lemma. *. Let G be a group with subgroups A, B and C such that $AC = CA$ and $BC = CB$. Then also $\langle A, B\rangle C = C\langle A, B\rangle$.*

Proof. Let $g \in \langle A, B\rangle$ and $c \in C$. By Lemma 1.1.2, it suffices to show that $g^c \in \langle A, B\rangle$. Since $g \in \langle A, B\rangle$, g can be written $g = a_1b_1 \ldots a_rb_r$ with $a_i \in A$ and $b_i \in B$ for all $b = 1 \ldots r$. Since by Lemma 1.1.2, $a_i^c \in AC$ and $b_i^c \in BC$ for all i, we obtain

$$g^c = a_1^cb_1^c \ldots a_r^cb_r^c \in \underset{\longleftarrow r\ times \longrightarrow}{(ACBC) \ldots (ACBC)} = \underset{\longleftarrow r\ times \longrightarrow}{(AB) \ldots (AB)}C$$

because C permutes with both A and B. Since clearly

$$\underset{\longleftarrow r\ times \longrightarrow}{(AB) \ldots (AB)} \subseteq \langle A, B\rangle,$$

we have $g^c \in \langle A, B\rangle C$. □

A group G is said to be *the product of its subgroups A and B* if $G = AB$. Sometimes such a group is also called *factorized by A and B*[1] or simply *factorized.* A subgroup S of $G = AB$ is called *prefactorized* if $S = (S \cap A)(S \cap B)$, or equivalently, if every $s \in S$ can be written $s = ab$ with $a \in A \cap S$ and $b \in B \cap S$. Following Wielandt [45], a subgroup S of G is called *factorized* if whenever $s = ab$ with $a \in A$ and $b \in B$, then $a \in S$ (and $b \in S$). Since every $g \in G$, and thus every element g of S, can be written $g = ab$ with $a \in A$ and $b \in B$, every factorized subgroup of G is prefactorized. It is also clear that every subgroup of G containing A or B is factorized; in particular G is factorized.

For some statements concerning groups which are the product of two of their subgroups, the following generalization of Dedekind's modular law is useful:

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[1] We will avoid this terminology wherever possible since this might lead to confusion when dealing with prefactorized subgroups S which are — as groups — factorized by $A \cap S$ and $B \cap S$. In the opinion of the author, it might have been preferable to use the terms *factorized* and *strongly factorized* instead of *prefactorized* and *factorized* respectively because of this problem of terminology. As we will see later, many results about factorized subgroups also hold for the subgroups now called prefactorized, which could be an indication that the concept of being prefactorized might be the more natural one when dealing with subgroups of factorized groups.

1.1.4 Lemma. *Let G be a group. If X, Y and U are subsets of G such that $U^{-1} = \{u^{-1} \mid u \in U\} \subseteq U$, then*

(i) *if $XU \subseteq X$, then $(X \cap Y)U = X \cap YU$ and*
(ii) *if $UX \subseteq X$, then $U(X \cap Y) = X \cap UY$.*

Proof. (i) The proof follows that usually given for Dedekind's modular law: Clearly, $(X \cap Y)U \subseteq XU \cap YU \subseteq X \cap YU$. Now let $x \in X \cap YU$ and write $x = yu$ with $y \in Y$ and $u \in U$. Then $xu^{-1} \in XU \cap Y \subseteq X \cap Y$ and therefore $x = xu^{-1}u \in (X \cap Y)U$. The proof of (ii) is similar. □

The following lemma shows in particular that prefactorized subgroups are factorized if and only if they contain $A \cap B$:

1.1.5 Lemma. *Suppose that the group G is the product of its subgroups A and B. Then the following statements are equivalent:*
(i) *S is factorized;* (ii) *$S = (S \cap A)(S \cap B)$ and $A \cap B \leq S$;* (iii) *$A \cap SB = A \cap S$;* (iv) *$A \cap SB \subseteq S$;* (v) *$AS \cap B = S \cap B$;* (vi) *$AS \cap B \subseteq S$;* (vii) *$S = (AS \cap B)(A \cap SB)$;* (viii) *$S = (AS \cap B)S$;* (ix) *$S = S(A \cap SB)$:* (x) *$S = AS \cap BS$;* (xi) *$S = SA \cap SB$.*

Proof. (i) ⇒ (ii). Since every factorized subgroup is prefactorized, it remains to show that $A \cap B \leq S$. For every $x \in A \cap B$, we have $1 = xx^{-1} \in S$ and, of course, $x \in A$ and $x^{-1} \in B$. So by the definition of a factorized subgroup, we have $x \in S$ and hence $A \cap B \leq S$.

(ii) ⇒ (iii). Since $S = (A \cap S)(B \cap S)$, we have $A \cap SB = A \cap (S \cap A)(S \cap B)B = A \cap (S \cap A)B$. By the modular law (or by Lemma 1.1.4), $A \cap (S \cap A)B = (S \cap A)(A \cap B) = S \cap A$ since also $A \cap B \subseteq S$.

(iii) ⇒ (iv). This is trivial.

(iv) ⇒ (i). If $s = ab \in S$ with $a \in A$ and $b \in B$, then $a = sb^{-1} \in A \cap SB \subseteq S$ whence S is factorized.

The implications (i) ⇒ (ii) ⇒ (v) ⇒ (vi) ⇒ (i) can be proved similarly.

To prove that (vii), (viii), (ix), (x) and (xi) are equivalent to S being factorized, suppose first that S is factorized. Then $S = (S \cap B)(A \cap S) \subseteq (AS \cap B)(A \cap BS) = S$, so (vii) holds. Also, (viii) and (ix) hold since $AS \cap B \subseteq S$ and $A \cap SB \subseteq S$ by (iv) and (vi) respectively. (x) and (xi) follow from (viii) and (ix) by Lemma 1.1.4.

Conversely, if one of the equations (vii), (viii), (ix), (x) and (xi) hold, then it is evident that at least one of the sets $AS \cap B$ or $A \cap SB$ is contained in S proving statements (iv) or (vi) respectively, which are both equivalent to S being factorized. □

Note. It is easy to see that for any subgroup S of $G = AB$, the set $A \cap SB$ equals the set $\{a \in A \mid ab \in S \text{ for some } b \in B\}$ and $AS \cap B = \{b \in A \mid ab \in S \text{ for some } a \in A\}$.

The next lemma studies the behaviour of factorized (prefactorized) subgroups in the subgroup lattice of a factorized group G. The statements about factorized subgroups can also be found in Chapter 1 of [4] from which also most proofs are derived. To see that the intersection of two prefactorized subgroups is not necessarily prefactorized, see Example 1.2.3.

1.1.6 Lemma. *Let the group G be the product of its subgroups A and B.*

(i) *If S is prefactorized (factorized) in G, then $T \leq S$ is prefactorized (factorized) with respect to the factorization $S = (S \cap A)(S \cap B)$ of S if and only if T is prefactorized (factorized) in G.*

(ii) *The intersection of any family of factorized subgroups of G is factorized.*

(iii) *The intersection of a prefactorized subgroup and a factorized subgroup is prefactorized.*

(iv) *If the subgroup S of G is the union of the prefactorized subgroups S_i where $i \in I$ of G, then S is prefactorized. It is factorized, provided one of the subgroups S_i is factorized.*

(v) *The product of two prefactorized subgroups one of which is normal in G is prefactorized. It is factorized, provided that either of the subgroups is factorized.*

(vi) *The product of any number of prefactorized normal subgroups is prefactorized. It is factorized if one of the normal subgroups is factorized.*

Proof. (i) The statement concerning prefactorized subgroups follows from the fact that by definition,

$$T = (S \cap A \cap T)(S \cap B \cap T) = (A \cap T)(B \cap T)$$

so that T is a prefactorized subgroup of G. In case S is a factorized subgroup of G, we have $A \cap B = (A \cap S) \cap (B \cap S)$; thus if T is factorized in S, then T contains $(A \cap S) \cap (B \cap S) = A \cap B$ so that by Lemma 1.1.5, T is a factorized subgroup of G.

(ii) Let $\{S_i \mid i \in I\}$ be a family of factorized subgroups of G. If $s = ab \in \bigcap S_i$, then $s \in S_i$ for all i, and hence $a \in S_i$ for all i by the definition of a factorized subgroup. Hence a is contained in $A \cap (\bigcap S_i)$, which shows that $\bigcap S_i$ is factorized.

(iii) Let S and P be a factorized and a prefactorized subgroup of G. Every $g \in P \cap S$ can be written $g = ab$ with $a \in A \cap P$ and $b \in P \cap B$ since P is prefactorized. Now S is factorized and also $g \in S$, so we have $a \in S$ and $b \in S$, that is, $a \in A \cap P \cap S$ and $b \in B \cap P \cap S$. This shows that $S \cap P$ is prefactorized.

(iv) Let $s \in S$, then $s \in S_i$ for some $i \in I$. Since S_i is prefactorized, we have $s = ab$ with $a \in A \cap S_i$ and $b \in B \cap S_i$, so in particular $a \in A \cap S$ and $b \in B \cap S$. This shows that S is prefactorized. If one of the S_i is factorized, it contains $A \cap B$ and so $A \cap B \leq S$ whence S is factorized.

(v) Let N and P be prefactorized subgroups of G with $N \trianglelefteq N$. Then

$$\begin{aligned} PN &= (P \cap A)(P \cap B)N \\ &= (P \cap A)N(P \cap B) \\ &= (P \cap A)(N \cap A)(N \cap B)(P \cap B) \\ &\leq (PN \cap A)(PN \cap B) \\ &\leq PN, \end{aligned}$$

which shows that PN is prefactorized. The statement about factorized subgroups follows as in (iv).

(vi) Let $\{N_i\}_{i \in I}$ be a family of prefactorized normal subgroups of G. By (v), the product of two prefactorized normal subgroups is prefactorized and since it is clearly normal, the statement is true for every finite index set I. For arbitrary index sets, the product of the N_i is the union of all products of a finite number of the N_i, so the full result follows from (iv). □

The following lemma shows in particular that if G is the product of its subgroups A and B, then also $G = A^x B^y$ for all x, $y \in G$.

1.1.7 Lemma (Wielandt [44]). *Let the group G be the product of its subgroups A and B. If A_0 and B_0 are normal subgroups of A and B respectively such that $A_0 B_0 = B_0 A_0$, then for every x and $y \in G$, there is a $z \in G$ such that $A_0^x = A_0^z$ and $B_0^y = B_0^z$ and thus $B_0^x B_0^y = B_0^y A_0^x = (A_0 B_0)^z$.*

Proof. Let x, $y \in G$ and write $xy^{-1} = a^{-1}b$ with $a \in A$ and $b \in B$. Let $z = ax = by$. Then $A_0^z = A_0^{ax} = A_0^x$ since $A_0 \trianglelefteq A$; similarly $B_0^Z = B_0^y$. So $A_0^x B_0^y) = (A_o B_o)^z$ is a subgroup of G which is equivalent to $A_0^x B_0^y = B_0^y A_0^x$ by Lemma 1.1.2. □

This has the following consequence:

1.1.8 Lemma. *If $G = AB = AC = BC$ where A, B and C are subgroups of G and C^g is factorized with respect to $G = AB$ for some $g \in G$, i.e. $C^g = (A \cap C^g)(B \cap C^g)$ and $C \geq A \cap B$, then $G = C$.*

Proof. Since $G = AC = CA$, we deduce from Lemma 1.1.7 that also $AC^g = C^g A$ which is conjugate to G, i.e. $G = AC^g$. Similarly, we obtain that $G = BC^g$ so that we may assume w.l.o.g. that C is factorized. Now by the modular law, $G = A(B \cap C) =$

$(A \cap C)B$, therefore, using the modular law again, $G = A(B \cap C) \cap (A \cap C)B = (A \cap (A \cap C)B)(B \cap C) = (A \cap C)(A \cap B)(B \cap C) \le C$. □

Note that the preceding lemma becomes false if we only have $C = (A \cap C)(B \cap C)$, that is, if C is only a prefactorized subgroup of $G = AB$: take any group $G \neq 1$ and put $A = B = G$ and $C = 1 \neq G$.

The following lemma will be needed later.

1.1.9 Lemma. *Let the group G be the product of its subgroups A and B and suppose that $\{N_i \mid i \in I\}$ is a set of factorized normal subgroups of G. Then*

$$\bigcap_{i \in I} AN_i = A\big(\bigcap_{i \in I} N_i\big).$$

Proof. Clearly, $\bigcap_{i \in I} AN_i$ contains $A\big(\bigcap_{i \in I} N_i\big)$. To prove the other inclusion, observe first that, since every subgroup of G that contains A or B is factorized, hence AN_i is factorized, $AN_i = A(B \cap N_i)$ whence $B \cap AN_i = (B \cap N_i)(A \cap B) = B \cap N_i$.

Now also $\bigcap_{i \in I} AN_i$ is factorized by Lemma 1.1.6, hence

$$\bigcap_{i \in I} AN_i = A\big(B \cap (\bigcap_{i \in I} AN_i)\big) = A\big(\bigcap_{i \in I}(B \cap AN_i)\big) = A\big(\bigcap_{i \in I}(B \cap N_i)\big) \le A\big(\bigcap_{i \in I} N_i\big).$$

□

1.2 Factorizers, prefactorizers, and quotient groups

For every subgroup S of the factorized group $G = AB$, the intersection of all factorized subgroups of G which contain S is factorized by Lemma 1.1.6. Clearly, this is the unique minimal factorized group that contains S. This subgroup is called the *factorizer* of S; we denote it with $X_G(S)$. If S is normal in G, its factorizer can be described as follows:

1.2.1 Lemma. *Let the group G be the product of its subgroups A and B. If N is a normal subgroup of G and $X = X_G(N)$ is the factorizer of N in G, then $AX = AN$ and $BX = BN$, hence $A \cap X = A \cap BN = A \cap BX$ and $B \cap X = B \cap AN = B \cap AX$ Thus $X = AN \cap BN$ has a triple factorization*

$$X = (AN \cap B)N = N(A \cap BN) = (AN \cap B)(A \cap BN).$$

Proof. Obviously, AN and BN are factorized subgroups of G containing N, therefore $X \le AN$ and $X \le BN$, from which it follows that $AX = AN$ and $BX = BN$. The remaining statements follow directly from this and from Lemma 1.1.5 □

The next lemma shows that homomorphic images of factorized groups inherit the factorization of G.

1.2.2 Lemma. *If the group G is the product of its subgroups A and B and N is a normal subgroup of G, then*

(i) *G/N is the product of its subgroups AN/N and BN/N.*

(ii) *If S is prefactorized in G, then SN/N is prefactorized in G/N.*

(iii) *If N is prefactorized (with respect to $G = AB$) and $N \leq S$, then S/N is a prefactorized subgroup of G/N if and only if S is a prefactorized subgroup of G.*

(iv) *If $N \leq S$, then S is a factorized subgroup of G if and only if S/N is factorized in G/N. In particular, $AN/N \cap BN/N = X_G(N)/N$.*

Proof. (i) This is trivial.

(ii) Clearly, $SN/N = (S \cap A)N/N \cdot (S \cap B)N/N \leq (SN/N \cap AN/N) \cdot (SN/N \cap BN/N)$ which is contained in SN/N. This shows that SN/N is prefactorized.

(iii) If S is prefactorized, it is clear by the preceding statement that S/N is prefactorized. Conversely, suppose that S/N is prefactorized, or equivalently, that $S = (S \cap AN)(S \cap BN)$. Then by the modular law, $S = (S \cap AN)(S \cap BN) = (S \cap A)N(S \cap B)$. Since N is prefactorized and $N \leq S$, we have $S = (S \cap A)(N \cap A)(N \cap B)(S \cap B) = (S \cap A)(S \cap B)$ which shows that S is prefactorized.

(iv) Suppose first that S is factorized. We already know that S/N is prefactorized. Moreover, $AN \cap BN \leq AS \cap SB = S$ by Lemma 1.1.5 and so $AN/N \cap BN/N \leq S/N$; thus by the same lemma, S/N is factorized.

Conversely, suppose that S/N is factorized, then by Lemma 1.1.5, S/N contains $AN/N \cap BN/N$ whence S contains the subgroup $AN \cap BN$. By Lemma 1.2.1, $AN \cap BN = X_G(N) = (AN \cap B)(A \cap NB)$ and so by the modular law,

$$\begin{aligned} S &= (S \cap AN)(S \cap BN) \\ &= (S \cap A)N(S \cap B) \\ &\leq (S \cap A)(A \cap NB)(AN \cap B)(S \cap B) \\ &= (S \cap A)(S \cap B), \end{aligned}$$

observing that $A \cap BN \leq A \cap S$ and $B \cap AN \leq B \cap S$. This also shows that $A \cap B \leq S$ and so by Lemma 1.1.5, S is factorized. $\square$

The definition of a factorizer cannot be extended to prefactorized subgroups, since the intersection of two prefactorized subgroups is not necessarily prefactorized, not even if the subgroups are normal, as the following simple example shows:

1.2.3 Example. Let V be a vector space of dimension 3 over some field F and let $\{x_1, x_2, x_3\}$ be a base of V. Let $A = \langle x_1, x_2\rangle$ and $B = \langle x_2, x_3\rangle$, then the subgroups $H_1 = \langle x_1, x_2 + x_3\rangle = \langle x_1\rangle \oplus \langle x_2 + x_3\rangle$ and $H_2 = \langle x_1 + x_2, x_3\rangle = \langle x_1 + x_2\rangle \oplus \langle x_3\rangle$ are prefactorized but their intersection $H = \langle x_1+x_2+x_3\rangle$ intersects A and B trivially, hence H cannot be prefactorized. If we choose F to be a field of prime order, H_1 and H_2 are minimal subject to containing H and being prefactorized while the factorizer of H is G by Lemma 1.2.1.

However if the group G is the product of its subgroups A and B and $H \leq G$, let

$$\mathcal{S} = \{S \leq G \mid H \leq S \text{ and } S \text{ is prefactorized}\}.$$

We call the minimal elements of $\mathcal{S}$ *prefactorizers* of H in G. Note, however, that unlike factorizers, prefactorizers need not exist if the group G is infinite.

1.2.4 Lemma. *Let the group G be the product of its subgroups A and B. for two subgroups A and B.*

(i) *If $H \leq G$ and S is a prefactorizer of H in G, then $S = X_S(H) \leq X_G(H)$.*

(ii) *If N is a normal subgroup of G and S is a prefactorizer of N in G, then S has a triple factorization $S = (A \cap S)(B \cap S) = (A \cap S)N = (B \cap S)N$.*

Proof. (i) If S is a prefactorizer of H in G, then $X_S(H)$, the factorizer of H with respect to the factorization $S = (S \cap A)(S \cap B)$ of S, is a prefactorized subgroup of G contained in S. Therefore we have $X_S(H) = S$ by the minimality of S. Since $X_G(H)$ is factorized, $X_G(H) \cap S$ is prefactorized by Lemma 1.1.6 and $H \leq X_G(H) \cap S \leq S$ whence $S = X_G(H) \cap S$ by the definition of S.

(ii) This follows directly from the fact that $S = X_S(N)$ and the characterization of $X_N(S)$ given in Lemma 1.2.1. $\square$

1.3 Classes of groups and factorizations

A class $\mathfrak{C}$ of groups is a class in the set-theoretical sense whose elements are groups and that satisfies the condition: if $G \in \mathfrak{C}$ and $H \cong G$, then $H \in \mathfrak{C}$. In other words, a class of groups is the union of isomorphy classes of groups.[1] Following Doerk and Hawkes [13], Chapter II, a closure operation C is a map

$$\mathrm{C}\colon \{\textit{group classes}\} \to \{\textit{group classes}\}$$

such that $\mathrm{C}\varnothing = \varnothing$ and for every class $\mathfrak{C}$ of groups, $\mathfrak{C} \subseteq \mathrm{C}\mathfrak{C}$ and $\mathrm{C}\mathfrak{C} = \mathrm{C}(\mathrm{C}\mathfrak{C})$ and if $\mathfrak{D}$ is a class of groups with $\mathfrak{C} \subseteq \mathfrak{D}$, then $\mathrm{C}\mathfrak{C} \subseteq \mathrm{C}\mathfrak{D}$.

We introduce the following closure operations on classes of groups

$$\begin{aligned}
\mathrm{Q}\mathfrak{C} &= \{G/N \mid G \in \mathfrak{C},\ N \trianglelefteq G\}\\
\mathrm{S}\mathfrak{C} &= \{H \mid \exists\, G \in \mathfrak{C} \text{ such that } H \leq G\}\\
\mathrm{S_n}\mathfrak{C} &= \{S \mid \exists\, G \in \mathfrak{C} \text{ such that } S \lhd\!\lhd G\}\\
\mathrm{R_0}\mathfrak{C} &= \{G \mid \exists\, N_1, \ldots, N_r \trianglelefteq H \text{ with } G/N_i \in \mathfrak{C} \text{ and } \textstyle\bigcap_{i=1}^r N_i = 1\}\\
\mathrm{D_0}\mathfrak{C} &= \{G \mid \exists\, G_1, \ldots, G_r \in \mathfrak{C} \text{ with } G = \textstyle\bigtimes_{i=1}^r G_i\}\\
\mathrm{N_0}\mathfrak{C} &= \{G \mid \exists\, S_1, \ldots, S_r \lhd\!\lhd H\colon\ S_i \in \mathfrak{C},\ G = \langle S_1, \ldots, S_r\rangle\}\\
\mathrm{E_\Phi}\mathfrak{C} &= \{G \mid G/N \in \mathfrak{C} \text{ for some } N \trianglelefteq G \text{ with } N \leq \Phi(G)\}
\end{aligned}$$

If C is a closure operation and $\mathfrak{C}$ is a class of groups, we say that $\mathfrak{C}$ is C-closed if $\mathrm{C}\mathfrak{C} = \mathfrak{C}$; if D is another closure operation, we define $\mathrm{CD}\mathfrak{C} = \mathrm{C}(\mathrm{D}\mathfrak{C})$; observe that the latter class need not be D-closed. Therefore we define $\langle \mathrm{C},\mathrm{D}\rangle\mathfrak{C}$ to be the smallest class of groups that is C-closed and D-closed. We also recall that $\mathrm{SD_0} = \langle \mathrm{S},\mathrm{D_0}\rangle$ and that a $\mathrm{SD_0}$-closed class is $\mathrm{R_0}$-closed.

A *formation* $\mathfrak{F}$ is a class of groups that is Q-closed and $\mathrm{R_0}$-closed. If $\mathfrak{C}$ is any class of groups, $\mathrm{QR_0}\mathfrak{C}$ is the smallest formation containing $\mathfrak{C}$ and $\mathrm{QSD_0}\mathfrak{C}$ is the smallest subgroup-closed formation containing $\mathfrak{C}$. A class $\mathfrak{C}$ of groups is said to be *saturated* if it is $\mathrm{E_\Phi}$-closed.

If $\mathfrak{C}$ is a class of groups, define

$$\mathrm{P}\mathfrak{C} = \{G \mid G/N \in \mathfrak{C} \text{ whenever } G/N \text{ is primitive for an } N \trianglelefteq G\}$$

(recall that a group is said to be primitive if it has a maximal subgroup whose core is trivial; see also Section 3.2). A class $\mathfrak{H}$ of finite groups is called a *Schunck class* if it is $\mathrm{P}\mathfrak{H} = \mathfrak{H}$, i.e. the finite group G belongs to $\mathfrak{H}$ if every primitive epimorphic

[1] Some authors impose the additional condition on a class of groups that it must contain the class of groups with one element. We do not make such an assumption.

image of G lies in $\mathfrak{H}$. Schunck classes are Q-closed and D_0-closed, and every saturated formation of finite groups is a Schunck class. It should be observed that P is not a closure operation, but PQ is; moreover for any class $\mathfrak{C}$ of groups, $\mathrm{PQ}\mathfrak{C}$ is the smallest Schunck class containing $\mathfrak{C}$.

The following simple lemma is basic when dealing with Schunck classes.

1.3.1 Lemma. *Let $\mathfrak{Z}$ be a Q-closed class of finite groups. If $\mathfrak{H}$ is a Schunck class, then a group of minimal order in $\mathfrak{Z} \setminus \mathfrak{H}$ is primitive.*

Proof. Let G be a group of minimal order that belongs to $\mathfrak{Z} \setminus \mathfrak{H}$. Since $\mathfrak{Z}$ is Q-closed, $G/N \in \mathfrak{Z}$ for every normal subgroup and so for every $N \neq 1$, we have $G/N \in \mathfrak{H}$ by the minimality of G. Since the group G does not belong to $\mathfrak{H}$, it must be primitive by the very definition of a Schunck class. $\square$

Let G be a finite group and $\mathfrak{C}$ a class of groups. If $\mathfrak{C}$ is R_0-closed, let $G^{\mathfrak{C}}$ denote the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{C}$. Then $G/G^{\mathfrak{C}} \in \mathrm{R}_0\mathfrak{C} = \mathfrak{C}$ and we call $G^{\mathfrak{C}}$ the $\mathfrak{C}$-residual of G; $G^{\mathfrak{C}}$ is obviously a characteristic subgroup of G.

Similarly, if $\mathfrak{C}$ is N_0-closed, then the subgroup generated by all subnormal $\mathfrak{C}$-subgroups of G is a (characteristic) $\mathfrak{C}$-subgroup of G which is called the $\mathfrak{C}$-radical of G and is denoted by $G_{\mathfrak{C}}$.

For any class of groups $\mathfrak{C}$, we define the characteristic $\mathrm{char}(\mathfrak{C})$ of $\mathfrak{C}$ to be the set of primes p such that $\mathfrak{C}$ contains a cyclic p-group, and for every set π of primes, we define

$$\mathfrak{C}_\pi = \{G \in \mathfrak{C} \mid \sigma(G) \subseteq \pi\}.$$

The following lemma shows in particular that the class of finite groups that are the product of (two) nilpotent subgroups is Q-closed and D_0-closed.

1.3.2 Lemma. *Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups and let*

$$\mathfrak{Z} = \{G \mid \exists\, A, B \leq G \text{ with } A \in \mathfrak{X} \text{ and } B \in \mathfrak{Y} \text{ such that } G = AB\}.$$

(i) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are Q-closed, then $\mathfrak{Z}$ is Q-closed.*
(ii) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are D_0-closed, then $\mathfrak{Z}$ is D_0-closed.*

Proof. (i) Let $G \in \mathfrak{Z}$, then G is the product of two subgroups A and B with $A \in \mathfrak{X}$ and $B \in \mathfrak{Y}$. Now if $N \trianglelefteq G$, then G/N is the product of its subgroups AN/N and BN/N by Lemma 1.2.2. Now by an isomorphism theorem, $AN/N \cong A/A \cap N \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}$ and similarly $BN/N \in \mathfrak{Y}$, proving that $G/N \in \mathfrak{Z}$.

(ii) Let $G = \bigtimes_{i=1}^r G_i$ and suppose that $G_i \in \mathfrak{Z}$ for $i = 1, \dots, r$. Then each G_i is the product of two subgroups A_i and B_i with $A_i \in \mathfrak{X}$ and $B_i \in \mathfrak{Y}$. Now $A = \bigtimes_{i=1}^r A_i \in \mathrm{D}_0\mathfrak{X} =$

$\mathfrak{X}$ and $B = \bigtimes_{i=1}^{r} B_i \in \mathrm{D}_0\mathfrak{Y} = \mathfrak{Y}$. Since the subset AB of $G = \bigtimes_{i=1}^{r} G_i$ clearly contains all G_i, G is the product of the $\mathfrak{X}$-group A and the $\mathfrak{Y}$-group B. □

Combining Lemma 1.3.1 and Lemma 1.3.2, we obtain

1.3.3 Lemma. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Q-closed classes of groups and let $\mathfrak{Z}$ be the class of finite groups that are the product of an $\mathfrak{X}$-group with a $\mathfrak{Y}$-group. If $\mathfrak{H}$ is a Schunck class, then a group of minimal order in $\mathfrak{Z} \setminus \mathfrak{H}$ is primitive.*

1.3.4 Lemma. *Let G be a finite group and suppose that $\mathfrak{C}$ is an R_0-closed class of groups. If X is a subgroup of the group G such that XN_1/N_1 and $XN_2/N_2 \in \mathfrak{C}$ for two normal subgroups N_1 and N_2 of G with $N_1 \cap N_2 = 1$, then $X \in \mathfrak{C}$.*

Proof. We have $XN_i/N_i \cong X/X \cap N_i \in \mathfrak{F}$ and $(X \cap N_1) \cap (X \cap N_2) = 1$, therefore $X \in \mathfrak{C}$. □

Remark. This can be used to show that there is even some weak form of R_0-closure of the product of an $\mathfrak{X}$-group and a $\mathfrak{Y}$-group in case the classes $\mathfrak{X}$ and $\mathfrak{Y}$ are R_0-closed: if the group G is the product of its subgroups A and B and there are normal subgroups N_1 and N_2 with $N_1 \cap N_2 = 1$ and such that G/N_i is the product of the $\mathfrak{X}$-subgroup AN_i/N_i and the $\mathfrak{Y}$-subgroup BN_i/N_i for $i = 1, 2$, then G is the product of the $\mathfrak{X}$-subgroup A and the $\mathfrak{Y}$-subgroup B.

A dual of the following statement due to Lockett can be found in [13], II.2.12:

1.3.5 Lemma. *Let G be a finite group and suppose that the class $\mathfrak{F}$ is a Fitting formation, i.e. a $\langle \mathrm{Q}, \mathrm{R}_0, \mathrm{S}_n, \mathrm{N}_0 \rangle$-closed class of groups.*

(i) *If N_1 and N_2 are two normal subgroups of G with $N_1 \cap N_2 = 1$ and $R_i/N_i \in \mathfrak{F}$ for $i = 1, 2$, then $R_1 \cap R_2 \in \mathfrak{F}$.*

(ii) *If R_1/N_1 and R_2/N_2 are the $\mathfrak{F}$-radicals of G/N_1 and G/N_2 respectively, then $R_1 \cap R_2$ is the $\mathfrak{F}$-radical of G.*

Proof. (i) We have

$$(R_1 \cap R_2)N_1/N_1 = (R_1 \cap R_2N_1)/N_1 \trianglelefteq R_1/N_1$$

so that

$$(R_1 \cap R_2)/(R_1 \cap R_2 \cap N_1) \cong (R_1 \cap R_2)N_1/N_1 \in \mathfrak{F}.$$

Similarly,

$$(R_1 \cap R_2)/(R_1 \cap R_2 \cap N_2) \in \mathfrak{F}$$

so that also $R_1 \cap R_2 \in \mathfrak{F}$ because $(R_1 \cap R_2 \cap N_1) \cap (R_1 \cap R_2 \cap N_2) = 1$.

(ii) Let R denote the $\mathfrak{F}$-radical of G. Then RN_i/N_i is a normal $\mathfrak{F}$-subgroup of G/N_i for $i = 1, 2$, therefore $R \leq R_1 \cap R_2$. On the other hand, the normal subgroup $R_1 \cap R_2$ is contained in $\mathfrak{F}$ by (i), proving the other inclusion. $\square$

Chapter 2
Factorizations of finite soluble groups

2.1 Hall subgroups and Hall systems

Finite soluble groups can be characterized by the fact that they possess Hall π-subgroups for every set π of primes (Hall [24]; see also Doerk and Hawkes [13], Sections I.3 and I.4). Note that we also consider G itself and the unit subgroup as Hall subgroups of G for the sets $\pm\!\not{}\mathbb{P}$ and $\varnothing$ of primes respectively.

If G is a finite soluble group, a set Σ of Hall subgroups of G is called a *Hall system* of G if Σ contains a Hall π-subgroup of G for every set of primes π and $HK = KH$ for all H, $K \in \Sigma$, we have $HK = KH$ (observe that this implies that Σ contains exactly one Hall π-subgroup for every set of primes π by Lemma 1.1.1).

Given a Hall p'-subgroup $G_{p'}$ of G for every prime p, it is easy to see that

$$G_\pi = \bigcap_{\pi \subseteq p'} G_{p'}$$

is a Hall π-subgroup of G and the set

$$\Sigma = \{G_\pi \mid \pi \text{ a set of primes}\}$$

is a Hall system of G. The set $\{G_{p'} \mid p \text{ a prime}\}$ is called a *complement basis* of Σ. Consequently, Hall systems always exist in finite soluble groups. Moreover, any two Hall systems are conjugate, i.e. if Σ and T are Hall systems, then there is $g \in G$ such that $\mathrm{T} = \Sigma^g = \{H^g \mid H \in \Sigma\}$.

Also, the set $\{G_p \mid p \text{ a prime}\}$ containing exactly one Sylow p-subgroup G_p of G for each prime is called a *Sylow basis* if $G_pG_q = G_qG_p$ whenever p and q are distinct primes. In this case, a Hall system of G can be obtained by defining the G_π's to be the product of all G_p where $p \in \pi$.

If S is a subgroup of the finite soluble group G, then a Hall system Σ is said to *reduce into* S if for every set of primes π, the intersection of the Hall π-subgroup $G_\pi \in \Sigma$ with S yields a Hall π-subgroup of S; in this case, the set

$$\Sigma \cap S = \{G_\pi \cap S \mid G_\pi \in \Sigma\}$$

is a Hall system of S. Conversely, it can be shown that every Hall system Σ^* of S can be extended to a Hall system Σ of G, i.e. every subgroup contained in Σ^* is contained in some subgroup contained in Σ.

Following Hall [25], a group is said to satisfy the property E_π (existence) if it possesses a Hall π-subgroup; it satisfies C_π (conjugacy) if it satisfies E_π and all its Hall π-subgroups are conjugate; finally, it satisfies D_π (dominance) if it has C_π and moreover every π-subgroup is contained in some Hall π-subgroup. Thus the finite soluble groups are precisely the finite groups that satisfy D_π for all sets π of primes.

The following lemma which relates the Sylow structure of a group G to a given factorization of G was first proved by Wielandt [44] for Sylow subgroups and Hall p'-groups of soluble groups:

2.1.1 Lemma. *Let $G = AB$ be a finite group satisfying D_π. If A_π and B_π are Hall π-subgroups of A and B respectively, then there are $a \in A$ and $b \in B$ such that $A_\pi^a B_\pi^b$ is a Hall π-subgroup of G. Furthermore, $A \cap B$ possesses Hall π-subgroups one of which is $A_\pi^a \cap B_\pi^b$.*

Proof. By the property D_π, the subgroup A_π is contained in a Hall π-subgroup G_π and also B_π is contained in a Hall π-subgroup which is conjugate to G_π by the property D_π, so we have $B_\pi \le G_\pi^g$ for a suitable $g \in G$. Write $g = ab^{-1}$, then $A_\pi^a \le G_\pi^a$ and $B_\pi^b \le G_\pi^{gb} = G_\pi^a$. Therefore, replacing A_π, B_π and G_π by suitable conjugates, we may suppose that A_π and B_π are contained in G_π.

By Lemma 1.1.1, the order of the group G is

$$|G| = \frac{|A||B|}{|A \cap B|};$$

thus the order of a Hall π-subgroup (which equals the π-part of $|G|$) is

$$|G_\pi| = \frac{|A_\pi||B_\pi|}{|A \cap B|_\pi}$$

where $|A \cap B|_\pi$ is the π-part of $|A \cap B|$. Now the order of $A_\pi \cap B_\pi$ is a π-number dividing $|A \cap B|_\pi$; hence

$$|G_\pi| \le \frac{|A_\pi||B_\pi|}{|A_\pi \cap B_\pi|} = |A_\pi B_\pi|.$$

Since we also have $A_\pi B_\pi \subseteq G_\pi$, we must have $A_\pi B_\pi = G_\pi$ as required. So $|G_\pi| = |A_\pi B_\pi|$, from which we deduce that $|A \cap B|_\pi = |A_\pi \cap B_\pi|$; therefore the subgroup $A_\pi \cap B_\pi$ is a Hall subgroup of $A \cap B$. $\square$

Recall that as a consequence of the Feit-Thompson theorem [14], every π-separable group satisfies D_π and that the class of π-separable groups is subgroup-closed so that every π-separable group satisfies the above lemma.

In the case of a soluble group, one is interested in Hall systems rather than single Hall subgroups. The following proposition states the existence of Hall systems in which every Hall subgroup is of the form just described.

2.1.2 Proposition. *If the soluble group G is the product of its subgroups A and B, then there is a Hall system of G of the form $\{A_\pi B_\pi \mid \pi$ a set of primes$\}$ which reduces into A and B.*

Proof. G is soluble, hence satisfies D_π for every set π of primes. By the preceding proposition, for every prime p there is a Hall p'-subgroup $G_{p'}$ of G of the form $G_{p'} = A_{p'}B_{p'}$ for suitable Hall p'-subgroups $A_{p'}$ and $B_{p'}$ of A and B respectively.

Extend the complement bases $\{G_{p'}\}$, $\{A_{p'}\}$ and $\{B_{p'}\}$ to Hall systems $\Sigma = \{G_\pi\}$, $\{A_\pi\}$ and $\{B_\pi\}$ of G, A and B respectively, then it is clear that $A_\pi \leq A \cap G_\pi$ and $B_\pi \leq B \cap G_\pi$ for every set π of primes; since A_π and B_π are Hall subgroups of A and B respectively, we have $A_\pi = A \cap G_\pi$ and $B_\pi = B \cap G_\pi$ and as in the proof of Lemma 2.1.1, $G_\pi = A_\pi B_\pi$. So Σ is the required Hall system. □

Remark. By induction on the number of factors, the last proposition as well as Lemma 2.1.1 can be extended to finite soluble groups that are the product of any (finite) number of subgroups.

2.2 Products of subgroups of coprime order

Let the group G be the product of its subgroups A and B. We establish some criteria for certain subgroups of G to be factorized or prefactorized, based on the orders or indices of A and B.

The next lemma is well-known:

2.2.1 Lemma. *If A and B are subgroups of the finite group G such that $|G : A|$ and $|G : B|$ are coprime, then $G = AB$.*

Proof. Under the hypothesis of the lemma, we have $|G : A \cap B| = |G : A| \cdot |G : B|$; together with the formula from Lemma 1.1.1, we obtain that $|AB| = |G|$ and thus we have $G = AB$. □

Recall that a subnormal subgroup S of a group G has subnormal defect n if n is the least integer such that there is a subnormal series of G of the form

$$G = S_0 \rhd S_1 \rhd \cdots \rhd S_n = S.$$

Such a chain can be obtained by recursively defining $S_0 = G$ and $S_{i+1} = S^{S_i}$ for all positive integers i (see Doerk and Hawkes [13], A.14.7 or Robinson [43], Chapter 13.1).

2.2.2 Lemma. *Let the group G be the product of its subgroups A and B. and suppose that $|G : A|$ and $|G : B|$ are finite.*

(i) *if $N \trianglelefteq G$ and $X = X_G(N)$ denotes its factorizer, then $|N : N \cap A| = |X : X \cap A|$ divides $|G : A|$ and $|N : N \cap B| = |X : X \cap B|$ divides $|G : B|$.*

(ii) *Every subgroup S of G contains a prefactorized subgroup $N \trianglelefteq S$ of finite index; moreover $N \leq A \cap B$.*

(iii) *If the indices of A and B are coprime, then every subnormal subgroup of G is prefactorized.*

Proof. (i) By Lemma 1.2.1, we have $AX = AN$ which shows that AX is a subgroup of G. Therefore

$$|X : X \cap A| = |AX : A| = |AN : A| = |N : N \cap A|$$

which divides $|G : A|$, similarly for B.

(ii) Let S be a subgroup of G. Since the indices of A and B are finite, so is $|G : A \cap B|$ and thus also $|G : (A \cap B)_G|$. Then also $|S : S \cap (A \cap B)_G|$ is finite; moreover $N = S \cap (A \cap B)_G$ is trivially prefactorized and so N is the required normal subgroup of S.

(iii) As a first case, assume that $S \trianglelefteq G$; then S/N is finite where $N = S \cap (A \cap B)_G$; moreover since N is prefactorized, by Lemma 1.2.2, S is a prefactorized subgroup of G/N if (and only if) S/N is prefactorized in G/N. Thus we may assume w.l.o.g. that $N = 1$ and that S is finite.

Now by part (i), $|S : S \cap A|$ divides $|G : A|$ and $|S : S \cap B|$ divides $|G : B|$. Therefore the indices of $A \cap S$ and $B \cap S$ are coprime and we have $S = (A \cap S)(B \cap S)$ by Lemma 2.2.1.

There remains the case when S is a nonnormal subnormal subgroup of defect $n > 1$ in G: we have already proved that $K = S^G \trianglelefteq G$ is prefactorized and that the indices of $K \cap A$ and $K \cap B$ are coprime. Since the subnormal defect of S in K is $n - 1$, by induction on the subnormal defect of S, the subgroup S is a prefactorized subgroup of N, hence is prefactorized in G by Lemma 1.1.6. □

For a periodic group G, i.e. a group in which every element has finite order, we define $\sigma(G)$ to be the set of primes that divide the order of an element of G. If G is finite, then clearly $\sigma(G)$ is the set of prime divisors of $|G|$. If π is a set of primes, the group G is called a π-group if $\sigma(G) \subseteq \pi$, that is, if the order of every element is divisible only by primes in π.

2.2.3 Lemma. *Let $G = AB$ be a (possibly infinite) group and let N be a normal subgroup of G. If X denotes the factorizer of N in G, then $\sigma(X/N) \subseteq \sigma(AN/N) \cap \sigma(BN/N)$.*

Proof. By Lemma 1.2.1, we have $X = (A \cap BN)N = (AN \cap B)N$ and so $X/N \leq AN/N$ and $X/N \leq BN/N$. Therefore $\sigma(X/N) \subseteq \sigma(AN/N) \cap \sigma(BN/N)$. □

2.2.4 Corollary. *Let the group G be the product of its subgroups A and B. If A and B have coprime orders, then every subnormal subgroup of G is factorized.*

Proof. If $N \trianglelefteq G$, this follows from lemma since we must have $X/N = 1$ and therefore $N = X_G(N)$ is factorized. If S is subnormal in G, the result follows by induction on the defect of S in G like in the proof of Lemma 2.2.2. □

Using transfinite induction, the last result can be extended from subnormal subgroups to descendant subgroups, observing that the intersection of arbitrarily many factorized subgroups is factorized by Lemma 1.1.6.

Under certain additional hypotheses, and in particular when G is finite and soluble, it is possible to extend Lemma 2.2.2, (iii) and Corollary 2.2.4 to a statement about all subgroups of the group G: there are factorized (prefactorized) conjugates for all subgroups of the finite group.

2.2.5 Lemma. *Let G be a finite group which is the product of its subgroups A and B with $(|A|, |B|) = 1$ and such that G satisfies D_π and $D_{\pi'}$ where π and π' denote the sets of prime divisors of $|A|$ and $|B|$ respectively. If S is a subgroup of G which satisfies E_π and $E_{\pi'}$, then S possesses a factorized conjugate S^g, i. e. $S^g = (S^g \cap A)(S^g \cap B)$ for a $g \in G$.*

More generally, if H is a factorized subgroup of G that satisfies D_π and $D_{\pi'}$ and contains S, then there is a $h \in H$ such that $S^h \leq H$ is factorized.

Proof. Clearly, A and B are Hall π and π'-subgroups of G. If S_π and $S_{\pi'}$ denote the Hall π- and π'-subgroups of S, then $S = S_\pi S_{\pi'}$. Now S_π and $S_{\pi'}$ are contained in Hall subgroups A^x and B^y of G where x, $y \in G$. By Lemma 1.1.7, there is a $g \in G$ with $A^x = A^{g^{-1}}$ and $B^y = B^{g^{-1}}$. Then we also have $S^g = S^g_\pi S^g_{\pi'}$ with $S^g_\pi \leq A$ and $S^g_{\pi'} \leq B$, hence $S^g = (S^g \cap A)(S^g \cap B)$. Note also that $S^g \geq A \cap B = 1$.

The second statement follows directly from the first, observing that $A \cap H$ and $B \cap H$ are π- and π'-groups respectively. □

Note that Lemma 2.2.5 applies in particular to all subgroups of π-separable groups G when π is the set of prime divisors of $|A|$. To extend Lemma 2.2.5 to the prefactorized case, we need the following

2.2.6 Lemma. *Let the group G be the product of its subgroups A and B. If $A \leq A^*$ and $B \leq B^*$ for two subgroups A^* and B^* of G, then every subgroup that is prefactorized with respect to the factorization $G = AB$ is also prefactorized with respect to the factorization $G = A^*B^*$ of G.*

Proof. Let S be a subgroup of G that is prefactorized with respect to the factorization $G = AB$, i.e. $S = (S \cap A)(S \cap B)$. Clearly, $S = (S \cap A)(S \cap B) \subseteq (S \cap A^*)(S \cap B^*) \subseteq S$, proving that $S = (S \cap A^*)(S \cap B^*)$. Application to the case $S = G$ shows in particular that G is the product of A^* and B^*. □

It is easy to see that the preceding lemma becomes false if we substitute 'factorized' for 'prefactorized': choose any group $G \neq 1$ and let $A = 1$ and $A^* = B = B^* = G$, then the unit subgroup is factorized with respect to $G = AB$ but not with respect to $G = A^*B^*$.

We can now extend Lemma 2.2.5 to a similar result for prefactorized subgroups; observe that the next lemma is true for every soluble group G that is the product of two subgroups which have coprime indices.

2.2.7 Lemma. *Let G be a finite soluble group which is the product of its subgroups A and B and such that $(|G : A|, |G : B|) = 1$. Let π be a set of primes such that A contains a Sylow p-subgroup of G for all $p \in \pi$ and B contains a Sylow p-subgroup of G for all $p \in \pi'$. If G satisfies D_π and $D_{\pi'}$, then for every subgroup S of G there is a prefactorized conjugate S^g, i. e. $S^g = (S^g \cap A)(S^g \cap B)$ for a $g \in G$.*

More generally, let H be a prefactorized subgroup of G such that $A \cap H$ and $B \cap H$ have coprime indices. If H satisfies D_π and $D_{\pi'}$ and contains S, then there is a $h \in H$ such that $S^h \leq H$ is prefactorized.

Proof. Since $(|G : A|, |G : B|) = 1$, the subgroups A and B contain Hall π- and π'-subgroups A_π and $B_{\pi'}$ respectively of G. Since the indices of A_π and $B_{\pi'}$ are coprime, we have $G = A_\pi B_{\pi'}$ by Lemma 2.2.1 and so every subgroup of S possesses a conjugate which is factorized with respect to the factorization $G = A_\pi B'_\pi$ by Lemma 2.2.7. So by Lemma 2.2.6, this conjugate is prefactorized with respect to the factorization $G = AB$.

The second statement follows from the first by considering H instead of G. □

2.3 Groups with many factorized subgroups

Let the finite soluble group G be the product of its subgroups A and B. We have seen in Lemma 2.2.5 and Lemma 2.2.7 that every subgroup S of G has a conjugate which is factorized (prefactorized) if the orders (indices) of A and B are coprime; more precisely, this conjugate can already be found in certain factorized (prefactorized) subgroups containing S.

Conversely, if G is a soluble group which is the product of two subgroups A and B in which every subgroup has a factorized (prefactorized) conjugate in every factorized (prefactorized) subgroup of G, then we will show that the orders (the indices) of A and B are coprime. Observe that for finite soluble groups, the condition that A and B have coprime indices is equivalent to the condition that A and B contain Hall π and π'-subgroups of G for a suitable set π of primes.

First, we deal with the nilpotent case:

2.3.1 Proposition. *Suppose that the finite nilpotent group G is the product of its subgroups A and B and that every subgroup of G has a prefactorized conjugate. Then A and B contain Hall π-and π'-subgroups of G for some set of primes π. In particular, if G is a p-group for some prime p, then $G = A$ or $G = B$.*

Proof. As a first step, we prove the proposition for p-groups G. Suppose that the proposition is false and let G be a counterexample. Clearly, if $G = A$, the proposition is fulfilled with $\pi = \pm\nvdash\mathbb{P}$; similarly if $G = B$ (let $\pi = \varnothing$). So we must have $A < G$ and $B < G$. Since G is nilpotent, A and B are contained in maximal normal subgroups A^* and B^* of G. By Lemma 2.2.6, every subgroup that is prefactorized with respect to the factorization $G = AB$ of G is also prefactorized with respect to the factorization $G = A^*B^*$ so that G is also a counterexample with respect to the latter factorization. Therefore we may suppose w.l.o.g. that A and B are maximal normal subgroups of G.

Let $N = A \cap B$, then G/N is the direct product of the two cyclic groups AN/N and BN/N, both of order p: $A/N = \langle a \rangle N/N$ and $B/N = \langle b \rangle N/N$ for suitable $a \in A$ and $b \in B$. Now the diagonal subgroup $D/N = \langle ab \rangle N/N$ is also a maximal normal subgroup of G/N which is not prefactorized in G/N since it intersects AN/N and BN/N trivially. Since N is prefactorized, by Lemma 1.2.2, D cannot be a prefactorized subgroup of G. So D, being a normal subgroup of G, does not have a prefactorized conjugate. This contradiction shows that we must have $G = A$ or $G = B$.

Now let G be any finite nilpotent group and suppose that P is a Sylow p-subgroup of G for some prime p. If S is a subgroup of P, then every conjugate of S in G is already conjugate to S in P since P is centralized by the Hall p'-subgroup of G. On the other hand, we know by Proposition 2.1.2 that $P = A_pB_p$ where A_p and B_p are the Hall p-subgroups of A and B respectively. So we may apply the first part to P and obtain that $P = A_p$ or $P = B_p$. Therefore every Sylow subgroup of G is contained in A or B. This shows that if π is the set of primes p such that A contains a Sylow p-subgroup of G, then A contains a Hall π-subgroup and B contains a Hall π'-subgroup of G. □

To analyse the case when every subgroup has a factorized conjugate, we need the following simple

2.3.2 Lemma. *Suppose that the finite group G is the product of its subgroups A and B. If A and B have coprime indices and $A \cap B = 1$, then A and B have coprime orders.*

Proof. This follows at once from Lemma 1.1.1. □

Then we obtain

2.3.3 Corollary. *Suppose that the finite nilpotent group G is the product of its subgroups A and B and that every subgroup of G has a factorized conjugate. Then A and B have coprime orders, hence are Hall subgroups of G.*

Proof. Since the unit subgroup is normal and thus factorized, by Lemma 1.1.5, we must have $A \cap B = 1$. So the result follows from Proposition 2.3.1 and Lemma 2.3.2. □

We formulate the result of this section for prefactorized subgroups first.

2.3.4 Proposition. *Suppose that the finite group G is the product of its subgroups A and B.*

(i) *If for every prefactorized subgroup H of G, every subgroup $S \leq H$ has a prefactorized conjugate in H, then A and B have coprime indices; moreover if H is a prefactorized subgroup of G, then $H \cap A$ and $H \cap B$ have coprime indices in H.*

(ii) *Suppose that the indices of A and B in G are coprime and that every subgroup of G satisfies D_π and $D_{\pi'}$ for a set π of primes such that A contains a Sylow p-subgroup of G for every $p \in \pi$ and B contain Sylow p-subgroups of G for all $p \in \pi'$. Let H be a prefactorized subgroup of G satisfying D_π and D'_π. If $|H : H \cap A|$ and $|H : H \cap B|$ are coprime and $S \leq H$, then there is a $h \in H$ such that $S^h \leq H$ is prefactorized.*

Proof. (i) Let p be a prime, then since every finite group satisfies D_p by Sylow's theorem, by Lemma 2.1.1, G possesses a prefactorized Sylow p-subgroup $P = (A \cap P)(B \cap P)$. Now by hypothesis, every subgroup of P has a prefactorized conjugate in P, and so by Proposition 2.3.1, P is contained in A or B. If π is the set of primes for which A contains a Sylow p-subgroup of G, then B contains a Sylow p-subgroup for all remaining primes $p \in \pi'$. So the index of A is a π'-number while that of B is a π-number.

Since every prefactorized subgroup H of G likewise satisfies the hypotheses of (i), it is clear that also $A \cap H$ and $B \cap H$ have coprime indices in H.

(ii) This has already been proved in Lemma 2.2.7. □

Unfortunately, the statements (i) and (ii) are not equivalent. However, it should be noted that (i) implies (ii) if the group G is finite and soluble since in this case, G satisfies D_π for every set of primes π.

In the case when A and B have coprime orders, a much more satisfactory result can be proved, which even holds for arbitrary finite groups.

2.3.5 Theorem. *Let the finite soluble group G be the product of its subgroups A and B. Then the following statements are equivalent:*

(i) *For every factorized subgroup H of G and every subgroup $S \leq H$, there is an $h \in H$ such that S^h is factorized.*

(ii) *The subgroups A and B of G have coprime orders and every subgroup of G satisfies D_π and $D_{\pi'}$, where π is the set of prime divisors of A.*

Proof. (i) $\Rightarrow$ (ii). From Proposition 2.3.4 and Lemma 2.3.2, we obtain that A and B have coprime orders. Since the property (i) is inherited by subgroups and the situation is completely symmetrical for the sets π and π', it remains to show that G itself satisfies D_π: it is clear that G possesses a Hall π-subgroup, namely A. If P is a π-subgroup of G, there is a conjugate P^g of P for some $g \in G$, with $P^g = (P^g \cap A)(P^g \cap B)$ and since B is a π'-group, we have $P^g \cap B = 1$ and so P^g is contained in A. So the conjugates of A are the maximal π-subgroups of G and every π-subgroup is contained in a conjugate of A; hence G satisfies D_π.

(ii) $\Rightarrow$ (i). This is the result of Lemma 2.2.5. □

For an additional result in the case when the finite soluble group G is the product of two nilpotent subgroups, see Section 3.3.

Chapter 3
Subgroups of products of two finite nilpotent groups

3.1 First results about products of nilpotent subgroups

The following theorem, known as the Kegel-Wielandt theorem, is probably the most important theorem about finite groups that are the product of two finite nilpotent subgroups; note that this theorem remains true for products of any finite number of nilpotent subgroups.

3.1.1 Theorem (Wielandt [45] and Kegel [33]). *Let the finite group G be the product of its nilpotent subgroups A and B. Then G is soluble.*

A proof of the Kegel-Wielandt theorem can also be found in [4], Section 2.4 and in [31], VI, § 4.

3.1.2 Proposition. *Let G be a finite group. If A, $B \leq G$ such that $AB^g = B^g A$ for all $g \in G$, then:*

(i) (*Kegel* [33]) *if $AB < G$, then $AB^G < G$ or $A^G B < G$; in particular, A or B is contained in a proper normal subgroup of G.*

(ii) (*Wielandt* [46]) $[A, B] \trianglelefteq A^{AB} \cap B^{AB} \lhd\lhd G$.

Proof. (i) Suppose that $AB^G = A^G B = G$. We have to show that $G = AB$. We proceed by induction on $|G : A| + |G : B|$, observing that the statement is trivial if $G = A = B$. Also, (i) is true if A is normal in G, we may assume that $A^x \neq A$ for some $x \in G$.

Since every conjugate of B permutes with both A and A^x, by Lemma 1.1.3, B also permutes with $A_1 = \langle A, A^x \rangle$. Then we have $G = A_1 B$ by induction hypothesis. Since $x \in G = AB^G$, we can write $x = ab^y$ with $a \in A$, $b \in B$ and $y \in G$. and w.l.o.g., we may assume that $a = 1$.

By Lemma 1.1.7, also $G = A_1 B^y = \langle A, A^x, B^y \rangle = AB^y$ and thus also $G = AB$.

(ii) If $G = AB$, then it is clear that $[A, B]$ and $A^{AB} \cap B^{AB}$ are normal subgroups of G and that $[A, B] \leq A^{AB} \cap B^{AB}$. Therefore we may suppose that $AB < G$.

From (i), we infer that $A^G B < G$ or $AB^G < G$; w.l.o.g. suppose the former. Then by finite induction, $[A, B] \trianglelefteq A^{AB} \cap B^{AB} \trianglelefteq A^G B$ and so also $[A, B] \trianglelefteq A^{AB} \cap B^{AB} \trianglelefteq A^G \trianglelefteq G$ which shows that the subgroups in question are subnormal in G. □

We mention an important special case of the previous proposition:

3.1.3 Corollary. *Let $G = AB$ be a finite group that satisfies D_π for a set π of primes. If A and B have normal Hall π-subgroups A_π and B_π, then:*

(i) *If G is not a π-group, then A_π or B_π is contained in a proper normal subgroup of G.*

(ii) *The subgroups $[A_\pi, B_\pi]$ and $A_\pi^{B_\pi} \cap B_\pi^{A_\pi}$ are subnormal π-subgroups of G.*

(iii) *If A_π and B_π are nilpotent, then the normal Hall π-subgroup $A_\pi \cap B_\pi$ of $A \cap B$, and thus every π-subgroup of $A \cap B$, is a subnormal subgroup of G.*

Proof. By Lemma 2.1.1, the group G possesses a Hall π-subgroup of the form $A_\pi^a B_\pi^b$ for some $a \in A$ and $b \in B$. Since the Hall π-subgroups of A and B are normal, we have $A_\pi^a B_\pi^b = A_\pi B_\pi$. Thus by Lemma 1.1.7, every conjugate of A_π permutes with every conjugate of B_π, and consequently $[A_\pi, B_\pi]$ and $A_\pi^{B_\pi} \cap B_\pi^{A_\pi}$ are subnormal π-subgroups of G by Proposition 3.1.2. □

3.1.4 Proposition (Pennington [40]). *Let $G = AB$ be a finite group satisfying D_π. If A and B have normal Hall π-subgroups, then $O_\pi(G) = \big(A \cap O_\pi(G)\big)\big(B \cap O_\pi(G)\big)$ and $A_\pi \cap B_\pi \leq O_\pi(G)$.*

Proof. Write $O = O_\pi(G)$. Since G satisfies D_π, by Lemma 2.1.1, there is a Hall π-subgroup of G of the form $H = A_\pi B_\pi$. We show that O is a factorized subgroup of $H = A_\pi B_\pi$ (which does not, however, imply that O is factorized in G because H is not necessarily factorized).

Consider the group G/O: since the normal Hall π-subgroups $A_\pi O/O$ and $B_\pi O/O$ are normal in AO/O and BO/O respectively, every conjugate of $A_\pi O/O$ permutes with $B_\pi O/O$ and so by Proposition 3.1.2, $A_\pi O/O \cap B_\pi O/O$ is a subnormal π-subgroup of G. Therefore $A_\pi O \cap B_\pi O \leq O$, showing that $O = A_\pi O \cap B_\pi O$. Thus by Lemma 1.2.1, O equals its factorizer in H, i.e. O is factorized in H as required. □

This leads to the following

3.1.5 Theorem (Amberg [1], Pennington [40]). *If G is the product of two finite nilpotent subgroups A and B, then $F(G)$, the Fitting subgroup of G, is factorized. Therefore the subgroup $A \cap B$ is subnormal in G.*

Proof. Since $F(G)$ is the product of all $O_p(G)$ and by Sylow's theorem, G satisfies $D_{\{p\}}$ for all primes p dividing the order of G, it is clear from the preceding Proposition 3.1.4 that $F(G)$ is prefactorized. Also, $A \cap B$ is nilpotent, hence is the direct product of all $A_p \cap B_p$ and so $F(G)$ also contains $A \cap B$. This also shows that $A \cap B \lhd\lhd F(G)$ is subnormal in G. □

We will see in Section 3.6 that the last two results can be generalized to radicals with respect to arbitrary Fitting formations.

Also, the fact that $A \cap B$, and hence every subgroup of $A \cap B$, is a subnormal subgroup of G has lead to the following result of Maier [36] and Wielandt [47]: they show that if the finite group G is the product of two subgroups A and B and $S \leq A \cap B$ such that $S \lhd\lhd A$ and $S \lhd\lhd B$, then $S \lhd\lhd G$. Further results in this direction can also be found in [38] and [11].

3.1.6 Corollary. *If the finite group G is the product of its nilpotent subgroups A and B and $N \trianglelefteq G$, then the factorizer and every prefactorizer of N is subnormal in G.*

Proof. Consider the factor group G/N. Then the subgroup $AN/N \cap BN/N$ is subnormal in G/N the preceding theorem. Therefore the subgroup $X = AN \cap BN$ is subnormal in G. On the other hand, by Lemma 1.2.1, X equals the factorizer of N in G.

For prefactorizers S, the result follows from the fact that $N \leq S \leq X$ and X/N is nilpotent. □

3.1.7 Corollary. *Let G be a finite group which is the product of its nilpotent subgroups A and B. If G possesses a nilpotent normal subgroup N such that $G = AN = BN$, then G is nilpotent.*

Proof. If N is a nilpotent normal subgroup of G, then its factorizer (with respect to the factorization $G = AB$) is contained in the factorized subgroup $F(G)$ of G. But by Lemma 1.2.1, the factorizer of N equals $AN \cap BN = G$ and so G is nilpotent. □

For generalizations of the last result to certain classes of infinite groups, we refer the reader to Section 6.3 of Amberg, Franciosi and de Giovanni [4]. We will also see in Corollary 3.5.2 that in fact N does not necessarily have to be normal in G.

The following proposition shows that every finite group G that is the product of two proper nilpotent subgroups has a proper factorized normal subgroup.

3.1.8 Proposition (Kegel [33]). *If the finite group G is the product of its nilpotent subgroups A and B and $A \neq B$, then A or B is contained in a proper normal subgroup of G.*

Proof. If $G = 1$, there is nothing to prove. Therefore suppose by finite induction that the proposition is true for all groups of smaller order than $|G|$ and let N be a minimal normal subgroup of G. If $AN \neq BN$, then by induction hypothesis, AN/N or BN/N is contained in a proper normal subgroup of G/N and therefore also A or B is contained in a proper normal subgroup of G.

Therefore we may assume that $AN = BN = G$, and since A, B and N are nilpotent, G is nilpotent by Corollary 3.1.7. Then, however, at least one of A and B is contained in a maximal subgroup of G which is normal in G. □

Kegel's result has been extended by Amberg, Franciosi and de Giovanni [3] to certain classes of (possibly infinite) groups G which are the product of their subgroups $A_1, \dots, A_r$ satisfying some nilpotency condition: if at least one of the subgroups A_i is properly contained in G, then some A_i is contained in a proper normal subgroup of G. This result holds in particular if the subgroups A_i are finite and nilpotent.

On the other hand, a similar result about minimal normal subgroups of a product of finite nilpotent subgroups does not hold: If p is any prime, J. D. Gillam [19] gives an example of a p-group P of order p^6 which is the product of two subgroups A and B but A and B do not contain normal subgroups of P. This also shows that the centre of P cannot be prefactorized: since $A \cap Z(P)$ and $B \cap Z(P)$ are normal subgroup of P, we must have $Z(P) \neq 1 = \big(A \cap Z(P)\big)\big(B \cap Z(P)\big)$

We mention a consequence Proposition 3.1.8 that might be of interest.

3.1.9 Proposition. *Let G be a finite group. If A and B are nilpotent subgroups of G such that $AB^g = B^gA$ for all $g \in G$, then $A \cap B$ is subnormal in G.*

Proof. If $A = B$ $(= G)$, the group G is nilpotent and hence every subgroup of G is subnormal. Therefore we may suppose that $A \neq B$ and by Proposition 3.1.8, one of the factors, say A, is contained in a normal subgroup N of G. Now by the modular law, for all $g \in G$,

$$A(N \cap B^g) = N \cap AB^g = N \cap B^gA(N \cap B^g)A$$

for all $g \in G$ which shows that A permutes with every conjugate of $B \cap N$. Therefore by induction on the order of G, $A \cap B = A \cap B \cap N$ is subnormal in N, hence in G. □

3.2 Primitive groups that are the product of two nilpotent subgroups

Let G be group that possesses a faithful representation as a a transitive permutation group on the set X. Recall that G is called *primitive* if we have $|Y| = 1$ for every subset Y of X such that $Yg = Y$ or $Yg \cap Y = \varnothing$ for all $g \in G$. By [43], 7.2.3, G is primitive if and only if for every $x \in X$, the stabilizer $M = \{g \in G \mid x^g = x\}$ of x is a maximal subgroup of G. Since G is transitive, the conjugates of M in G are precisely the stabilizers of the elements of X, so it is clear that $M_G = 1$. On the other hand, if an arbitrary group G possesses a maximal subgroup M with $M_G = 1$, then the permutation representation of G on the cosets of M is clearly faithful and has M as a stabilizer, hence G is primitive by the theorem stated above.

The next simple lemma gives a sufficient condition for a finite group to be primitive. Theorem 3.2.2 below will show that this condition is also necessary:

3.2.1 Lemma. *Let G be a finite group. If G possesses a maximal subgroup $M < G$ that supplements every minimal normal subgroup of G, then G is primitive and M is a stabilizer of G.*

Proof. Suppose that $M_G \neq 1$, then M_G contains a minimal normal subgroup N of G. But then $N \leq M = MN = G$, contradicting the maximality of M. □

The class of finite primitive groups can be divided into three disjoint subclasses as the following theorem (Baer [6], see also Doerk and Hawkes [13], A.15.2) shows:

3.2.2 Theorem. *Let G be a finite group. Then the following statements are equivalent:*

(i) *G is primitive with stabilizer M;*

(ii) *G satisfies one of the following statements:*

  (1) *G has a unique minimal normal subgroup N; moreover N is abelian, $N = C_G(N)$ and N is complemented by M.*

  (2) *G has a unique minimal normal subgroup N; N is non-abelian, $C_G(N) = 1$ and N is supplemented by M. Furthermore, if V is a minimal supplement to N, then $N \cap V \leq \Phi(V)$,*

  (3) *G has exactly two isomorphic non-abelian minimal normal subgroups N and N^*. Moreover $C_G(N) = N^*$ and $C_G(N^*) = N$ and $N \cong N^* \cong NN^* \cap M$. M complements N and N^*, and if $V < G$ supplements N and N^*, then V complements N and N^*. Also, $M \cap NN^*$ is a (normal) subgroup of M isomorphic with N (and N^*).*

If G is a finite soluble group, clearly every minimal normal subgroup of G is abelian, so a finite primitive soluble group must belong to the class of groups described in (1) above.

Primitive finite soluble groups can be characterized in several ways:

3.2.3 Lemma. *Let G be a finite soluble group and suppose that N is a minimal normal subgroup of G whose order is divisible by the prime p. Then the following statements are equivalent:*

(i) *G is primitive;*
(ii) *$N = C_G(N)$;*
(iii) *$N = O_p(G)$ for the prime p and $O_{p'}(G) = 1$;*
(iv) *$N = F(G)$;*
(v) *$\Phi(G) = 1$ and N is the only minimal normal subgroup of G.*

Proof. Note first that N is elementary abelian of exponent p since G is soluble.

(i) $\Rightarrow$ (ii): This follows directly from Theorem 3.2.2 since N is abelian.

(ii) $\Rightarrow$ (iii): If $N = C_G(N)$ and $N^* \neq N$ is a minimal normal subgroup, then $[N.N^*] \leq N \cap N^* = 1$, which shows that $N^* \leq C_G(N) = N$, a contradiction. Therefore N is the unique minimal normal subgroup of G. So if $O_{p'}(G) \neq 1$, we must have $N \leq O_{p'}(G)$ which is, of course, impossible, N being a p-group. Therefore $O_{p'}(G) = 1$. Let $O = O_p(G)$, then $N \leq O$; moreover O is nilpotent and so $Z(O) \neq 1$. $Z(O)$ is characteristic in O, hence normal in G so that $N \leq Z(O)$ and so $O \leq C_G(N) = N$, proving that $N = O = O_p(G)$.

(iii) $\Rightarrow$ (iv): Clearly, the Hall p'-subgroup of $F(G)$ is characteristic in $F(G)$, so it is normal in G and hence contained in $O_{p'}(G) = 1$. Therefore $F(G)$ is a p-group and $F(G) = O_p(G) = N$.

(iv) $\Rightarrow$ (v): $\Phi(G)$ is a characteristic subgroup of G properly contained in $F(G) = N$. Therefore we must have $\Phi(G) = 1$. Clearly, every minimal normal subgroup of the soluble group G is abelian, therefore contained in the Fitting subgroup of G. Since $F(G) = N$ is itself a minimal normal subgroup, it follows that N is the unique minimal normal subgroup of G.

(v) $\Rightarrow$ (i): Since $\Phi(G) = 1$, there is a maximal subgroup M of G which does not contain N. If $M_G \neq 1$, the normal subgroup M_G must contain the unique minimal normal subgroup N of G. But then $N \leq M$, a contradiction. So we must have $M_G = 1$ and G is primitive. □

We will now analyse primitive groups G which are the product of two finite nilpotent subgroups. The main result, namely that if G is non-nilpotent, then one of these subgroups is a Sylow p-subgroup and the other a Hall p'-subgroup, is due to Gross [22].

First, we prove a generalization of this result which is based on the proof of Gross' statement that can be found in [4], Lemma 2.5.2. For the definition of a π-separable group, we refer the reader to Section 4.1.

3.2.4 Lemma. *Let the π-separable group G be the product of its subgroups A and B and suppose that A and B have normal Hall π'-subgroups. Furthermore, suppose that G does not contain nontrivial normal π'-subgroups and let $O = O_\pi(G)$. Then*

(i) $C_G(O) \leq O$;

(ii) *if O is the unique minimal normal subgroup of G, then A or B is a π-group;*

(iii) *if in addition A and B have normal Hall π-subgroups, then AO/O and BO/O are Hall π- and Hall π'-subgroups of G/O and O is factorized;*

(iv) *if in addition, O is abelian and $O < G$, then A and B are Hall π- and Hall π'-subgroups of G. Thus every subgroup of G has a factorized conjugate (in particular, every normal subgroup of G is factorized), and if $G = AC = BC$ for some $C \leq G$, then $G = C$.*

Proof. (i) This is a result of Hall and Higman [26], Lemma 1.2.3.

(ii) Let $A_{\pi'}$ and $B_{\pi'}$ be the Hall π'-subgroups of A and B respectively, then $A_{\pi'}B_{\pi'}$ is a Hall π'-subgroup of G by Lemma 2.1.1, because every π-separable group satisfies D_π and $D_{\pi'}$, and since $A_{\pi'}$ and $B_{\pi'}$ are normal subgroups of A and B respectively, we have $A_{\pi'}B^g_{\pi'} = B^g_{\pi'}A_{\pi'}$ for all $g \in G$ by Lemma 1.1.7. Now by Proposition 3.1.2, $[A^g_{\pi'}, B^g_{\pi'}]$ is a subnormal π'-subgroup of G and as such it is contained in $O_{\pi'}(G) = 1$. Therefore also $[A^G_{\pi'}, B^G_{\pi'}] = 1$. If, say, B is not a π-group, then $B^G_{\pi'} \neq 1$, thus $O \leq B^G_{\pi'}$ and $[A_{\pi'}, O] \leq [A^G_{\pi'}, B^G_{\pi'}] = 1$. Therefore, $A_{\pi'} \leq C_G(O) \leq O$ and A must be a π-group.

(iii) Suppose w.l.o.g. that A is a π-group. As above, G has a Hall π-subgroup of the form $A_\pi B_\pi = AB_\pi$ by Lemma 2.1.1. Now $B_\pi \trianglelefteq B$ and therefore $B^G_\pi = B^A_\pi$ which is contained in the π-group AB_π. So B^G_π is a normal π-subgroup of G, and B_π must be contained in O. Hence AO/O is a π-group and BO/O is a π'-group. Moreover by Corollary 2.2.4, $1 = O/O$ is a factorized subgroup of G/O and so by Lemma 1.2.2, O is factorized.

(iv) Since $O_{\pi\pi'}(G)/O$ is a π'-group, it is contained in $BO/O = B_{\pi'}O/O$. Now if O is abelian, then both O and $B_{\pi'}$ centralize $B_\pi \leq O$ and so $O_{\pi\pi'}(G)$ centralizes B_π: Therefore $B_\pi \leq Z\big(O_{\pi\pi'}(G)\big) \leq C_G(O) = O$. By the minimality of O, we have either $Z\big(O_{\pi\pi'}(G)\big) = 1$ or $Z\big(O_{\pi\pi'}(G)\big) = C_G(O)$. From the latter, it follows directly that $O = O_{\pi\pi'}(G) = G$, contrary to our assumption that $O < G$. So we must have $Z\big(O_{\pi\pi'}(G)\big) = 1$ and hence $B_\pi = 1$. So by Lemma 1.1.1, A and B must be Hall subgroups of G. The remaining statements follow from Lemma 2.2.5 and Lemma 1.1.8. $\square$

We summarize the results of Lemma 3.2.4 for primitive groups that are the product of two nilpotent subgroups A and B

3.2.5 Lemma. *Let the finite soluble group $G = AB$ be the product of its nilpotent subgroups A and B. Suppose that G is primitive and denote with N the unique minimal normal subgroup of G. Then*

(i) *$N = F(G) = O_p(G)$ is an elementary abelian p-group for some prime p and $O_{p'}(G) = 1$;*

(ii) *A or B is a Sylow p-subgroup of G containing N; if $A \neq B$, then the other is a Hall p'-subgroup. In particular, $A \cap B = 1$.*

(iii) *If $F_2/N = F(G/N)$, then F_2/N is a p'-group and every prime divisor of $|B|$ divides already $|F_2|$.*

(iv) *If A and B are proper subgroups of G, then A and B are maximal nilpotent subgroups of G.*

(v) *If $A \neq B$, then every subgroup of G possesses a factorized conjugate.*

(vi) *If $A \neq B$, then $G = BC = CA$ for a subgroup C, then $G = C$.*

Proof. First, we deal first with the case when G is nilpotent. Since then a maximal subgroup of G is normal, a primitive nilpotent group is cyclic of prime order. So either $G = A = B$ or $A = 1$ and $B = G$ or $A = G$ and $B = 1$. In all three cases, it is easy to see that the lemma holds. Therefore we may suppose from now on that A, B and N are properly contained in G.

(i) follows directly from Lemma 3.2.3.

(ii), (v) and (vi) follow directly from Lemma 3.2.4 with $\pi = \{p\}$, since we have already excluded the case when $G = N$.

(iii) If P/N is a Sylow p-subgroup of F_2/N, then P is a normal p-subgroup of G since F_2/N is nilpotent. This shows that $P = N$ and F_2/N is a p'-group. So F_2/N is contained in the Hall p'-subgroup BN/N of G/N. Therefore $Z(BN/N) \leq Z(F_2/N) \leq C_{G/N}(F_2/N) \leq F_2/N$. Since $Z(BN/N) \cong Z(B)$ is the direct product of the centres of the primary components of BN/N, this shows that every prime divisor of B divides F_2/N.

(iv) Suppose that A is contained in a proper nilpotent subgroup H of G, then $G = HB$ and, applying the same arguments to $G = HB$, we obtain $H \cap B = 1$. So by Lemma 1.1.1, we must have $|A| = |H|$ and thus $A = H$, similarly for B. Therefore A and B are maximal nilpotent subgroups of G. □

For further results about the structure of G, see also Lemma 4.2.1.

3.3 Pronormal and abnormal subgroups

Recall that a subgroup P of the finite group G is called *pronormal* if P and P^g are conjugate in the subgroup $\langle P, P^g\rangle$ for every $g \in G$. If $P \leq H \leq G$, then, of course, P is also a pronormal subgroup of H.

Pronormal subgroups of finite soluble groups can also be characterized in the following way:

3.3.1 Theorem (Mann [39]). *A subgroup P of a finite soluble group G is pronormal if and only if each Hall system of G reduces into exactly one conjugate of P.*

A proof of this can also be found in [13], I.6.6.

3.3.2 Proposition. *Let G be a finite soluble group which is the product of its nilpotent subgroups A and B. If S is a prefactorized subgroup of G, then the Hall system*

$$\Sigma = \left\{A_\pi B_\pi \mid \pi \subseteq \pm\nvdash\mathbb{P}\right\}$$

reduces into S. Moreover, Σ is the only Hall system of G that reduces into every prefactorized subgroup of G. In particular Σ is the only Hall system of G consisting entirely of prefactorized Hall subgroups.

Proof. By Proposition 2.1.2, Σ is a Hall system of G. Since S is the product of its nilpotent subgroups $S \cap A$ and $S \cap B$, also the subgroup S possesses a Hall system of the form

$$\left\{S_\pi = (S \cap A)_\pi (S \cap B)_\pi \mid \pi \subseteq \pm\nvdash\mathbb{P}\right\}$$

where $(S \cap A)_\pi$ and $(S \cap B)_\pi$ are the Hall π-subgroups of $S \cap A$ and $S \cap B$ respectively. Since A_π and B_π are the unique Hall π-subgroups of A and B respectively, we must have $(S \cap A)_\pi \leq A_\pi$ and $(S \cap B)_\pi \leq B_\pi$. This shows that for every set π of primes, the Hall π-subgroup $S_\pi = (S \cap A)_\pi (S \cap B)_\pi$ of S is contained in the π-subgroup $S \cap A_\pi B_\pi$, and therefore that $S_\pi = S \cap A_\pi B_\pi$. Since this is true for every set π of primes, we have shown that Σ reduces into S.

Now suppose that Σ^* is another Hall system of G reducing into every prefactorized subgroup of G. If π is a set of primes, then Σ^* reduces into the Hall subgroup $A_\pi B_\pi$ of G. Therefore $A_\pi B_\pi$ must be contained in a Hall π-subgroup $H \in \Sigma^*$ and so $A_\pi B_\pi \in \Sigma^*$. Continuing like this for every set π of primes, we have $\Sigma = \Sigma^*$ as required. $\square$

The next proposition is a direct consequence of Proposition 3.3.2 and Theorem 3.3.1.

3.3.3 Proposition. *Let the finite group G be the product of its nilpotent subgroups A and B. If P is a pronormal subgroup of G, then P has a unique prefactorized conjugate.*

Proof. If P^g and P^h are prefactorized conjugates of P for $g, h \in G$, then by Proposition 3.3.2, the Hall system

$$\Sigma = \big\{(A_\pi B_\pi \mid \pi \subseteq \pm\nvdash\mathbb{P}\big\}$$

reduces into both P^g and P^h. Since P is pronormal, we must have $P^g = P^h$ by Theorem 3.3.1. $\square$

A subgroup S of G is called *abnormal* if $g \in \langle S, S^g \rangle$ for every $g \in G$. Of course, every abnormal subgroup of G is pronormal in G. Also, if $S \leq H \leq G$, then, S is abnormal in H. The following lemma shows in particular that an abnormal subgroup cannot be contained in a proper normal subgroup of G.

Remark. By induction on the number of factors, it can be proved that Proposition 3.3.2 and Proposition 3.3.3 even holds for groups G that are the product of finitely many finite nilpotent subgroups.

3.3.4 Lemma. *Let S be an abnormal subgroup of the (possibly infinite) group G. If $S \leq K \trianglelefteq H \leq G$, then $H = K$. Thus $S = N_G(S)$.*

Proof. For all $h \in H$, we have $S^h \leq K^h = K$ and therefore $h \in \langle S, S^h \rangle \leq K$. So we have $H = K$. The second statement follows from the fact that $S \trianglelefteq N_G(S)$. $\square$

3.3.5 Proposition. *Let $G = AB$ be the product of the finite nilpotent subgroups A and B. Then every abnormal subgroup of G possesses exactly one factorized conjugate.*

Proof. Let S be an abnormal subgroup of G. If $S = G$, then there is nothing to prove, so suppose that $S < G$ and let M be a maximal subgroup of G which contains S. By Lemma 3.3.4, M cannot be normal in G and therefore $M_G < M < G$ and G/M_G cannot be nilpotent. Hence we may apply Lemma 3.2.5 to show that M/M_G possesses a factorized conjugate, say, M^g/M_G. So by Lemma 1.2.2, also M^g is factorized in G. Now S^g is clearly an abnormal subgroup of M^g and by induction on the order of G, the subgroup S^g possesses a factorized conjugate in M^g which is also factorized in G by Lemma 1.1.6 and is clearly conjugate in G to S as required. The uniqueness of this conjugate follows from Proposition 3.3.3. $\square$

Since every maximal nonnormal subgroup of a group G is abnormal, we also have

3.3.6 Corollary. *Suppose that the group G is the product of its finite nilpotent subgroups A and B. Then every maximal nonnormal subgroup of G has a unique factorized conjugate.*

The following results shows that in general, one cannot expect every conjugate of a subgroup to be prefactorized or factorized.

3.3.7 Corollary. *If the finite group G is the product of its subgroups A and B and every maximal subgroup of G is prefactorized, then G is nilpotent.*

Proof. Let M be a maximal subgroup of G. Since M is a pronormal subgroup of G, M has a unique prefactorized conjugate by Proposition 3.3.3. Since on the other hand, every conjugate of G is a maximal subgroup, hence prefactorized, it follows that $M \trianglelefteq G$. So every maximal subgroup of G is normal and G must be nilpotent. □

This result can be used to characterize those products of two finite nilpotent subgroups in which every subgroup is factorized or prefactorized.

3.3.8 Proposition. *Let G be a finite soluble group which is the product of its nilpotent subgroups A and B. Then the following statements are equivalent:*

(i) *Every subgroup of the nilpotent group G has a factorized (prefactorized) conjugate.*

(ii) *G is nilpotent; its subgroups A and B have coprime orders (coprime indices).*

(iii) *Every subgroup of G is factorized (prefactorized).*

Proof. (i) ⇒ (ii): This follows immediately from Proposition 2.3.1 and Corollary 2.3.3 respectively.

(ii) ⇒ (iii). Obviously, every subgroup of G is subnormal, hence this follows from Lemma 2.2.2 and Corollary 2.2.4 respectively.

(iii) ⇒ (i): We have to show that G is nilpotent. But this follows at once from Corollary 3.3.7 □

The next lemma shows that abnormal subgroups can be characterized as the normalizers of the pronormal subgroups:

3.3.9 Lemma. *Let G be a (possibly infinite) group. Then the normalizer of a pronormal subgroup of G is abnormal in G. Therefore a subgroup of G is abnormal if and only if it is pronormal and self-normalized.*

Proof. Suppose that P is a pronormal subgroup of G and let $g \in G$. We have to show that $g \in H = \langle N_G(P), N_G(P)^g \rangle$. Since $\langle P, P^g \rangle \leq H$, there is an $h \in H$ such that $P^h = P^g$. Hence $gh^{-1} \in N_G(P)$ and consequently $g \in \langle N_G(P), H \rangle = H$, proving that $N_G(P)$ is abnormal.

From this, it follows that a self-normalized pronormal subgroup is abnormal. On the other hand, we have already proved in Lemma 3.3.4 that abnormal subgroups are self-normalized. □

A factorized (abnormal) subgroup clearly contains $A \cap B$, hence we obtain

3.3.10 Corollary. *Let the group G be the product of its nilpotent subgroups A and B. If P is any pronormal subgroup of G, then it is normalized by some conjugate of $A \cap B$.*

Suppose that the group G is the product of its nilpotent subgroups A and B and that P is a prefactorized pronormal subgroup of G. Then $N_G(P)$ is abnormal in G, and therefore we know that some conjugate of $N_G(P)$ is factorized. The next proposition shows that $N_G(P)$ itself is factorized (and therefore no other conjugate of $N_G(P)$ can be factorized). This means that the investigation whether a pronormal subgroup P of G has a prefactorized (factorized) conjugate may be reduced to finding the factorized conjugate $N_G(P)^g$ of its normalizer and to checking whether the conjugate P^g of P that lies in $N_G(P)^g = N_G(P^g)$ is a prefactorized (factorized) normal subgroup of $N_G(P)^g$.

3.3.11 Proposition. *Let the group G be the product of its subgroups A and B and let P be a pronormal subgroup of G. Then the following statements are equivalent:*

(i) *P is prefactorized;*

(ii) *$N_G(P)$ is factorized and P is a prefactorized (normal) subgroup of $N_G(P)$.*

Proof. Suppose that P is a prefactorized pronormal subgroup of G. Since $N_G(P)$ is abnormal in G, by Proposition 3.3.5, there is a $g \in G$ such that $N_G(P)^g = N_G(P^g)$ is factorized. Now by Proposition 3.3.2, the Hall system Σ defined there reduces into $N_G(P^g)$ and since $P^g \trianglelefteq N_G(P^g)$, Σ reduces into P^g as well.

On the other hand, Σ also reduces into P by Proposition 3.3.2 since P is prefactorized, and so we must have $P^g = P$ by Proposition 3.3.3, showing that the normalizer $N_G(P) = N_G(P^g)$ of P is factorized.

The other implication follows directly from Lemma 1.1.6. □

3.3.12 Corollary. *Let the group G be the product of its subgroups A and B. If P is a pronormal subgroup of G such that $N_G(P)$ is factorized, then the factorizer $X_G(P)$ of P can be written $X_G(P) = N_A(P)P \cap N_B(P)P$ and it possesses a triple factorization*

$$X_G(P) = A^*P = B^*P = A^*B^*$$

where $P \trianglelefteq X$, $A^* = N_A(P) \cap N_B(P)P \leq A$ and $B^* = N_A(P)P \cap N_B(P) \leq B$.

Proof. $N_G(P)$ is factorized, so we have $X_G(P) \leq N_G(P)$ which shows that $X_G(P)$ is also the factorizer of P in $N_G(P) = N_A(P)N_B(P)$. Since $P \trianglelefteq N_G(P)$, the corollary now follows directly from Lemma 1.2.1. □

It also follows from Proposition 3.3.11 that the system normalizer of the Hall system Σ is factorized:

3.3.13 Corollary. *Suppose that the group G is the product of its finite nilpotent subgroups and let Σ be the Hall system defined in Proposition 3.3.2. Then $N_G(H)$ is factorized for all $H \in \Sigma$ and also the system normalizer*

$$N_G(\Sigma) = \bigcap_{H \in \Sigma} N_G(H)$$

of Σ is factorized.

Proof. Since every Hall subgroup $H \in \Sigma$ is prefactorized, it follows from Proposition 3.3.11 that $N_G(H)$ is factorized for all $H \in \Sigma$. The system normalizer $N_G(\Sigma)$ of Σ is factorized since by Lemma 1.1.6, the intersection of any number of factorized subgroups is factorized. □

Question. Recall that the hypercentre of a finite soluble group equals the intersection of its system normalizers. Is the hypercentre of a finite group G that is the product of two nilpotent subgroups prefactorized? (This question is motivated by the fact that a corresponding result holds for $\mathcal{F}$-injectors: if an $\mathcal{F}$-injector of a group G that is the product of two nilpotent subgroups is factorized (prefactorized), then also its core, the $\mathcal{F}$-radical of G, is factorized (prefactorized); see Proposition 3.6.4 for details.)

On the other hand, by an example of Heineken [28], the hypercentre does not necessarily contain $A \cap B$ and hence it is not factorized. Also, an example of Gillam [19] of a finite p-group G which is the product of two subgroups A and B which do not contain normal subgroups of G shows that it is possible to have $A \cap Z(G) = 1$ and $B \cap Z(G) = 1$ whence the centre $Z(G)$ of G is not necessarily prefactorized.

3.4 Projectors

Let $\mathfrak{X}$ be a class of groups. An $\mathfrak{X}$-subgroup X of a group G is $\mathfrak{X}$-*maximal* in G if for every $\mathfrak{X}$-subgroup Y of G with $X \leq Y$, it follows that $X = Y$. A subgroup P of G is called an $\mathfrak{X}$-projector if PN/N is an $\mathfrak{X}$-maximal subgroup of G/N for every normal subgroup N of G.

Recall from Section 1.3 that a class $\mathfrak{H}$ of finite groups is called a *Schunck class* if $G \in \mathfrak{H}$ whenever all primitive homomorphic images of G lie in $\mathfrak{H}$. The Schunck classes of finite soluble groups are precisely the classes for which every finite soluble group G possesses an $\mathfrak{H}$-projector. In this case, the $\mathfrak{H}$-projectors of G are conjugate; see e.g. [13], III.3.10, 3.21, and by [13], III.3.22, if P is an $\mathfrak{H}$-projector of G and $P \leq H \leq G$, then P is an $\mathfrak{H}$-projector of H as well. This also shows that $\mathfrak{H}$-projectors are pronormal.

In view of Proposition 3.3.11, we examine first the case when a projector is normal in G; cf. Blessenhohl and Gaschütz [10].

3.4.1 Lemma. *Let G be a finite soluble group and suppose that $\mathfrak{H}$ is a Schunck class such that an $\mathfrak{H}$-projector P is normal in G. Then $P = O^{\pi'}(G)$ where π is the characteristic of $\mathfrak{H}$.*

Proof. Suppose first that a prime $p \in \pi$ divides the order of G/P. Then G/P has a subgroup of order p which belongs to $\mathfrak{H}$ and so $1 = P/P$ is not $\mathfrak{H}$-maximal in G/P, contradicting the fact that P is an $\mathfrak{H}$-projector. Therefore G/P is a π'-group and $O^{\pi'}(G) \leq P$.

Now suppose that $O^{\pi'}(G) < P$ and let M be a maximal normal subgroup of P containing $O^{\pi}(G)$. Then P/M is a π'-group and since G (and hence P) is soluble, P/M is cyclic of prime order $p \in \pi'$. But $P/M \in \mathrm{Q}\mathfrak{H} = \mathfrak{H}$ whose characteristic is π, hence does not contain cyclic p-groups. This contradiction shows that $P = O^{\pi'}(G)$. $\square$

We will also make use of the following property of saturated formations.

3.4.2 Proposition. *If $\mathfrak{F}$ is a saturated formation of characteristic π, then every group $G \in \mathfrak{F}$ is a π-group.*

Proof. See e.g. Doerk and Hawkes [13], IV.4.3. $\square$

This result is false for arbitrary Schunck classes, e.g. for the classes of finite π-perfect (soluble) groups, i.e. the class of finite (soluble) groups G for which $O^{\pi}(G) = G$, if $\varnothing \subsetneqq \pi \subsetneqq \pm\nvdash\mathbb{P}$.

For saturated formations, the next result has been proved by Heineken [28]:

3.4.3 Theorem. *Let $\mathfrak{H}$ be a Schunck class and suppose that the group G is the product of its finite nilpotent subgroups A and B.*

(i) *If* $\mathrm{char}(\mathfrak{H})$ *contains $\sigma(A) \cap \sigma(B)$, then G has a unique factorized $\mathfrak{H}$-projector.*

(ii) *If $\mathfrak{H}$ is a saturated formation, then G has a unique prefactorized $\mathfrak{H}$-projector.*

Proof. Note first that in both cases, the uniqueness of the factorized or prefactorized projector follows from Proposition 3.3.3 since an $\mathfrak{H}$-projector is pronormal. Therefore it remains to prove the existence of such projectors in G.

Let P be an $\mathfrak{H}$-projector of G. Since $N_G(P)$ is an abnormal subgroup of G, by Proposition 3.3.5, it has a factorized conjugate $N_G(P)^g = N_G(P^g)$ where $g \in G$. Therefore by Proposition 3.3.11, it is enough to show that P^g is a prefactorized subgroup of $N_G(P^g)$, hence we may assume that $G = N_G(P^g)$ and $P \trianglelefteq G$. Then we have $P = O^{\pi'}(G)$ by the preceding Lemma 3.4.1, where $\pi = \mathrm{char}(\mathfrak{H})$.

Now under the hypothesis of (i), the groups AP/P and BP/P have coprime orders since π contains all common prime divisors of A and B. Therefore by Corollary 2.2.4, $1 = P/P$ is a factorized subgroup of G/P and so by Lemma 1.2.2, P is factorized in G.

In the case of (ii), P is a π-group by Proposition 3.4.2 since $\mathfrak{H}$ is a saturated formation. On the other hand, since G/P is a π'-group, P is the unique Hall π-subgroup of G, which must be prefactorized by Lemma 2.1.1. $\square$

Let the group G be the product two nilpotent subgroups A and B. The following example shows that none of the following subgroups of G is necessarily prefactorized: G', $G^{\mathfrak{N}}$, $[A, B]$, $O^\pi(G)$ for all sets $\pi \neq \varnothing$ of primes. (Observe that since all these subgroups are normal, there cannot exist prefactorized conjugates, and, except possibly for $[A, B]$, no automorphism of G can map these subgroups to a prefactorized subgroup).

Since for every finite group G, the subgroup $O^\pi(G)$ is the unique $\mathfrak{H}$-projector of G where $\mathfrak{H}$ is the Schunck class of π-perfect groups, the example also shows that statement (ii) of Theorem 3.4.3 becomes false when $\mathfrak{H}$ is a Schunck class that is not a saturated formation.

Observe also that the subgroups A and B are abelian in the following

3.4.4 Example. Let p and q be distinct primes and let N be a q-dimensional vector space over $F = GF(p)$, written additively. Define an automorphism α of N of order q by

$$\alpha\colon (x_1, \ldots, x_q) \mapsto (x_q, x_1, \ldots, x_{q-1}).$$

Now let G be the semidirect product of N by $Q = \langle\alpha\rangle$, then G is (isomorphic with) the standard wreath product of a group of order p with a group of order q.

Let D be the diagonal subgroup

$$D = \big\{(x, \ldots, x) \mid x \in F\big\}$$

of N. Clearly, α centralizes D (in fact, $D = Z(G)$) and so $A = D \times Q$ is an abelian subgroup of G. Next, let

$$B = \big\{(x_1, \ldots, x_{q-1}, 0) \mid x_i \in F\big\},$$

then B is a $q-1$-dimensional F-subspace of N. Since D has F-dimension 1 and clearly B and D intersect trivially, we have $N = DB$ and thus $G = QDB = AB$, and since $B \leq N$, we have $A \cap B = QD \cap N \cap B = D(Q \cap N) \cap B = D \cap B = 1$.

By [13], A.18.4, the derived subgroup G' of G equals

$$M = \Big\{(x_1, \ldots, x_q) \mid x_i \in F,\ \sum_{i=1}^{q} x_i = 0\Big\}$$

and $O^p(G) = Q^G = MQ$. Since obviously $N = O^p(G)$, we also have that $G^{\mathfrak{N}} = O^p(G) \cap O^q(G) = MQ \cap N = M(Q \cap N) = M = G'$ and $[A, B] = [Q, B] = M$ because $[A, B] \leq G' = N$ and $[Q, B] \leq [A, B]$ contains the elements of the form

$$(\ldots, -1, 1, \ldots)$$

which generate M.

Since clearly $A \cap M \leq N$, we have $A \cap M = A \cap N \cap M = QD \cap N \cap M = D(Q \cap N) \cap M = D \cap M$; thus

$$A \cap M = \bigl\{(x, \ldots, x) \mid x \in F, q \cdot x = 0\bigr\}.$$

Since $q \neq p$, we have $x = 0$ whenever $q \cdot x = 0$; thus $A \cap M = 1$. Furthermore, $B \cap N$ equals the set

$$\Bigl\{(x_1, \ldots, x_{q-1}, 0) \mid x_i \in F,\ \sum_{i=1}^{q-1} x_i = 0\Bigr\}$$

which has dimension $q - 2$ while the dimension of N itself is $q - 1$. This shows that $(M \cap A)(M \cap B) = M \cap B < M$ and so M, the derived subgroup of G as well as its nilpotent residual, is not prefactorized.

Also, if $O^p(G) = MQ$ were prefactorized, then also its Sylow p-subgroup M would have to be prefactorized by Lemma 2.1.1. So MQ cannot be prefactorized either.

3.5 $\mathfrak{H}$-maximal subgroups

The results about $\mathfrak{H}$-projectors obtained in Section 3.4 can also be obtained via a more general result about $\mathfrak{H}$-maximal subgroups. The following result will also be used in Section 3.6 to prove certain results for saturated Fitting formations.

3.5.1 Theorem. *let $\mathfrak{H}$ be a Schunck class and suppose that the group G is the product of its nilpotent subgroups A and B. If H is an $\mathfrak{H}$-maximal subgroup of G, then:*

(i) *If* $\operatorname{char}(\mathfrak{H})$ *contains $\sigma(A) \cap \sigma(B)$, then H possesses a factorized conjugate in G.*

(ii) *Without any hypothesis on the characteristic of $\mathfrak{H}$, if $\mathfrak{H}$ is a saturated formation, then H has a prefactorized conjugate in G.*

Proof. (i) If $1 \neq N$ denotes a proper normal subgroup of G, then the $\mathfrak{H}$-group HN/N is contained in an $\mathfrak{H}$-maximal subgroup Y/N of G/N. By induction on the order of G, there is a $g \in G$ such that Y^g/N, and therefore Y^g, is factorized. Then $H^g \leq Y^g$ is also an $\mathfrak{H}$-maximal subgroup of Y^g, so if $Y^g < G$, by induction on $|G|$ again, H^g possesses a factorized conjugate in Y^g which is also a factorized subgroup of G by Lemma 1.1.6. Therefore we may suppose that for all normal subgroups $N \neq 1$, the factor group G/N is an $\mathfrak{H}$-subgroup. Of course, we may also exclude the case when $G \in \mathfrak{H}$ since then $H = G$ is factorized. But then by Lemma 1.3.1, G must be primitive.

Next, consider the case when $A = B$ $(= G)$: since G is primitive and nilpotent, G is cyclic of prime order p, say, and p is a common prime divisor of A and B, hence $\mathfrak{H}$ contains G, a case that has already been treated.

So we are left with the case when $A \neq B$. Then, however, H possesses a factorized conjugate in G by Lemma 3.2.5 and the proof of (i) is complete.

(ii) Let $\pi = \mathrm{char}(\mathfrak{H})$. Since $\mathfrak{H}$ is a saturated formation, H is a π-group by Proposition 3.4.2, and, replacing H by a suitable conjugate if necessary, we may suppose that H is contained in the Hall subgroup $A_\pi B_\pi$ of G. Now $\sigma(A_\pi) \cap \sigma(B_\pi) \leq \pi = \mathrm{char}(\mathfrak{H})$ whence H, being also an $\mathfrak{H}$-maximal subgroup of $A_\pi B_\pi$, is a factorized subgroup of $A_\pi B_\pi$ by part (i). Thus it is a prefactorized subgroup of G by Lemma 1.1.6. □

Observe also that, $\mathfrak{H}$-projectors being in particular $\mathfrak{H}$-maximal subgroups, Example 3.4.4 shows that Theorem 3.5.1, (ii) cannot be extended to arbitrary Schunck classes.

Remark. The result of Theorem 3.4.3 can also be obtained combining Theorem 3.5.1 (existence) and Proposition 3.3.3 (uniqueness).

Note also that there may well be more than one factorized $\mathfrak{H}$-maximal subgroup and that these need not even be isomorphic: consider the symmetric group of degree 3 which is the product of a cyclic group A of order 2 and a cyclic group B of order 3. Let $\mathfrak{H} = \mathfrak{N}$, then A and B are maximal nilpotent subgroups of G which are factorized but not isomorphic.

As a corollary to Theorem 3.5.1, we obtain the following result which has been proved by Fransman [17] and Amberg and Fransman [5] for Schunck classes containing all finite nilpotent groups. The latter result in turn generalizes a result of Peterson [42] for saturated formations.

3.5.2 Corollary. *Suppose that G is a finite group with subgroups A, B and C where A and B are nilpotent and $C \in \mathfrak{H}$. If $G = AB = AC = BC$, then $G \in \mathfrak{H}$, provided that $\sigma(A) \cap \sigma(B) \subseteq \mathrm{char}(\mathfrak{H})$.*

Proof. Let D be an $\mathfrak{H}$-maximal subgroup of G containing C. Then also $AD = BD = G$. Now by Theorem 3.5.1, D^g is factorized for some $g \in G$ and by Lemma 1.1.8, $G = D \in \mathfrak{H}$. □

This result becomes false as soon as we drop the condition $\sigma(A) \cap \sigma(B) \subseteq \mathrm{char}(\mathfrak{H})$: let $G = A = B$ be a cyclic group of order p for some prime p and suppose that $\mathfrak{H}$ is a class of groups (not necessarily a Schunck class) whose characteristic does not contain p.

Then $G \notin \mathfrak{H}$ and therefore the unit subgroup is the only $\mathfrak{H}$-subgroup of G, and of course $G = A \cdot 1 = B \cdot 1 = AB$.

3.6 Injectors and radicals

A *Fitting set* $\mathcal{F}$ is a set of subgroups of the group G which satisfies

(FS1) If $M \in \mathcal{F}$, then $M^g \in \mathcal{F}$ for all $g \in G$,

(FS2) If M, $N \in \mathcal{F}$ and M and N normalize each other, then $MN \in \mathcal{F}$.

(FS3) If $G \in \mathcal{F}$ and $N \trianglelefteq G$, then $N \in \mathcal{F}$.

Let $\mathfrak{F}$ be a *Fitting class*, i.e. an $\langle \mathrm{S_n}, \mathrm{N_0} \rangle$-closed class of finite groups, then it is easy to see that the set $\{S \leq G \mid S \in \mathfrak{F}\}$ is a Fitting set of the group G. Moreover, if $\mathcal{F}$ is a Fitting set of G and $H \leq G$, then the set $\mathcal{F}_H = \{S \leq H \mid S \in \mathcal{F}\}$ is a Fitting set of S.

Let $\mathcal{F}$ be a set of subgroups of the group G that satisfies (FS1) and (FS2). In analogy to the $\mathfrak{F}$-radical defined in Section 1.3, the $\mathcal{F}$*-radical* of G is the subgroup of G generated by all subnormal $\mathcal{F}$-subgroups of G. To simplify notation, if $H \leq G$, we will denote the $\mathcal{F}_H$-radical of H again with $H_{\mathcal{F}}$.

3.6.1 Lemma. *Let $\mathcal{F}$ be a set of subgroups of the finite group G satisfying* (FS1) *and* (FS2) *above. Then:*

(i) *Every subnormal $\mathcal{F}$-subgroup of G is contained in a normal $\mathcal{F}$-subgroup of G.*

(ii) *$G_{\mathcal{F}}$ is the unique maximal normal $\mathcal{F}$-subgroup of G.*

(iii) *If $S \lhd\lhd G$, then $S_{\mathcal{F}} \leq S \cap G_{\mathcal{F}}$, and if $\mathcal{F}$ is a Fitting set and $S \lhd\lhd G$, then $S_{\mathcal{F}} = S \cap G_{\mathcal{F}}$.*

Proof. (i) If $S \trianglelefteq G$, (i) is trivially true. Therefore by induction on the subnormal defect of S, the subgroup S is contained in a normal $\mathcal{F}$-subgroup R of $S^G \trianglelefteq G$. Since all G-conjugates of R are normal $\mathcal{F}$-subgroups of S^G by (FS1) and their product R^G is an $\mathcal{F}$-subgroup by (FS2), R^G is a G-invariant $\mathcal{F}$-subgroup containing S as required.

(ii) By (i), $G_{\mathcal{F}}$ is generated by all normal $\mathcal{F}$-subgroups of G and so it is an $\mathcal{F}$-subgroup by (FS2).

(iii) $S_{\mathcal{F}}$ is a subnormal $\mathcal{F}$-subgroup of G by (ii), therefore $S_{\mathcal{F}} \leq S \cap G_{\mathcal{F}}$. If $\mathcal{F}$ is a Fitting set, by (FS3), $S \cap G_{\mathcal{F}}$ is a subnormal subgroup of $G_{\mathcal{F}}$ and so $S \cap G_{\mathcal{F}} \in \mathcal{F}$. Therefore $S \cap G_{\mathcal{F}}$ is a normal $\mathcal{F}$-subgroup of S and as such contained in $S_{\mathcal{F}}$. ◻

An $\mathcal{F}$*-injector* I is a subgroup I of G such that $I \cap S$ is $\mathcal{F}$-maximal in S for all subnormal subgroups S of G.

By mere definition, it is clear that for every subnormal subgroup S of G, the subgroup $I \cap S$ is an $\mathcal{F}_S$-injector of S.

The following proposition was originally proved for Fitting classes by Fischer, Gaschütz and Hartley [15]. For a proof, see e.g. Doerk and Hawkes [13], VIII.2.9 and 2.13.

3.6.2 Proposition. *Let $\mathcal{F}$ be a Fitting set of the finite soluble group G. Then G possesses exactly one conjugacy class of $\mathcal{F}$-injectors. If I is an $\mathcal{F}$-injector of G and $I \leq H \leq G$, then I is also an $\mathcal{F}_H$-injector of H. In particular, the $\mathcal{F}$-injectors of G are pronormal subgroups of G.*

The next lemma shows that if $\mathcal{F}$-injectors exist and form a single conjugacy class, then their core equals the $\mathcal{F}$-radical.

3.6.3 Lemma. *Let G be a finite soluble group. Then the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of G equals the intersection of all $\mathcal{F}$-injectors of G.*

Proof. Clearly, the intersection of all $\mathcal{F}$-injectors of G is a normal $\mathcal{F}$-subgroup of G, hence it is contained in $G_{\mathcal{F}}$. On the other hand, if I is an $\mathcal{F}$-injector, $I \cap G_{\mathcal{F}}$ is an $\mathcal{F}$-injector of $G_{\mathcal{F}}$, therefore $I \cap G_{\mathcal{F}} = G_{\mathcal{F}}$. This shows that $G_{\mathcal{F}}$ is contained in every $\mathcal{F}$-injector of G. □

Next, we show that the $\mathcal{F}$-radical of a finite group G which is the product of two nilpotent subgroups is always prefactorized (factorized) if an injector of G is prefactorized (factorized).

3.6.4 Proposition. *Let G be the product of its finite nilpotent subgroups A and B and let $\mathcal{F}$ be a Fitting set of G such that G possess a prefactorized (factorized) $\mathcal{F}$-injector I. Then the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of G is prefactorized (factorized).*

Proof. We prove the proposition by induction on the order of G. Observe that the case $G = 1$ is trivial.

Let $X = AG_{\mathcal{F}} \cap BG_{\mathcal{F}}$ be the factorizer of $G_{\mathcal{F}}$ in G. X is a factorized subnormal subgroup of G by Corollary 3.1.6, therefore $I \cap X$ is a prefactorized (factorized) $\mathcal{F}$-injector of X by Lemma 3.6.1. Now $G_{\mathcal{F}} = X \cap G_{\mathcal{F}} = X_{\mathcal{F}}$ which shows that the $\mathcal{F}$-radical of G coincides with that of X. So if $X < G$, by induction hypothesis, we must have $X_{\mathcal{F}} = G_{\mathcal{F}}$ which is prefactorized (factorized).

In the other case, we have $G = X = AG_{\mathcal{F}} = BG_{\mathcal{F}}$ and hence $G/G_{\mathcal{F}}$ is nilpotent. This shows that $I/G_{\mathcal{F}} \lhd\lhd G/G_{\mathcal{F}}$ is subnormal in $G/G_{\mathcal{F}}$. Thus I is a subnormal $\mathcal{F}$-subgroup of G and therefore I is contained in $G_{\mathcal{F}}$. So $G_{\mathcal{F}} = I$ is prefactorized (factorized). □

If, with the notation of the preceding Proposition 3.6.4, the $\mathcal{F}$-injector I is contained in a factorized subgroup H of G, then I is also an $\mathcal{F}_H$-injector of H by Proposition 3.6.2. This shows that $H_{\mathcal{F}}$ is prefactorized (factorized) for all such subgroups H. If, on the other hand, $H_{\mathcal{F}}$ is prefactorized (factorized) for all factorized subgroups containing an $\mathcal{F}$-injector of G, the following proposition shows that G has a unique factorized $\mathcal{F}$-injector.

3.6.5 Proposition. *Let G be the product of its finite nilpotent subgroups A and B and let $\mathcal{F}$ be a Fitting set of G. Let I denote an $\mathcal{F}$-injector of G and define*

$$\mathcal{S} = \{S \leq G \mid S \leq G \text{ is factorized and contains a conjugate of } I\}.$$

Then the following statements are equivalent:

(i) *Every $S \in \mathcal{S}$ contains a prefactorized (factorized) $\mathcal{F}$-injector.*

(ii) *For every $S \in \mathcal{S}$, the $\mathcal{F}$-radical $S_{\mathcal{F}}$ is prefactorized (factorized).*

Proof. By Proposition 3.6.4, we have already seen that whenever S possesses a prefactorized (factorized) $\mathcal{F}$-injector, then the corresponding radical $S_{\mathcal{F}}$ must be prefactorized (prefactorized). This proves the necessity of our condition.

Conversely, suppose that the proposition is true for all groups of order smaller than $|G|$ (observe that the statement is trivial if $G = 1$) and let I be an $\mathcal{F}$-injector of G. Since I is pronormal by Proposition 3.6.2, by Proposition 3.3.11, we may assume w.l.o.g. that $N_G(I)$ is factorized and thus that $N_G(I) \in \mathcal{S}$. Now if $N_G(I) < G$, then by induction hypothesis, I is prefactorized (factorized) in $N_G(I)$, hence also in G by Lemma 1.1.6. In the other case, when $N_G(I) = G$, we have $I \trianglelefteq G$ and thus $I = G_{\mathcal{F}}$ which is prefactorized (factorized) by hypothesis. □

For Fitting classes, we obtain the following result:

3.6.6 Corollary. *Let $\mathfrak{V}$ be an* s*-closed class of finite groups and let $\mathfrak{F}$ be a Fitting class. Then the following statements are equivalent:*

(i) *For every group $G \in \mathfrak{V}$, the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ is prefactorized (factorized).*

(ii) *Every group $G \in \mathfrak{V}$ that is the product of two nilpotent subgroups has a unique prefactorized (factorized) $\mathfrak{F}$-injector*

Proof. Suppose that the group $G \in \mathfrak{V}$ is the product of its nilpotent subgroups A and B. Let $\mathcal{F} = \{S \leq G \mid S \in \mathfrak{F}\}$, then $\mathcal{F}$ is a Fitting set of G, and since also all subgroups of G belong to $\mathfrak{V}$, the equivalence of (i) and (ii) follows directly from the equivalence of the corresponding statements of Proposition 3.6.5. □

Since we know from Theorem 3.1.5 that the Fitting subgroup (and thus $F_n(G)$ for all $n \geq 1$) is factorized, we obtain the following result about nilpotent injectors:

3.6.7 Corollary. *Let the group G be the product of its finite nilpotent subgroups A and B. Then G has a factorized $\mathfrak{N}^k$-radical and a unique factorized $\mathfrak{N}^k$-injector where*

$\mathfrak{N}^k$ is the Fitting class of groups of nilpotent length $\leq k$; in particular G has a unique factorized $\mathfrak{N}$-injector.

Proof. By Theorem 3.1.5, $F(G)$, and hence $F_k(G)$ for every nonnegative integer k, is factorized for every finite group G that is the product of two nilpotent subgroups. Now F_k is the $\mathfrak{N}^k$-radical of G, and so by the preceding proposition (with $\mathfrak{V}$ the class of all finite groups, say), G has a unique factorized $\mathfrak{N}^k$-injector. □

The following example[1] shows that even when $\mathfrak{F}$ is a Fitting class and the finite soluble group G is the product of two cyclic groups A and B, its $\mathfrak{F}$-radical is not necessarily prefactorized and G need not have prefactorized $\mathfrak{F}$-injectors. Observe also that in the following example, every subgroup of A permutes with every subgroup of B.[2] The second part of the example shows that for Fitting sets $\mathcal{F}$, a (normally embedded) $\mathcal{F}$-injector is not necessarily prefactorized (factorized), even when the $\mathcal{F}$-radical is prefactorized (factorized).

3.6.8 Example. Let G be a finite soluble group and

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

be a principal series of G. Every p-chief factor G_i/G_{i-1} can be regarded as a vector space over $GF(p)$, the field with p elements, on which every $g \in G$ acts as a nonsingular linear transformation $\lambda_i(g)$. Let

$$\Delta(g) = \prod \det\bigl(\lambda_i(g)\bigr)$$

where the product is taken over all i such that G_i/G_{i-1} is a p-group. Δ is a homomorphism from G to the multiplicative group of $GF(p)$. (Note that this homomorphism does not depend on the choice of the principal series of G by the Jordan-Hölder theorem). Then $\mathfrak{D}(p)$, the class of finite soluble groups G such that $\Delta(g) = 1$ for all $g \in G$, is a normal Fitting class, i.e. a Fitting class such that every finite soluble group has a (unique) normal $\mathfrak{D}(p)$-injector, as has been shown by Blessenhohl and Gaschütz, [10]; see also Hawkes [27] or Doerk and Hawkes [13], IX.2.14 (b).

Let S and S^* denote the symmetric groups on the sets $\{1, 2, 3\}$ and $\{1^*, 2^*, 3^*\}$. Put $G = S \times S^*$, then the $\mathfrak{D}(3)$-injector of G is the normal subgroup

$$D = \langle (123), (1^*2^*3^*), (12)(1^*2^*) \rangle$$

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[1] I wish to thank K. Doerk for pointing out the relevance of the following example in the context of products of nilpotent groups.

[2] Maier [37] calls such subgroups A and B totally permutable; this is, of course, a property much stronger than just the permutability of A and B.

which has index 2 in G.

Now let $A = \langle(123), (1^*2^*)\rangle$ and $B = \langle(1^*2^*3^*), (12)\rangle$. It is easy to see that A and B are cyclic groups of order 6 and that $A \cap B = 1$.

Therefore $|AB| = 36 = |G|$ and so $G = AB$. Finally, $A \cap D$ and $B \cap D$ are both cyclic of order 3 and so D cannot be factorized (or prefactorized, which is the same in this case).

For the second example, let $P = \langle(12)(1^*2^*)\rangle$ be a Sylow 2-subgroup of D and let $\mathcal{F} = \{1, P^g \mid g \in G\}$. Then it follows from the fact that P is normally embedded in G or simply by direct calculation that $\mathcal{F}$ is a Fitting set of G with injector P and radical $G_{\mathcal{F}} = 1$. Thus $G_{\mathcal{F}} = 1$ is a factorized subgroup of G.

On the other hand, since $N_G(P) = \langle(12), (1^*2^*)\rangle$ is factorized, by Proposition 3.3.11, if G had a factorized $\mathcal{F}$-injector, then it would have to be contained in $N_G(P)$. So P would have to be prefactorized, which is evidently not the case.

Question. Is there an example of a Fitting class $\mathfrak{F}$ and a finite group G which is the product of two nilpotent subgroups such that G does not have a prefactorized (factorized) $\mathfrak{F}$-injectors but nevertheless $G_{\mathfrak{F}}$ is prefactorized (factorized)?

To obtain a further result in the case when $\mathfrak{F}$ is a saturated Fitting formation, we have to employ the results of Section 3.5. Then we obtain the following proposition, whose first statement has been proved by Amberg and Fransman [5] in the case when $\mathfrak{H}$ contains all finite nilpotent groups.

3.6.9 Proposition. *Let the finite group G be the product of its nilpotent subgroups A and B and let $\mathfrak{H} = \mathrm{N}_0\mathfrak{H}$ be a Schunck class whose characteristic contains $\sigma(A) \cap \sigma(B)$ (an N_0-closed saturated formation). Then*

(i) *$G_{\mathfrak{H}}$ is factorized (prefactorized).*

(ii) *If G admits $\mathfrak{H}$-injectors, then every $\mathfrak{H}$-injector has a factorized (prefactorized) conjugate.*

Proof. (i) Suppose first that $\mathfrak{H}$ is a Schunck class whose characteristic contains $\sigma(A) \cap \sigma(B)$. Then the factorizer X of $R = G_{\mathfrak{H}}$ has a triple factorization

$$X = (A \cap BR)R = (AR \cap B)R = (AR \cap B)(A \cap BR)$$

by Lemma 1.2.1, and since $(AR \cap B)$ and $(A \cap BR)$ are nilpotent and $R \in \mathfrak{H}$, we have $X \in \mathfrak{F}$ by Corollary 3.5.2. On the other hand, by Corollary 3.1.6, X is subnormal in G, hence $X \leq R$ and therefore $R = X$ is factorized.

Next, let $\mathfrak{H}$ be a saturated formation and put $\pi = \mathrm{char}(\mathfrak{X})$. Let $A_\pi B_\pi$ be the prefactorized Hall π-subgroup of G. Since $\mathfrak{H}$ is a saturated formation, every $\mathfrak{H}$-group is a π-group, hence $R = G_{\mathfrak{H}}$ must be contained in the prefactorized subgroup $A_\pi B_\pi$ of G. Hence $A_\pi B_\pi$ contains a prefactorizer X of R, and by Lemma 1.2.4, $X = A^*R = B^*R = A^*B^*$ for suitable subgroups A^* and B^* of A_π and B_π respectively. Now we have $\sigma(A^*) \cap \sigma(B^*) \subseteq \pi = \mathrm{char}(\mathfrak{H})$, hence by Corollary 3.5.2, we have that $X \in \mathfrak{H}$. Since every prefactorizer of G is subnormal in G by Corollary 3.1.6, it follows that $X = R$ as in the first part.

(ii) The second part follows directly from Section 3.5 since $\mathfrak{H}$-injectors are in particular $\mathfrak{H}$-maximal subgroups. □

Remark. Proposition 3.6.9, (ii) becomes false if $\mathrm{N}_0 H = H$ is only a Schunck class but not a saturated formation by Example 3.4.4, for the classes of π-perfect groups are also N_0-closed.

Note also that we do not claim that a (pre)factorized injector be unique in the preceding proposition. However, this is the case when $\mathfrak{F}$ is a saturated Fitting formation, i.e. a Fitting class that is also a saturated formation. Perhaps it is also worth noting that saturated Fitting formations include s-closed Fitting classes, for these are saturated formations by a theorem of Bryce and Cossey [7], [8]. An outline of their proof can also be found in [13], Chapter XI.

3.6.10 Corollary. *Let the finite group G be the product of its nilpotent subgroups A and B and suppose that $\mathfrak{F}$ is both a Fitting class and a Schunck class (a saturated Fitting formation whose characteristic contains $\sigma(A) \cap \sigma(B)$). Then:*

(i) *$G_{\mathfrak{F}}$ is prefactorized (factorized).*

(ii) *G has a unique prefactorized (factorized) $\mathfrak{F}$-injector.*

Proof. It follows directly from Proposition 3.6.9 that $G_{\mathfrak{F}}$ is prefactorized (factorized). Therefore the second statement follows from Corollary 3.6.6, taking $\mathfrak{V}$ to be the class of all finite soluble groups. □

Chapter 4

Structural properties of a product of two finite nilpotent groups

4.1 Fundamental results about the p-length of a p-soluble group

A group G is said to be π-separable for a set π of primes if the series

$$1 \trianglelefteq O_{\pi'}(G) \trianglelefteq O_{\pi'\pi}(G) \trianglelefteq O_{\pi'\pi\pi'}(G) \trianglelefteq \dots$$

reaches G after a finite number of steps. Then the number of nontrivial π-factors in that series of G is called the π-length of G. The group G is called π-soluble if the π-factors in the above series are soluble. If $\pi = \{p\}$, the group G is also called p-separable (p-soluble), and we also write $l_p(G)$ instead of $l_{\{p\}}(G)$.

Bounds on the p-length of a finite p-soluble group in terms of certain invariants of its Sylow p-subgroups will play an important role in the sequel. If G is any finite p-soluble group for the prime p and P is a Sylow p-subgroup of G, define the integers $b_p(G)$, $c_p(G)$, $d_p(G)$ and $e_p(G)$ as follows: let $p^{b_p}(G)$ be the order of P, $c_p(G)$ its nilpotency class, $d_p(G)$ the derived length of P and $p^{e_p}(G)$ its exponent. $l_p(G)$ will denote the p-length of G.

The following observations are the basis for the bounds on the p-length of G that we will cite below:

4.1.1 Theorem. *Let G be a finite p-soluble group, where p is a prime. Then:*

(i) (*Hall and Higman* [26]) *If $c_p(G) > 0$, then $c_p(G/O_{p'p}(G) < c_p(G)$.*

(ii) (*Hall and Higman* [26]) *If $p \neq 2$ and $d_p(G) > 0$, then $d_p(G/O_{p'p}(G) < d_p(G)$.*

(iii) (*Berger and Gross* [9]) *If $d_2(G) > 0$, then $d_2(G/O_{2'22'2}(G) < d_2(G)$. If the Sylow 2-subgroups or the Sylow 3-subgroups of G are abelian and $d_2(G) > 0$, then already $d_2(G/O_{2'2}(G) < d_2(G)$.*

(iv) (*Hall and Higman* [26]) *$e_p(G/O_{p'p}(G) < e_p(G)$, provided that $e_p(G) > 0$ and G satisfies one of the following conditions holds:*

(a) *$p \neq 2$ and p is not a Fermat prime;*

(b) *p is an odd Fermat prime and the Sylow 2-subgroups of G are abelian;*

(c) $p = 2$ *and the Sylow* q*-subgroups of* G *are abelian for all Mersenne primes* q.

The following bounds on the p-length $l_p(G)$ of a finite p-soluble group G in terms of the structure of a Sylow p-subgroup are known. Unless otherwise noted, the bounds are due to Hall and Higman [26].

4.1.2 Theorem. *Let the finite group* G *be a* p*-soluble group. Then:*

$$\text{(i)}\quad b_p(G) \geq \begin{cases} \dfrac{p^{l_p}-1}{p-1} & \textit{if } p \textit{ is odd and not a Fermat prime*;} \\ \dfrac{(p-2)^{l_p+1} - l_p(p-3) - p + 2}{(p-3)^2} & \textit{if } p \textit{ is a Fermat prime} > 3; \\ 2^{l_p-1} + l_p - 1 & \textit{if } p = 3; \\ \frac{1}{2} l_p(l_p+1) & \textit{in any case.} \end{cases}$$

$$\text{(ii)}\quad c_p(G) \geq \begin{cases} p^{l_p-1} & \textit{if } p \textit{ is odd and not a Fermat prime*;} \\ \dfrac{(p-2)^{l_p}-1}{(p-3)} & \textit{if } p \textit{ is a Fermat prime} > 3; \\ \min\{l_p, 2^{l_p-1}\} & \textit{if } p = 3; \\ l_p & \textit{in any case.} \end{cases}$$

$$\text{(iii)}\quad d_p(G) \geq \begin{cases} l_p & \textit{if } p \geq 3\textit{*;} \\ l_p & \textit{if } p = 2 \textit{ and } d_3(G) \leq 1 \ \textit{(Berger and Gross [9]);} \\ \min\{l_p, \frac{1}{2} l_p + 1\} & \textit{if } p = 2 \ \textit{(Berger and Gross [9]).} \end{cases}$$

$$\text{(iv)}\quad e_p(G) \geq \begin{cases} l_p & \textit{if } p \textit{ is neither 2 nor a Fermat prime*;} \\ \frac{1}{2} l_p & \textit{if } p \textit{ is an odd prime*;} \\ \min\{l_p, \frac{1}{2}(l_p+1)\} & \textit{if } p = 2 \ \textit{(Gross [21]).} \end{cases}$$

Here $[x]$ denotes the greatest integer $\leq$ the real number x.

In the same paper, Hall and Higman also show that the inequalities marked * are best possible in the sense that for every integer n, there is a group G of order $\geq n$ such that the bound is attained. In our context, it is of interest that the examples furnished by Hall and Higman are groups whose order is divisible by only two primes, whence they are the product of their Sylow subgroups and thus products of two nilpotent subgroups.

Observe that all functions of l_p in the previous theorem are increasing when $l_p \geq 0$ so that the inequalities indeed bound l_p in terms of $b_p(G)$, $c_p(G)$, $d_p(G)$ and $e_p(G)$.

4.2 Bounds on the π-lengths of products of two finite nilpotent subgroups

We recall that the classes of π-separable groups of π-length $\leq k$ form saturated s-closed Fitting formations for every nonnegative integer k.[1] So we will frequently encounter primitive groups as minimal counterexample. In view of Lemma 3.2.5, it is no surprise that the case when G is the product of two nilpotent subgroups of coprime order is of special importance.

4.2.1 Lemma. *Let the finite group G be the product of its nilpotent subgroups A and B. If $(|A|,|B|) = 1$ and $O_{\pi'}(G) = 1$ where π is the set of prime divisors of $|A|$, then:*

(i) $O_{\pi'\pi} = O_\pi(G) = F(G)$.

(ii) *Let $F_0 = 1$ and $F_{k+1}/F_k = F(G/F_k)$ for $k > 0$. Then*

$$F_{k+1}/F_k = \begin{cases} O_\pi(G/F_k) & \text{if } k \text{ is even} \\ O_{\pi'}(G/F_k) & \text{if } k \text{ is odd;} \end{cases}$$

moreover every prime divisor of $|AF_k/F_k|$ divides $|F_{k+1}/F_k|$ if k is even and every prime divisor of $|BF_k/F_k|$ divides $|F_{k+1}/F_k|$ if k is odd.

(iii) $n(G) = l_\pi(G) + l_{\pi'}(G)$*; moreover either* $n(G) = 2l_p(G) = 2l_{p'}(G)$ *or* $n(G) = 2l_p(G) - 1 = 2l_{p'}(G) + 1$.

Proof. (i) If p divides the order of $F(G)$, then a Sylow p-subgroup of $F(G)$ is a proper normal subgroup of G. Since $O_{\pi'}(G) = 1$, we must have $p \in \pi$ and $F(G) \leq O_\pi(G)$. On the other hand, $O_\pi(G)$ is contained in the nilpotent Hall π-subgroup A of G and is therefore nilpotent. It follows that $F(G) = O_\pi(G) = O_{\pi'\pi}(G)$.

(ii) Clearly, $O_\pi(G) > 1$ since G is soluble by the Kegel-Wielandt theorem (our Theorem 3.1.1). Now $G/O_\pi(G)$ does not contain nontrivial normal π-subgroups and, exchanging π and π' (observe that the set π' is the set of primes that do divide the order of B plus the primes that do *not* divide the order of G, so that we may assume that π' contains exactly the prime divisors of B), we may suppose that the statement is true for $G/F(G)$. The first part of the statement follows since by part (i), $O_\pi(G) = F(G)$.

For the second statement, observe that $F_1 \leq A$ and thus $Z(A) \leq C_G(F_1)$ which is contained in F_1. Now $Z(A)$ is the product of the (nontrivial) centres of the primary components of A, and so every prime divisor of $|A|$ divides already the order of $Z(A) \leq F_1$. The general statement follows by considering G/F_k instead of G and exchanging A and B if k is odd.

[1] The proof is straightforward except possibly for saturation, a proof of which can be obtained easily from the proof given by Robinson [43], 9.3.4 for $\pi = \{p\}$.

(iii) This follows immediately from (ii), since the Fitting series

$$1 = F_0 \lhd F_1 \lhd \cdots \lhd F_n = G$$

coincides with the upper π-series which equals the upper π'-series.

$$1 \trianglelefteq O_{\pi'}(G) \lhd O_{\pi'\pi}(G) \lhd O_{\pi'\pi\pi'} \lhd \cdots \lhd G.$$

$\square$

We will soon see that (iii) remains true for some prime p also when we remove the condition $(|A|, |B|) = 1$. This will be proved in Section 4.3. The dual of the previous lemma is likewise true:

4.2.2 Lemma. *Let the finite group G be the product of its nilpotent subgroups A and B. If $(|A|, |B|) = 1$ and $O^{\pi'}(G) = 1$ where π is the set of prime divisors of $|A|$, then:*

(i) $O^{\pi'\pi} = O^{\pi}(G) = G^{\mathfrak{N}}$.

(ii) *Set $L_0 = G$ and for $k > 0$, define $L_{k+1} = (L_k)^{\mathfrak{N}}$. Then*

$$L_k/L_{k-1} = \begin{cases} O^{\pi}(L_k) & \text{if } k \text{ is odd} \\ O^{\pi'}(L_k) & \text{if } k \text{ is even} \end{cases}.$$

Proof. (i) Obviously, $G = AO^{\pi}(G)$ so that $G/O^{\pi}(G) \cong A/A \cap O^{\pi}(G)$ is nilpotent. On the other hand, $G/G^{\mathfrak{N}}$ must be a π-group because $O^{\pi'}(G) = 1$ which proves the other inclusion.

(ii) Note first that every normal subgroup of G is factorized by Corollary 2.2.4 so that in particular $O^{\pi}(G)$ is factorized (this can be seen more easily in this case observing that $B \leq O^{\pi}(G)$). Now $O^{\pi}\big(O^{\pi}(G)\big) = O^{\pi}(G)$ which shows that $O^{\pi}(G)$ satisfies the hypotheses of this lemma for the set π' (which we may assume to contain exactly the prime divisors of B; cf. the remark in the proof of Lemma 4.2.1). The full statement now follows by induction on k. $\square$

It is easy to see that one can construct from the inequalities stated in Theorem 4.1.2 functions $f\colon \mathfrak{N} \to \pm\nvdash\mathbb{N}_0$ which satisfy

(BP1) for all finite soluble groups G and all primes p, $l_p(G) \leq f(G_p)$, where G_p is a Sylow p-subgroup of G.

(BP2) $f(P/N) \leq f(P)$ for all finite p-groups P and $N \trianglelefteq P$ where p a prime.

4.2.3 Theorem. *Let f be a function satisfying* (BP1) *and* (BP2) *above, and suppose that the finite group G is the product of its nilpotent subgroups A and B. Then $l_\pi(G) \leq \max\big\{f(A_p), f(B_p) \mid p \in \pi\big\}$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Let $k = \max_{p\in\pi}\{f(A_p), f(B_p)\}$, and denote with $\mathfrak{X}$ the class of nilpotent groups H satisfying $f(H_p) \le k$ for all $p \in \pi$. Then we have A, $B \in \mathfrak{X}$; moreover, using (BP2), it is easy to see that the class $\mathfrak{X}$ is Q-closed.

If $\mathfrak{H}$ denotes the Schunck class of finite groups satisfying $l_\pi(G) \le k$, then a finite group of minimal order that is a product of two $\mathfrak{X}$-subgroups but does not lie in $\mathfrak{H}$ is primitive by Lemma 1.3.3. So G must be primitive with unique minimal normal subgroup N of exponent p and $l_\pi(G) > l_\pi(G/N)$. This shows that $p \in \pi$. Since in view of (BP1), the theorem is trivially true if $G = A = B$, we may also assume by Lemma 3.2.5 that A is a Sylow p-subgroup of G and that B is a Hall p'-subgroup.

Since $O_{\pi\pi'}/O_\pi(G)$ is a π'-group and $B_{\pi'}$ is a Hall π'-subgroup of G, we have

$$O_{\pi\pi'}/O_\pi(G) \le B_{\pi'}O_\pi(G)/O_\pi(G).$$

Now $B_{\pi'}$ is centralized by B_π because B is nilpotent, and since moreover

$$C_G\big(O_{\pi\pi'}(G)/O_\pi(G)\big) \le O_{\pi\pi'}(G)/O_\pi(G)$$

(see e.g. [26], Lemma 1.2.3) we must have $B_\pi \le C_G\big(O_{\pi\pi'}(G)/O_\pi(G)\big) \le O_{\pi\pi'}(G)/O_\pi(G)$; consequently $B_\pi \le O_\pi(G)$. This shows that $AO_\pi(G)/O_\pi(G)$ is a p-group with $p \in \pi$ and $BO_\pi(G)/O_\pi(G)$ is a π'-group. In particular, the order of $G/O_\pi(G)$ is divisible only by p and primes in π', hence every p-series is also a π-series and viceversa, and in particular, $l_p\big(G/O_\pi(G)\big) = l_\pi\big(G/O_\pi(G)\big)$. Since moreover $O_{p'p}(G) = N \le O_\pi(G)$, it follows that

$$l_\pi(G) \le 1 + l_p\big(G/O_\pi(G)\big) \le l_p(G) \le f(A_p) \le k$$

by (BP1). This final contradiction proves the theorem. □

The following bounds on l_p follow directly from Theorem 4.1.2 and Theorem 4.2.3 with $\pi = p$:

4.2.4 Corollary. *Let the finite group G be the product of its nilpotent subgroups A and B. Write l_p for $l_p(G)$ and let $b_p = \max\{b_p(A), b_p(B)\}$, $c_p = \max\{c_p(A), c_p(B)\}$, $d_p = \max\{d_p(A), d_p(B)\}$ and $e_p = \max\{e_p(A), e_p(B)\}$, then:*

$$\text{(i)}\ \ b_p \ \ \ge \ \ \begin{cases} \dfrac{p^{l_p}-1}{p-1} & \textit{if } p \textit{ is odd and not a Fermat prime*;} \\ \dfrac{(p-2)^{l_p+1} - l_p(p-3) - p + 2}{(p-3)^2} & \textit{if } p \textit{ is a Fermat prime} > 3; \\ 2^{l_p-1} + l_p - 1 & \textit{if } p = 3; \\ \frac{1}{2}l_p(l_p+1) & \textit{in any case.} \end{cases}$$

$$\text{(ii)}\ c_p \quad \geq \quad \begin{cases} p^{l_p-1} & \textit{if } p \textit{ is odd and not a Fermat prime*;} \\ \dfrac{(p-2)^{l_p}-1}{(p-3)} & \textit{if } p \textit{ is a Fermat prime} > 3; \\ \min\{l_p, 2^{l_p-1}\} & \textit{if } p = 3; \\ l_p & \textit{in any case.} \end{cases}$$

$$\text{(iii)}\ d_p \quad \geq \quad \begin{cases} l_p & \textit{if } p \geq 3\textit{*;} \\ l_p & \textit{if } p = 2 \textit{ and } d_3(G) \leq 1; \\ \min\{l_p, \frac{1}{2}l_p + 1\} & \textit{if } p = 2. \end{cases}$$

$$\text{(iv)}\ e_p \quad \geq \quad \begin{cases} l_p & \textit{if } p \textit{ is neither } 2 \textit{ nor a Fermat prime*;} \\ \frac{1}{2}l_p & \textit{if } p \textit{ is an odd prime*;} \\ \min\{l_p, \frac{1}{2}(l_p+1)\} & \textit{if } p = 2. \end{cases}$$

Observe that by our remark after Theorem 4.1.2, the inequalities marked * are again best-possible.

It is clear that the bounds for $p = 2$ above are very bad compared with the results obtained for other primes. However, using the fact that $l_p(G) \leq l_{p'}(G)+1$, we can obtain better results if we use information about the Hall p'-subgroups of A and B.

In order to obtain handy bounds on π-length, we have to simplify the formulas given in Theorem 4.1.2

4.2.5 Lemma. *Let the group G be a finite p-soluble group of p-length l_p, where p is an odd prime. Then*

(i) $b_p(G) \geq 2^{l_p-1} + l_p - 1$;
(ii) $c_p(G) \geq \min\{l_p, 2^{l_p-1}\}$;
(iii) $d_p(G) \geq l_p$;
(iv) $e_p(G) \geq \frac{1}{2}l_p$.

Proof. (i) If p is odd and not a Fermat prime, then by Theorem 4.1.2 we have

$$b_p(G) \geq \frac{p^{l_p}-1}{p-1} = \sum_{i=0}^{l_p-1} p^i \geq 2^{l_p-1} + (l_p-1)\cdot 1,$$

whereas if p is a Fermat prime > 3, we have

$$\begin{aligned} \frac{(p-2)^{l_p+1} - l_p(p-3) - p + 2}{(p-3)^2} &= \frac{1}{p-3}\Big(\sum_{i=0}^{l_p}(p-2)^i - (l_p-1)\Big) \\ &\geq \frac{1}{p-3}\Big((p-2)^{l_p} + (p-2)\cdot(l_p-1) + 1 - (l_p-1)\Big) \\ &\geq (p-3)^{l_p-1} + l_p - 1 \geq 2^{l_p-1} + l_p - 1. \end{aligned}$$

The case $p = 3$ being obvious, it follows that the first formula holds for all odd primes.

The other statements should be clear. □

Define $b_\pi(G) = \max\{0, b_p(G) \mid p \in \pi\}$ and similarly $c_\pi(G)$, $d_\pi(G)$, $e_\pi(G)$ (note that if G is nilpotent, then $c_\pi(G)$ and $d_\pi(G)$ are the class and derived length of a Hall π-subgroup of G), then we obtain the following bounds on the π-length of a group G which is the product of two finite nilpotent subgroups.

4.2.6 Corollary. *Let the finite group G of π-length l_π be the product of its nilpotent subgroups A and B, let π be a set of primes, and define $b_\pi = \max\{b_\pi(A), b_\pi(B)\}$, $c_\pi = \max\{c_\pi(A), c_\pi(B)\}$, $d_\pi = \max\{d_\pi(A), d_\pi(B)\}$ and $e_\pi = \max\{e_\pi(A), e_\pi(B)\}$. Then the following inequalities hold:*

(i) $b_\pi \geq 2^{l_\pi - 1} + l_\pi - 1$ *if* $2 \notin \pi$ *and* $b_{\pi'} \geq 2^{l_\pi - 2} + l_\pi - 2$ *if* $2 \in \pi$;

(ii) $c_\pi \geq \min\{l_p, 2^{l_\pi - 1}\}$ *if* $2 \notin \pi$ *and* $c_{\pi'} \geq \min\{l_p - 1, 2^{l_\pi - 2}\}$ *if* $2 \in \pi$;

(iii) $d_\pi \geq l_\pi$ *if* $2 \notin \pi$ *and* $d_{\pi'} \geq l_\pi - 1$ *if* $2 \in \pi$;

(iv) $e_\pi \geq \frac{1}{2} l_\pi$ *if* $2 \notin \pi$ *and* $e_{\pi'} \geq \frac{1}{2}(l_\pi - 1)$ *if* $2 \in \pi$.

Proof. (i). Suppose first that $2 \notin \pi$. Then for every finite p-group P, define $f(P)$ by

$$2^{f(P)-1} + f(P) - 1 = b_p(P).$$

Then f satisfies (BP1) by the preceding lemma, and it satisfies (BP2) because b_p satisfies it and the function $2^{x-1}+x+1$ is strictly increasing for $x \geq 0$. Therefore by Theorem 4.2.3, $l_\pi(G) \leq \max_{p\in\pi}\{f(A_p), f(B_p)\}$ and so

$$\begin{aligned} 2^{l_\pi - 1} + l_\pi - 1 &\leq \max_{p\in\pi}\{2^{f(A_p)-1} + f(A_p) - 1, 2^{f(B_p)-1} + f(B_p) - 1\} \\ &= \max_{p\in\pi}\{b_p(A), b_p(B)\} = b_\pi. \end{aligned}$$

If $2 \in \pi$, we use the fact that $l_\pi - 1 \leq l_{\pi'}$; thus

$$2^{l_\pi - 2} + l_\pi - 2 \leq 2^{l_{\pi'} - 1} + l_{\pi'} - 1 \leq b_{\pi'}.$$

The proof of the other statements is similar. □

4.3 Connections between Fitting length and π-lengths

If G is a soluble group of Fitting length $n = n(G)$, then it is always possible to obtain from its Fitting series a series whose factors are π- and π'-groups with $\frac{n}{2}$ π-factors if n is even and $\frac{(n+1)}{2}$ π-factors if n is odd. So $2l_\pi(G) \leq n+1$ for every set of primes π. Since also $2l_{\pi'} \leq n+1$, we also have $l_\pi + l_{\pi'} \leq n+1$. The next theorem shows that products of two finite nilpotent groups have the property that $n(G) \leq 2l_p(G)$ and $n(G) \leq l_p(G) + l_{p'}(G)$ for at least one prime p.

4.3.1 Theorem. *Let the group G be the product of its finite nilpotent subgroups A and B. Then*

(i)
$$n(G) \leq 2 \max_{p\in\pm\nvdash\mathbb{P}} \{l_p(G)\},$$

(ii)
$$n(G) \leq 2 \max_{p\in\pm\nvdash\mathbb{P}} \{l_{p'}(G)\} + 1 \quad \textit{and}$$

(iii)
$$n(G) \leq \max_{p\in\pm\nvdash\mathbb{P}} \{l_p(G) + lp'(G)\}.$$

Proof. Suppose that the group G is a minimal counterexample for one of the above inequalities. Since the classes of groups H such that

$$2 \max_{p\in\pm\nvdash\mathbb{P}} \{l_p(H)\} \leq 2 \max_{p\in\pm\nvdash\mathbb{P}} \{l_p(G)\}$$

$$2 \max_{p\in\pm\nvdash\mathbb{P}} \{l_p(H)\} \leq 2 \max_{p\in\pm\nvdash\mathbb{P}} \{l_{p'}(G)\} + 1$$

or

$$\max_{p\in\pm\nvdash\mathbb{P}} \{l_p(H) + l_{p'}(H)\} \leq \max_{p\in\pm\nvdash\mathbb{P}} \{l_p(G) + l_{p'}(G)\}$$

are Q-closed and the classes of groups H such that $n(G) \leq k$ form Schunck classes, the group G can be assumed primitive by Lemma 1.3.1. Now by Lemma 4.2.1, we have $n(G) \leq 2l_p(G)$, $n(G) \leq 2l_{p'}(G)$ and $n(G) \leq l_p(G) + l_{p'}(G)$ where p is the exponent of the unique minimal normal subgroup of G. So the theorem is true also for primitive groups.

□

The last theorem generalizes a result of R. Maier [35]:

4.3.2 Corollary. *Let G be a finite group G. Then the following statements are equivalent:*

(i) *G is the product of its nilpotent subgroups A and B and $l_p(G) \leq 1$ for all primes p.*

(ii) *G is metanilpotent.*

Proof. (i) $\Rightarrow$ (ii). This follows directly from Theorem 4.3.1 that $n(G) \leq 2$, in other words, G is metanilpotent.

(ii) $\Rightarrow$ (i). Since the class of finite nilpotent groups forms a Schunck class, the soluble group G has a nilpotent projector P (see Section 3.4). Now since $G/F(G)$ is nilpotent, $G = PF(G)$ where P is a nilpotent subgroup, whence G is the product of two nilpotent subgroups. The statement about the p-length is trivial. □

Remark. There are finite soluble groups G satisfying $l_p(G) \leq 1$ for all primes p which are not metanilpotent and therefore do not admit a factorization by two nilpotent subgroups, e.g. if G is the regular wreath product of three groups of orders p, q and r where p, q and r are distinct primes. The group G is also an example of a group satisfying

$$n(G) \leq \max_{p \in \pm\nvdash\mathbb{P}} \left\{ l_p(G) + l_{p'}(G) \right\}$$

but which is not the product of two nilpotent subgroups.

Question. Does every finite soluble group satisfying

$$n(G) \leq 2 \max_{p \in \pm\nvdash\mathbb{P}} \left\{ l_p(G) \right\} \quad \textit{or} \quad n(G) \leq 2 \max_{\pi \subseteq \pm\nvdash\mathbb{P}} \left\{ l_\pi(G) \right\}$$

admit a factorization by two nilpotent subgroups?

Recall that a group is called *modular* if its subgroup lattice is modular. Modular finite nilpotent groups can also be characterized as follows:

4.3.3 Lemma. *Let G be a finite nilpotent group. Then G is modular if and only if $AB = BA$ for all subgroups A and B of G.*

Proof. Let G be a modular nilpotent group. We show that $AB = BA$ for all subgroups A and B of G. Suppose that this is false and let G be a minimal counterexample. Then G has subgroups A and B such that AB is not a subgroup of G (cf. Lemma 1.1.2), A and B are proper subgroups of G and we have $G = \langle A, B \rangle$. Now let M be a maximal subgroup of G that contains A, then $M = M \cap \langle A, B \rangle = \langle A, M \cap B \rangle$ by the modularity of G, and by the minimality of G, we have $M = A(B \cap M)$. Now $A \cap B \leq M$ and since G is nilpotent, the index $|G : M| = |B : B \cap M| = p$ is a prime. So Lemma 1.1.1 yields

$$|AB| = \frac{|A| \cdot |B|}{|A \cap B|} = p \cdot \frac{|A| \cdot |B \cap M|}{|A \cap B \cap M|} = p \cdot |M| = |G|$$

and therefore we have $G = AB$. This contradiction shows that in a modular nilpotent group any two subgroups permute.

Conversely, suppose that every two subgroups of the group G permute. We have to show that the subgroup lattice of G is modular, i.e. that for arbitrary subgroups A, B and C of G with $A \leq C$, we have $\langle A, B \cap C \rangle = \langle A, B \rangle \cap C$. But this follows directly from the 'usual' modular law of group theory: since any two subgroups of G permute, we have $AB = \langle A, B \rangle$ and $\langle A, B \cap C \rangle = A(B \cap C)$, and hence by the modular law, $\langle A, B \cap C \rangle = A(B \cap C) = AB \cap C = \langle A, B \rangle \cap C$. □

This can be used to prove that finite nilpotent modular groups having a complemented abelian maximal normal subgroup are themselves abelian. This follows at once from the following

4.3.4 Lemma (R. Maier [35]). *Let P be a modular group of order p^n and suppose that N an abelian maximal normal subgroup of P which possesses a complement C Then P is abelian.*

Proof. By the maximality of N, we have $|C| = p$. Let $1 \neq x \in N$. then $\langle x \rangle$ is a subgroup of order p and $\langle x \rangle C$ is a subgroup of order p^2 by the modularity of P. Thus $\langle x \rangle C$ is abelian so that x commutes with every element of C and P must be abelian. □

4.3.5 Theorem (R. Maier [35]). *Let the finite group $G = AB$ be the product of the nilpotent modular subgroups A and B. Then G is metanilpotent.*

Proof. Let G be a counterexample of minimal order. Since the class of metanilpotent groups forms a saturated formation and the class of modular nilpotent groups is Q-closed, G is a primitive group by Lemma 1.3.3. Denote with N its unique minimal normal subgroup, then $N = F(G)$ is elementary abelian of prime exponent p. Also, we have $A \neq B$ and by Proposition 3.1.8, A or B is contained in a maximal normal subgroup M of G which must be factorized.

Since G is a minimal counterexample, M is metanilpotent, hence $M = F_2(G)$. By Lemma 3.2.5, w.l.o.g. A is a Sylow p-subgroup of G and $F_2(G)/N$ is a p'-group and it is easy to see that G/M is a cyclic p-group, whence $AM = G$. Therefore $AM/M \cong A/A \cap M = A/N$ and since N has a complement C in G, we have $A = A \cap CN = (A \cap C)N$ which shows that $A \cap C \cong A/N$ is cyclic of order p.

So N is an abelian maximal normal subgroup of A which has a complement $A \cap C$ in A, and by the modularity of A, the subgroup A is abelian by Lemma 4.3.4. But then we have $A \leq C_G(N) = N$. This final contradiction shows that G must be metabelian. □

4.4 Bounds on Fitting length and derived length

Bounds on the Fitting length of a group G which is the product of its finite nilpotent subgroups A and B can be obtained using the bounds on the p-lengths of G obtained in Section 4.2 and the inequalities

$$n(G) \leq 2 \cdot \max_{p \in \pm\nvdash\mathbb{P}} \bigl\{l_p(G)\bigr\}$$

$$n(G) \leq 2 \cdot \max_{p \in \pm\nvdash\mathbb{P}} \bigl\{l_{p'}(G)\bigr\} + 1$$

$$n(G) \leq \max_{p \in \pm\nvdash\mathbb{P}} \bigl\{l_p(G) + l_{p'}(G)\bigr\}$$

established in Section 4.3. Slightly better bounds on the nilpotent length of G, still based on the bounds in Theorem 4.1.2, are available via a closer analysis of the series described in Lemma 4.2.1,

Because of the great number of possibilities, we will restrict ourselves to bounds on $n(G)$ in terms of $d_p(A)$ and $d_p(G)$ for all primes p since the methods used there also lead to estimates in terms of b_p, c_p and e_p.

4.4.1 Theorem. *Let the finite group G be the product of its nilpotent subgroups A and B. Then*

(i)

$$n(G) \leq 2d(A) + \max\bigl\{0,\ 2d_2(A) - 4\bigr\} + 1 \quad \textit{and}$$

$$n(G) \leq 2d(B) + \max\bigl\{0,\ 2d_2(B) - 4\bigr\} + 1;$$

(ii)

$$n(G) \leq d(A) + d(B) + \max\bigl\{0,\ d_2(A) - 2,\ d_2(B) - 2\bigr\};$$

(iii)

$$n(G) \leq 2\max\bigl\{d_{2'}(A),\ d_{2'}(B)\bigr\} + 1;$$

(iv) (*Berger and Gross* [9])

$$n(G) \leq \max\bigl\{d(A) + d(B),\ \tfrac{4}{3}\bigl(d(A) + d(B)\bigr) - 1\bigr\}.$$

Proof. Suppose that the theorem is false and let the group $G = AB$ be a minimal counterexample to one of the inequalities. The classes of finite groups of nilpotent length $\leq k$ clearly form saturated formations, hence Schunck classes, and the class of nilpotent groups H with $d_p(H) \leq d_p(A)$ for every prime p is Q-closed, similarly for B. Since in addition the above functions are nondecreasing with $d_p(A)$ and $d_p(B)$ for all primes p, the minimal counterexample G must be primitive by Lemma 1.3.3. Thus it suffices to prove the above inequalities in case G is primitive.

As all of the above inequalities hold for nilpotent groups G, we can also assume that $G \neq A$ and $G \neq B$ and thus we infer from Lemma 3.2.5 that G has a unique minimal normal subgroup N of prime exponent p, that w.l.o.g. A is a Sylow p-subgroup of G and that B is a Hall p'-subgroup.

Since A is a Sylow p-subgroup of G, by Theorem 4.1.2, we have

$$l_p(G) \leq \begin{cases} d_p(A) & \text{if } p \text{ is odd} \\ \max\bigl\{d_2(A), 2d_2(A) - 2\bigr\} & \text{if } p = 2. \end{cases}$$

Therefore we have

$$(*) \qquad l_p(G) \leq d(A) + \max\bigl\{0, d_2(A) - 2\bigr\}$$

in any case.

Similarly, by Theorem 4.2.3, we obtain

$$\begin{aligned} l_{p'}(G) &\leq \max_{q \in \pm\nvdash\mathbb{P}\setminus\{p\}} \bigl\{d_q(A), d_q(B), 2d_2(A) - 2, 2d_2(B) - 2\bigr\} \\ (**) \qquad &= \max_{q \in \pm\nvdash\mathbb{P}\setminus\{p\}} \bigl\{d_q(B), 2d_2(B) - 2\bigr\} \\ &= d(B) + \max\bigl\{0, d_2(B) - 2\bigr\}, \end{aligned}$$

observing that A is a p-group.

Now by Lemma 4.2.1 and $(*)$,

$$\begin{aligned} n(G) &\leq 2l_p(G) \leq 2d(A) + \max\bigl\{0, 2d_2(A) - 4\bigr\} \\ &\quad \leq 2l_p(G) \leq 2d(A) + \max\bigl\{0, 2d_2(A) - 4\bigr\} + 1. \end{aligned}$$

By $(**)$,

$$n(G) \leq 2l_{p'}(G) + 1 \leq 2d(B) + \max\bigl\{0, 2d_2(B) - 4\bigr\} + 1.$$

This proves (i).

Next, again by Lemma 4.2.1, $n(G) \leq l_p(G) + l_{p'}(G)$ and so by $(*)$ and $(**)$,

$$n(G) \leq d(A) + \max\bigl\{0, d_2(A) - 2\bigr\} + d(B) + \max\bigl\{0, d_2(B) - 2\bigr\}.$$

Since either A or B is a $2'$-group, we have $d_2(A) = 0$ or $d_2(B) = 0$ and hence

$$n(G) \leq d(A) + d(B) + \max\bigl\{0, d_2(A) - 2, d_2(B) - 2\bigr\},$$

proving (ii).

To see that (iii) is true, we have to consider the cases $p = 2$ and $p \neq 2$ separately. Assume first that $p = 2$ and hence that B is a $2'$-group. Then by $(**)$

$$n(G) \leq 2l_{p'}(G) + 1 \leq 2d(B) + \max\{0, 2d_2(B) - 4\} + 1 = 2d_{2'}(B) + 1.$$

If $p \neq 2$, A is a $2'$-group and therefore we have

$$n(G) \leq 2l_p(G) \leq 2d(A) + \max\{0, 2d_2(A) - 4\} + 1 = 2d_{2'}(A) \leq 2d_{2'}(A) + 1$$

by $(*)$. Thus (iii) holds.

To prove (iv), suppose first that $d(A) \leq 2$ and $d(B) \leq 2$. Then by (ii), $n(G) \leq d(A) + d(B)$ and (iv) holds. Since we have also excluded the cases $G = A$ and $G = B$, we may assume that $d(A) > 0$ and $d(B) > 0$, hence that $d(A) + d(B) \geq 3$. Then $d(A) + d(B) \leq \frac{4}{3}\big(d(A) + d(B)\big) - 1$ and thus it remains to show that

$$n(G) \leq \tfrac{4}{3}\big(d(A) + d(B)\big) - 1.$$

Consider next the case when $p = 2$ and $2d(A) - 2 \leq d(B) + 1$. Then

$$2d(A) - 2 + d(B) \leq \tfrac{4}{3}\big(d(A) + d(B)\big) - 1$$

and since we also have $d(A) \leq d(B) + 1$, we obtain that

$$n(G) = l_p(G) + l_{p'}(G) \leq \max\{d(A), 2d(A) - 2\} + d(B) \leq \tfrac{4}{3}\big(d(A) + d(B)\big) - 1.$$

If $p = 2$ and $2d(A) - 2 > d(B)$, then we have

$$2d(B) + 1 \leq \tfrac{4}{3}\big(d(A) + d(B)\big) - 1$$

and hence

$$n(G) \leq 2l_{p'}(G) + 1 \leq 2d(B) + 1 \leq \tfrac{4}{3}\big(d(A) + d(B)\big) - 1.$$

Similarly, if $p \neq 2$ and $2d(B) - 2 \leq d(A) + 1$, then

$$n(G) = l_p(G) + l_{p'}(G) \leq d(A) + \max\{d(B), 2d(B) - 2\} \leq \tfrac{4}{3}\big(d(A) + d(B)\big) - 1$$

and finally if $p > 2$ and $2d(B) - 2 > d(A) + 1$. then

$$n(G) \leq 2l_p(G) \leq 2d(A) \leq \tfrac{4}{3}\big(d(A) + d(B)\big) - 1.$$

□

Remark. The difference of the above inequalities is essentially due to the irregular behaviour of the prime 2, for if A and B have odd order, then clearly $n(G) \leq d(A)+d(B)$. Since in fact $l_2(G) \leq d_2(G)$ if $d_3(G) \leq 1$ by Theorem 4.1.2, it can be checked easily that if $d_3(A) \leq 1$ when $d_2(B) \geq 2$ and $d_3(B) \leq 1$ if $d_2(A) \geq 2$, then we obtain

$$n(G) \leq d(A) + d(B) \qquad \textit{and} \qquad n(G) \leq 2d(A) + 1.$$

These inequalities, and thus the results of Theorem 4.4.1 for groups of odd order, are best-possible: if p and q are distinct primes, and C_p and C_q denote cyclic groups of order p and q respectively, then a group of the form $G = C_p \wr C_q \wr \ldots \wr C_p \wr C_q \wr C_p$ where C_q occurs k times satisfies $n(G) = 2k + 1$. Moreover, it is the products of its Sylow p- and q-subgroups which have derived lengths $k + 1$ and k respectively by [26], Theorem 3.5.1.

In this context, it should also be mentioned that Berger and Gross [9] conjecture that $l_2(G) \leq d(G)$ in any case which would, as has been remarked by Berger and Gross themselves, imply that $n(G) \leq d(A) + d(B)$ and $n(G) \leq 2d(A) + 1$ always.

Also, (ii) improves a result of Gross [22] who shows that $n(G) \leq d(A) + d(B)$ if $c_2(A) \leq 3$ and $c_2(B) \leq 3$.

A bound on the Fitting length similar to (i), however based on the inequalities in terms of c_p instead of those involving d_p has been obtained by Heineken [29].

Next, we will obtain some information about the derived length of a finite group G which is the product of its nilpotent subgroups A and B in terms of the derived lengths or classes of A and B.

The main problem is that hardly anything is known about the derived length of G if G itself is nilpotent. In fact, the only nontrivial result seems to be Itô's theorem which states that the derived length of a product of two abelian groups has derived length at most 2 (Itô [32]).

In this context, we also mention that the class of the nilpotent group G is not bounded by the classes (or derived lengths) of the subgroups A and B: in fact, for every nonnegative integer n, there exist groups G of order p^{2n} which are the product of two abelian subgroups A and B with $A \trianglelefteq G$ (and thus $d(G) \leq 2$) such that $c(G) = n$, see Dickenschied [12], *Beispiel 7.1* for details of the construction.

However, it is possible to obtain bounds on the derived length of certain quotient groups of a group G which is the product of two nilpotent subgroups A and B, such as $G/F(G)$ and $G/\Phi(G) \cap O_\pi(G)$, where π is the set of common prime divisors of $|A|$ and $|B|$.

To see that groups G that are minimal subject to $d\big(G/\Phi(G)\big) = k$ are primitive, in view of Lemma 1.3.1, the following lemma is useful:

4.4.2 Lemma. *The class of finite soluble groups G that satisfy $d\big(G/\Phi(G)\big) \leq n$ equals the class of finite groups G such that $G^{(n-1)}$ is nilpotent, i.e. the class of groups* *such that $d\big(G/F(G)\big) \le n-1$; this class is* S-, Q-, D_0- *and* E_Φ-*closed; in other words, it is a subgroup-closed saturated Formation.*

Proof. We show first that the two classes are equal: if $G^{(n-1)}$ is nilpotent, it is contained in the Fitting subgroup $F(G)$ of G. Since $F(G)/\Phi(G)$ is abelian, we have $G^{(n)} \le \Phi(G)$ and $G/\Phi(G)$ has derived length $\le n$. On the other hand, if $G^{(n)} = (G^{(n-1)})' \le \Phi(G)$, then $G^{(n-1)}\Phi(G)/\Phi(G) \le F\big(G/\Phi(G)\big) = F(G)/\Phi(G)$ and so $G^{(n-1)} \le F(G)$, hence it is nilpotent.

That the class in question is closed with respect to subgroups, homomorphic images and finite direct products can be checked easily using the second definition. Saturation is obvious from the first. □

If the group G is the product of its nilpotent subgroups A and B, this can be used to reduce the search for a bound on $d\big(G/\Phi(G)\big)$ to finding bounds in the special case when A and B have coprime orders.

4.4.3 Proposition. *Suppose that $f\colon \mathfrak{N} \times \mathfrak{N} \to \pm\nvdash\mathbb{N}_0$ is a function satisfying*

(i) *$f(A,B) = f(A^*,B^*)$ if $A \cong A^*$ and $B \cong B^*$ for all A, $B \in \mathfrak{N}$;*

(ii) *$f(A/M, B/N) \le f(A,B)$ for all finite nilpotent groups A and B and for all $M \trianglelefteq A$, $N \trianglelefteq B$ and*

(iii) *if the finite group G is the product of its nilpotent subgroups A and B and A and B have coprime order, then $d\big(G/\Phi(G)\big) \le f(A,B)$.*

Then we have $d\big(G/\Phi(G)\big) \le f(A,B)$ for every group G that is the product of its nilpotent subgroups A and B.

Proof. Suppose that the group G is a counterexample of minimal order to the proposition. By the minimality of G, if $1 \ne N \trianglelefteq G$, then we have

$$d\big((G/N)/\Phi(G/N)\big) \le f(AN/N, BN/N) = f(A/A\cap N, B/B\cap N) \le f(A,B)$$

by (i) and (ii). Therefore every proper epimorphic image belongs to the Schunck class described in Lemma 4.4.2 for $n = f(A,B)$ but G does not belong to that class. Thus G must be primitive by Lemma 1.3.1. If $A = B$ $(= G)$, then G is cyclic of prime order, and since $G = A \cdot 1$, we have

$$1 = d\big(G/\Phi(G)\big) = d(G) \le f(A,1) \le f(A,B).$$

Thus we may assume that $A \ne B$. But now by Lemma 3.2.5, A and B have coprime orders and so by (iii), we have

$$d\big(G/\Phi(G)\big) \le f(A,B).$$

This final contradiction proves the proposition. □

For every $n \in \pm\nvdash\mathbb{N}$, consider the classes of groups G such that $G^{(n)}$ is a nilpotent π-group. Although these classes are not saturated, groups G that are minimal subject to not belonging to one of these classes and which are the product of two nilpotent subgroups A and B still have a structure very similar to that of a primitive product of two nilpotent subgroups if π is chosen to be the set of common prime divisors of $|A|$ and $|B|$:

4.4.4 Lemma. *Suppose that the finite group G is the product of its nilpotent subgroups A and B and let $\pi = \sigma(A) \cap \sigma(B)$. If there is an integer n such that $(G/K)^{(n)}$ is a nilpotent π-group for all normal subgroups $K \neq 1$ of G but $G^{(n)}$ is not a nilpotent π-group, then G has a unique minimal normal subgroup N of prime exponent p; furthermore $G^{(n)} = N$, $F(G)$ is a p-group, and (w.l.o.g.) A and B are a Sylow p-group of G and a Hall p'-subgroup of G respectively. In particular, $\pi = \varnothing$.*

Proof. Suppose first that G has two distinct minimal normal subgroups N and N^* and let $H/N = (G/N)^{(n)}$ and $H^*/N^* = (G/N^*)^{(n)}$. Then $G^{(n)}$ is contained in $H \cap H^* = (H \cap H^*)/(N \cap N^*)$ which is a nilpotent normal π-group by Lemma 1.3.5 and the fact that H/N and H^*/N^* are nilpotent π-groups. Therefore G must have a unique minimal normal subgroup N which is an elementary abelian p-group for some prime p because G is soluble, and also the Fitting subgroup of G must be a p-group.

Next, we show that w.l.o.g. A is a Sylow p-subgroup of G and that B a Hall p'-subgroup: assume first that $\Phi(G) \neq 1$. Then we must have $N \leq \Phi(G)$, and since $(G/N)^{(n)}$ is nilpotent, we have $G^{(n)}\Phi(G)/\Phi(G) \leq F\big(G/\Phi(G)\big) = F(G)/\Phi(G)$. Therefore we have $G^{(n)} \leq F(G)$ and thus $G^{(n)}$ is a nilpotent p-group. Therefore $p \notin \pi$ and we may suppose w.l.o.g. that p divides $|A|$ but not $|B|$. In particular, B is a p'-group and A contains a Sylow p-subgroup A_p of G whence $F(G) \leq A_p$. Now $A_{p'}$ centralizes A_p since A is nilpotent and therefore $A_{p'} \leq C_G\big(F(G)\big) \leq F(G)$. But $F(G)$ is a p-group, from which we deduce that $A_{p'} = 1$. Hence A is a p-group and we have already observed that B is a p'-group. Then, however, it follows from Lemma 1.1.1 that A and B must be a Sylow p- and Hall p'-subgroups of G, and the lemma is proved in this case.

If $\Phi(G) = 1$, G is primitive by Lemma 3.2.3. If $A = B$ $(= G)$, then G is a nilpotent π-subgroup; therefore we must have $A \neq B$ and so by Lemma 3.2.5, A and B are a Sylow p- and a Hall p'-subgroup of G.

Therefore we have $\pi = \{p\} \cap p' = \varnothing$ in both cases. Since $(G/N)^{(n)}$ is a π-group, we have $(G/N)^{(n)} = 1$ and hence $G^{(n)} \leq N$. Since $G^{(n)} \neq 1$ by hypothesis, we must have $N = G^{(n)}$ by the minimality of G. □

Now we are ready to prove the bounds on the derived length of $G/O_\pi(G) \cap \Phi(G)$.

4.4.5 Theorem. *Let the group $G = AB$ be the product of its finite nilpotent subgroups A and B. If π is the set of primes that divide the orders of both A and B, then $G^{(n)}$ is a nilpotent π-subgroup contained in $\Phi(G)$, where*

$$n = \max\bigl\{c_{\sigma'}(A),\ \tfrac{1}{2}d_\sigma(A)\bigl(d_\sigma(A)+1\bigr)\bigr\} + \max\bigl\{c_{\tau'}(B),\ \tfrac{1}{2}d_\tau(B)\bigl(d_\tau(B)+1\bigr)\bigr\},$$

and σ and τ are arbitrary sets of odd primes.

Proof. We show first by way of contradiction that $G^{(n)}$ is a nilpotent π-group, so suppose that G is a counterexample of minimal order. If $1 \neq N$ is a normal subgroup of G, then we have

$$\begin{aligned} n \geq{} & \max\bigl\{c_{\sigma'}(AN/N),\ \tfrac{1}{2}d_\sigma(AN/N)\bigl(d_\sigma(AN/N)+1\bigr)\bigr\} \\ & + \max\bigl\{c_{\tau'}(BN/N),\ \tfrac{1}{2}d_\tau(BN/N)\bigl(d_\tau(BN/N)+1\bigr)\bigr\} \end{aligned}$$

whence $(G/N)^{(n)}$ is a nilpotent π-group for all normal subgroups $N \neq 1$ of G, and so by Lemma 4.4.4, G has a unique minimal normal subgroup N of prime exponent p, w.l.o.g. A is a Sylow p-subgroup and B is a Hall p'-subgroup of G. Moreover, $G^{(n)} \leq N$ and also $F = F(G)$ is a p-group.

If $p \notin \sigma$, consider the group G/Z where $Z = Z(F)$: Clearly,

$$1 < Z(A) \leq Z = Z(F) \leq C_G(F) \leq F,$$

therefore $c_p(A/Z) < c_p(A)$. Since A is a p-group, we have

$$\max\bigl\{c_{\sigma'}(A),\ \tfrac{1}{2}d_\sigma(A)\bigl(d_\sigma(A)+1\bigr)\bigr\} = c_p(A) = c(A)$$

and similarly

$$c(AZ/Z) = \max\bigl\{c_{\sigma'}(AZ/Z),\ \tfrac{1}{2}d_\sigma(AZ/Z)\bigl(d_\sigma(AZ/Z)+1\bigr)\bigr\}.$$

Therefore

$$\begin{aligned} n-1 \geq{} & \max\bigl\{c_{\sigma'}(AZ/Z),\ \tfrac{1}{2}d_\sigma(AZ/Z)\bigl(d_\sigma(AZ/Z)+1\bigr)\bigr\} \\ & +\ \max\bigl\{c_{\tau'}(BZ/Z),\ \tfrac{1}{2}d_\tau(BZ/Z)\bigl(d_\tau(BZ/Z)+1\bigr)\bigr\} \end{aligned}$$

which shows that already $(G/Z)^{(n-1)} = 1$. Since Z is abelian, we must have $G^{(n)} = 1$. This contradiction shows that we must have $p \in \sigma$; in particular, p is odd.

Since $F = O_{p'p}(G)$ and $p \neq 2$, we have $d_p(A/F) = d(A/F) \leq d(A) - 1 = d_p(A) - 1$ by Theorem 4.1.1 and therefore

$$\max\bigl\{c_{\sigma'}(AF/F),\ \tfrac{1}{2}d_\sigma(AF/F)\bigl(d_\sigma(AF/F)+1\bigr)\bigr\} = \tfrac{1}{2}d(A/F)\bigl(d(A/F)+1\bigr).$$

Thus

$$\begin{aligned} n - d(A) \geq \max\bigl\{c_{\sigma'}(AF/F),\ \tfrac{1}{2}d_\sigma(AF/F)\bigl(d_\sigma(AF/F)+1\bigr)\bigr\} \\ +\ \max\bigl\{c_{\tau'}(BF/F),\ \tfrac{1}{2}d_\tau(BF/F)\bigl(d_\tau(BF/F)+1\bigr)\bigr\} \end{aligned}$$

This shows that G/F has derived length $\leq n - d(A)$, yielding that

$$d(G) \leq d(F) + d(G/F) \leq n$$

since $F \leq A$. This final contradiction shows that $G^{(n)}$ is a nilpotent π-group.

It remains to show that $G^{(n)} \leq \Phi(G)$ for every finite group G that is the product of two nilpotent subgroups A and B. But since our first result implies that $G^{(n)} = 1$ if A and B have coprime orders, this follows at once from Proposition 4.4.3. □

Remark. That we have to treat the prime 2 differently is essentially due to the fact that for a Sylow 2-subgroup P of the finite soluble group G, it is possible to have

$$d\bigl(PO_{2'2}(G)/O_{2'2}(G)\bigr) = d(P).$$

However, this can only happen when P is non-abelian and also the Sylow 3-subgroups of G are non-abelian (cf. Theorem 4.1.1). Transferring these considerations to the proof of the theorem, we obtain that the theorem also holds for arbitrary sets of primes σ and τ, provided that A has an abelian Sylow 3-subgroup if the Sylow 2-subgroup of B is non-abelian and viceversa, i.e. if $d_3(A) \leq 1$ whenever $d_2(B) \geq 2$ and $d_3(B) \leq 1$ if $d_2(A) \geq 2$.

If we set $\sigma = \tau = \varnothing$ in the preceding theorem, we obtain the following

4.4.6 Corollary (Gross [22] and Pennington [40]). *If the finite group G is the product of its nilpotent subgroups A and B of classes c and d, then $G^{(c+d)}$ is a nilpotent π-group contained in the Frattini subgroup of G; in particular $G/\Phi(G)$ has derived length $\leq c + d$.*

4.4.7 Corollary. *If the group G is the product of its nilpotent subgroups A and B of coprime order, then $d(G) \leq c(A) + c(B)$.*

The irregular behaviour of the prime 2 can also be compensated by considering a Hall $2'$-subgroup of G instead of a Sylow 2-subgroup. The key to this is the observation that most statements of Theorem 4.1.1 can be extended to statements about a nilpotent Hall π-subgroup of a group G if we replace p by π and p' by π'. We state some of the most important consequences.

4.4.8 Lemma (Berger and Gross [9]). *Let G be a finite soluble group.*

(i) *If $O_{\pi'\pi}(G)/O_{\pi'}(G)$ is nilpotent, then $O_{\pi'\pi}(G) = \bigcap_{p\in\pi} O_{p'p}(G)$.*

(ii) *If H is a nilpotent Hall π-subgroup of G and $c(H) > 0$, then*

$$c\big(HO_{\pi'\pi}/O_{\pi'\pi}(G)\big) < c(H);$$

if $2 \notin \pi$ and $d(H) > 0$, then also

$$d\big(HO_{\pi'\pi}(G)/O_{\pi'\pi}(G)\big) < d(H).$$

Proof. (i) Since $O_{\pi'\pi}(G)/O_{\pi'}(G)$ is nilpotent, it is clear that $O_{\pi'\pi}(G) \leq O_{p'p}(G)$ for all $p \in \pi$. Since $O_{\pi'}(G) \leq O_{p'p}(G)$ for all primes $p \in \pi$, $O_{\pi'}(G)$ must be contained in $O_{p'}(G)$ for all such primes. On the other hand, $\bigcap_{p\in\pi} O_{p'}(G)$ is a normal π'-subgroup of G, therefore it is contained in $O_{\pi'}(G)$. This shows that $O_{\pi'}(G) = \bigcap_{p\in\pi} O_{p'}(G)$. Now a π'-element contained in $\bigcap_{p\in\pi} O_{p'p}(G)$ is contained in $O_{p'}(G)$ for all $p \in \pi$, whence it is contained in $O'_{\pi}(G)$. This shows that $\big(\bigcap_{p\in\pi} O_{p'p}(G)\big)/O_{\pi'}(G)$ is a π-group and therefore $\bigcap_{p\in\pi} O_{p'p}(G) \leq O_{\pi'\pi}(G)$. This proves (i).

(ii) Denote with H_p the normal Sylow p-subgroup of H. For all primes p, consider the canonical homomorphisms

$$\alpha_p\colon H_p \longrightarrow H_pO_{p'p}(G)/O_{p'p}(G).$$

These homomorphisms induce a homomorphism

$$\alpha\colon H = \bigtimes_{p\in\pi} H_p \longrightarrow D = \bigtimes_{p\in\pi} H_pO_{p'p}(G)/O_{p'p}(G)$$

whose kernel is $H \cap \big(\bigcap_{p\in\pi} O_{p'p}(G)\big) = H \cap O_{\pi'\pi}(G)$ by part (i). This, together with an isomorphism theorem, shows that

$$\begin{aligned} c\big(HO_{\pi'\pi}(G)/O_{\pi'\pi}(G)\big) &= c\big(H/H \cap O_{\pi'\pi}(G)\big) \\ &\leq c(D) = \max_{p\in\pi}\big\{c\big(H_pO_{p'p}(G)/O_{p'p}(G)\big)\big\} \end{aligned}$$

Now by Theorem 4.1.1, for every prime p,

$$c\big(H_pO_{p'p}(G)/O_{p'p}(G)\big) < c(H_p)$$

because the H_p are Sylow p-subgroups of G. This shows that

$$\begin{aligned} c\big(HO_{\pi'\pi}(G)/O_{\pi'\pi}(G)\big) &\leq \max_{p\in\pi}\big\{c\big(H_pO_{p'p}(G)/O_{p'p}(G)\big)\big\} \\ &< \max_{p\in\pi}\big\{c(H_p)\big\} = c(H) \end{aligned}$$

as required.

If $p \neq 2$ and $H_p \neq 1$, also

$$d\big(H_pO_{p'p}(G)/O_{p'p}(G)\big) < d(H)$$

so that by the same argument,

$$d\big(HO_{\pi'\pi}(G)/O_{\pi'\pi}(G)\big) < d(H)$$

if $2 \notin \pi$. $\square$

The following theorem seems rather similar to the preceding Theorem 4.4.5; however its results are in terms of the derived lengths of A and B only and do not involve $c_2(A)$ or $c_2(B)$.

4.4.9 Theorem. *Let the group $G = AB$ be the product of its finite nilpotent subgroups A and B. If π is the set of primes that divide the orders of both A and B, then $G^{(n)}$ is a nilpotent π-subgroup contained in $\Phi(G)$, where*

$$\begin{aligned} n = {} & \tfrac{1}{2}d(A)\big(d(A)+1\big) + \tfrac{1}{2}d(B)\big(d(B)+1\big) \\ & + \max\big\{\tfrac{1}{2}d_2(A)\big(d_2(A)+1\big), \tfrac{1}{2}d_2(B)\big(d_2(B)+1\big)\big\}. \end{aligned}$$

Proof. Like in the proof of Theorem 4.4.5, in view of Proposition 4.4.3, it suffices to show that $G^{(n)}$ is a nilpotent π-group. Suppose that this is false and let G be a counterexample of minimal order. Thus if $N \trianglelefteq G$, then

$$\begin{aligned} n \geq {} & \tfrac{1}{2}d(AN/N)\big(d(AN/N)+1\big) + \tfrac{1}{2}d(BN/N)\big(d(BN/N)+1\big) \\ & + \max\big\{\tfrac{1}{2}d_2(AN/N)\big(d_2(AN/N)+1\big), \tfrac{1}{2}d_2(BN/N)\big(d_2(BN/N)+1\big)\big\}. \end{aligned}$$

Therefore by Lemma 4.4.4, w.l.o.g. A is a Sylow p-subgroup of G containing the unique minimal normal subgroup N of G and also $F = F(G) \leq A$; moreover B is a Hall p'-subgroup of G. We may also assume that $d(A) = d_p(A) > 0$ and $d(B) > 0$ since otherwise $G = B$ or $G = A$ and in these cases the theorem is obviously true.

Consider first the case when $p \neq 2$. Then we have $d(AF/F) < d(A)$ by Theorem 4.1.1 and therefore

$$\begin{aligned} n - d(A) \geq {} & \tfrac{1}{2}d(AF/F)\big(d(AF/F)+1\big) + \tfrac{1}{2}d(BF/F)\big(d(BF/F)+1\big) \\ & + \tfrac{1}{2}d_2(B)\big(d_2(B)+1\big). \end{aligned}$$

This shows that $d(G/F) \leq n - d(A)$ and since $F \leq A$, we have

$$d(G) \leq d(F) + d(G/F) \leq d(A) + n - d(A) = n.$$

This contradiction shows that we must have $p = 2$.

Therefore we have

$$\tfrac{1}{2}d(AF/F)\big(d(AF/F)+1\big)+\tfrac{1}{2}d_2(AF/F)\big(d_2(AF/F)+1\big)=d_2(A)\big(d_2(A)+1\big).$$

Now by Lemma 4.2.1, $F_3 = F_3(G) = O_{2'22'2}(G)$ and by Theorem 4.1.1, we have

$$d(AF_3/F_3) < d(A);$$

moreover, applying Lemma 4.4.8 with $\pi = 2'$, we have

$$d(B) \geq d(BF/F) > d(BF_2/F_2) \geq d(BF_3/F_3)$$

where $F_2 = F_2(G)$. Hence

$$\begin{aligned} d(G/F_3) &\leq d_2(AF_3/F_3)\big(d_2(AF_3/F_3)+1\big)+\tfrac{1}{2}\big(d(BF_3/F_3)(d(BF_3/F_3)+1\big) \\ &\leq n - 2d(A) - d(B) \end{aligned}$$

and since by Lemma 4.2.1, $F \leq A$, the group F_2/F is isomorphic with a section of B and F_3/F_2 is isomorphic with a section of A, we obtain

$$\begin{aligned} d(G) &\leq d(F) + d(F_2/F) + d(F_3/F_2) + d(G/F_3) \\ &\leq d(A) + d(B) + d(A) + n - 2d(A) - d(B) \\ &= n. \end{aligned}$$

This final contradiction proves the theorem. □

Remark. As in the remark after Theorem 4.4.5, the bound can be improved significantly if $d_3(A) \leq 1$ whenever $d_2(B) \geq 2$ and $d_3(B) \leq 1$ if $d_2(A) \geq 2$. In this case, we already obtain that $G^{(n)} \leq \Phi(G) \cap O_\pi(G)$ where

$$n \geq \tfrac{1}{2}d(AN/N)\big(d(AN/N)+1\big)+\tfrac{1}{2}d(BN/N)\big(d(BN/N)+1\big).$$

There is yet another possibility to gain some information about the derived length of a product of two nilpotent subgroups:

4.4.10 Theorem. *Let the finite group G be the product of its nilpotent subgroups A and B. Then $G/F(G)$ has derived length at most k where*

$$k = \big(d(A) + \max\{0, d_2(A) - 1\}\big)\big(d(B) + \max\{0, d_2(B) - 1\}\big).$$

Proof. Suppose that G is a counterexample of minimal order. Then we have

$$d\big((G/N)/F(G/N)\big) \leq k$$

for all normal subgroup $N \neq 1$ of G but $d\big(G/F(G)\big) > k$. Since the class of groups H satisfying $d\big(H/F(H)\big) \leq k$ forms a saturated formation, hence is a Schunck class, by

Lemma 4.4.2 G must be primitive. Since our statement is trivial if G is nilpotent, we may also assume that $d(A) > 0$, $d(B) > 0$ and also that $A \neq B$, hence by Lemma 3.2.5, w.l.o.g. A is a Sylow p-subgroup of G and B is a Hall p'-subgroup, where p is the exponent of the unique minimal normal subgroup $F = F(G)$ of G.

Suppose first that $p \neq 2$, or that $d(AF/F) < d(A)$ if $p = 2$. Observe that we also have $d(A/F) < d(A)$ if $p \neq 2$ by Theorem 4.1.1 since $F = O_{p'p}(G)$. Therefore if we let $F_2 = F_2(G)$, then

$$\begin{aligned} d(G/F_2) &\leq d(AF/F)\big(d(BF/F) + \max\{0, d_2(BF/F) - 1\}\big) \\ &\leq \big(d(A) - 1\big)\big(d(B) + \max\{0, d_2(B) - 1\}\big) \\ &\leq k - d(B). \end{aligned}$$

Since by Lemma 4.2.1, $F_2/F \leq BF/F$, we have

$$d(G/F) \leq d(F_2/F) + d(G/F_2) \leq n.$$

This contradiction shows that we must have $p = 2$ and $d(AF/F) = d(A)$. This also shows that $d(A) \geq 2$ because if we had $d(A) = 1$, then A would be abelian, hence $A \leq C_G(F) = F$ which would imply that $d(AF/F) = 0 < d(A)$.

Now by Lemma 4.2.1, $F_2 = O_{22'}(G)$ and $F_3 = F_3(G) = O_{2'2}(G)$ and so by Theorem 4.1.1, $d(AF_3/F_3) < d(A)$; moreover by Lemma 4.4.8, $d(BF_3/F_3) < d(B)$. If $F_4 = F_4(G)$, then we have

$$\begin{aligned} d(G/F_4) &\leq \big(2d(AF_3/F_3) - 1\big)\big(d(BF_3/F_3)\big) \\ &\leq \big(2d(A) - 1 - 2\big)\big(d(B) - 1\big) \\ &\leq \big(2d(A) - 1\big)\big(d(B) - 1\big) - (2d(A) - 1) - 2d(B) + 2 \\ &\leq \big(2d(A) - 1\big)\big(d(B) - 1\big) - d(A) - d(B) - \big(d(B) - 1\big) \end{aligned}$$

Since $F_2/F \leq BF/F$, $F_3/F_2 \leq AF_2/F_2$ and $F_4/F_3 \leq BF_3/F_3$ by Lemma 4.2.1, we have

$$\begin{aligned} d(G/F) &\leq d(F_2/F) + d(F_3/F_2) + d(F_4/F_3) + d(G/F_4) \\ &\leq d(B) + d(A) + \big(d(B) - 1\big) \\ &\qquad + (2d(A) - 1)\big(d(B) - 1) - d(A) - d(B) - (d(B) - 1\big) \\ &= k. \end{aligned}$$

Thus we have reached a final contradiction which proves the theorem. ◻

Remark. Again, if we have $d_3(A) \leq 1$ in case $d_2(B) \geq 2$ and $d_3(B) \leq 1$ if $d_2(A) \geq 2$ in the preceding theorem, then we obtain that $d\big(G/F(G)\big) \leq d(A)d(B)$, a result that has already been obtained by Gross [22] under the hypothesis that A and B have coprime orders.

4.4.11 Corollary. *If the finite group G is the product of its nilpotent subgroups A and B, then has derived length at most*

$$d\big(G/\Phi(G)\big) \leq \big(d(A)+\max\{0,d_2(A)-1\}\big)\big(d(B)+\max\{0,d_2(B)-1\}\big)+1$$

and if $\pi=\sigma(A)\cap\sigma(B)$ and

$$k=\big(d(A)+\max\{0,d_2(A)-1\}\big)\big(d(B)+\max\{0,d_2(B)-1\}\big)+\max\{d(A),d(B)\}\big),$$

then $G^{(k)}$ is a nilpotent π-group contained in the Frattini subgroup of G.

Proof. The first statement follows directly from the fact that $F(G)/\Phi(G)$ is abelian.

For the second statement, let N be the π-component of $F=F(G)$. Since $F=(A\cap F)(B\cap F)$ by Theorem 3.1.5, we have $F/N=(A\cap F)N/N(B\cap F)N/N$. By the definition of π, $(A\cap F)N/N$ and $(B\cap F)N/N$ have coprime orders, and since F/N is nilpotent, $F/N=(A\cap F)N/N\times(B\cap F)N/N$ whence

$$d(F/N)\leq\max\{d\big((A\cap F)N/N\big),\ d\big(B\cap F)N/N\big)\}\leq\max\{d(A),\ d(B)\}.$$

This shows that $G^{(k)}$ is a nilpotent π-group. Since the corollary is obviously true if $G=1$, we may also suppose that $\max\{d(A),d(B)\}\geq 1$, and therefore by the first part, also $G^{(k)}\leq\Phi(G)$. □

The following theorem describes the special case when one of the factors is abelian.

4.4.12 Theorem. *Let the finite group G be the product of an abelian group A and a nilpotent group B. Then*

(i) (*Franciosi, de Giovanni, Heineken and Newell* [16]) *$AF(G)\trianglelefteq G$;*
(ii) *$n(G)\leq 3$;*
(iii) *$d\big(G/F(G)\big)\leq d(B)$;*
(iv) *$G^{(n)}\leq\Phi(G)\cap O_\pi(G)$ where $\pi=\sigma(A)\cap\sigma(B)$ and $n=\max\{2d(B),1\}$.*

Proof. (i). If $G=1$, this is clearly true, so suppose by finite induction that (i) is true for all groups of smaller order than G. If $G/\Phi(G)$ satisfies the theorem, so does G because $F(G)/\Phi(G)=F\big(G/\Phi(G)\big)$. Therefore we must have $\Phi(G)=1$.

Consider first the case when G possesses two distinct minimal normal subgroups N^* and N^{**}. Denote with F^*/N^* and F^{**}/N^{**} the Fitting subgroups of G/N^* and G/N^{**}, then $F^*\cap F^{**}=F$ by Lemma 1.3.5 and since by Theorem 3.1.5, F^* and F^{**} are factorized, by Lemma 1.1.9, $AF^*\cap AF^{**}=AF$, which is a normal subgroup of G since by induction hypothesis, AF^* and AF^{**} are normal subgroups of G.

Therefore we may assume that G possesses a unique minimal normal subgroup N which is an elementary abelian p-group for a prime p, and, because $\Phi(G) = 1$, the group G is primitive. Since G can be assumed non-nilpotent, A or B is a p-group containing $N = F$ while the other is a p'-group by Lemma 3.2.5. If $N = F \leq A$, then $A \leq C_G(F) = F$ and thus $A = F$, proving that $AF = F \trianglelefteq G$.

There remains the case when $F \leq B$. Then the p'-group $F_2/F = F(G/F)$ is contained in the Hall p'-subgroup AF/F of G/F and therefore $AF_2 \leq AF$. The other inclusion $AF \leq AF_2$ is trivial, and by the minimality of G, the subgroup $AF_2/F = AF/F$ is normal in G/F, or equivalently, $AF \trianglelefteq G$. This final contradiction proves (i).

(ii) follows directly from the fact that the series

$$1 \lhd F(G) \trianglelefteq AF(G) \trianglelefteq G$$

has nilpotent factors. The same result about $n(G)$ can be obtained as described in the remark after Theorem 4.4.1, observing that we have $d_2(A) \leq 1$ and $d_3(A) \leq 1$ since A is abelian.

The remaining statements follow from the remark after Theorem 4.4.10. □

4.5 Properties of the Fitting quotient group

4.5.1 Proposition. *Let the finite group G be the product of its nilpotent subgroups A and B. Then:*

(i) *$G/[A,B]$ is nilpotent. The factorizer of $[A,B]$ is $A^G \cap B^G = (A \cap B^G)(B \cap A^G)$.*

(ii) *Let $A_0 = A$, $B_0 = B$ and $G_0 = G$ and for all $i > 0$, define $A_{i+1} = A_i \cap B_i^{G_i}$ and $B_{i+1} = B_i \cap A_i^{G_i}$. Then $G_i = A_iB_i$ is a factorized subgroup of G for all i and $G_{i+1} \trianglelefteq G_i$; moreover G_i/G_{i+1} is nilpotent. If n is the smallest integer such that $G_n = G_{n+1}$, then $G_n = A \cap B$. Thus the series*

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = A \cap B$$

consists of factorized subnormal subgroups of G such that G_i/G_{i+1} is a nilpotent group.

Proof. (i) First observe that $[A,B] \trianglelefteq \langle A,B \rangle$ for arbitrary subgroups A and B of G. Now $A^G = A[A,B]$ and $B^G = B[A,B]$ which shows that $G/[A,B]$ is the product of the normal nilpotent subgroups $A[A,B]/[A,B]$ and $B[A,B]/[A,B]$ and is therefore nilpotent. By Lemma 1.2.1, the factorizer of $[A,B]$ is $A[A,B] \cap B[A,B] = A^G \cap B^G = (A \cap B^G)(B \cap A^G)$.

(ii) Suppose by finite induction that $G_i = A_iB_i$ where $A_i \leq A$ and $B_i \leq B$. (This is clearly true if $i = 0$). Then G_i satisfies the hypothesis of the preceding part; therefore G_i/G_{i+1} is a nilpotent group, $G_{i+1} = A_{i+1}B_{i+1}$ and $A_{i+1} \cap B_{i+1} = A_i \cap B_i = A \cap B$. as required.

Now suppose that $G_n = G_{n+1}$. Since $G_{n+1} = A_n^{G_n} \cap B_n^{G_n}$, we must have $A_n^{G_n} = G_n$ and $B_n^{G_n} = G_n$ and so neither A_n nor B_n is contained in a proper normal subgroup of G_n. So it follows from Proposition 3.1.8 that $G_n = A_n = B_n$ and so $G_n = A_n \cap B_n \leq A \cap B$ proving that $G_n = A \cap B$. □

Recall that in Example 3.4.4, we have shown that the subgroup $[A, B]$ above is not necessarily prefactorized.

4.5.2 Corollary. *Let the finite group G be the product of its nilpotent subgroups A and B. If n is the length of the series defined in Proposition 4.5.1, then G has Fitting length at most $n + 1$ and derived length at most*

$$n \cdot \max\big\{d(A), d(B)\big\} + \min\big\{d(A), d(B)\big\}.$$

Proof. The series

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = A \cap B \unrhd 1$$

introduced in Proposition 4.5.1 has nilpotent factors (clearly also $A \cap B$ is nilpotent) and its length is $n + 1$. This is clearly an upper bound for the Fitting length of G.

To obtain the bound on the derived length, consider the factor group G_i/G_{i+1} for some i: $G_i = A_iB_i = A_i^{G_i}B_i$; thus

$$G_i/A_i^{G_i} \cong B_i/B_i \cap A_i^{G_i} = B_i/B_{i+1}$$

whose derived length is $\leq d(B)$ and $G_i/B_i^{G_i} \cong A_i/A_{i+1}$ which has derived length $\leq d(A)$. Since $G_{i+1} = A_i^{G_i} \cap B_i^{G_i}$, it follows that $d(G_i/G_{i+1}) \leq \max\big\{d(A), d(B)\big\}$. Clearly, $d(A \cap B) \leq \min\big\{d(A), d(B)\big\}$ and also the second statement follows. □

We have already proved in Proposition 3.1.2 that if G is the product of its nilpotent subgroups A and B, then the commutator subgroups $[A_p, B_p]$ are subnormal p-subgroups and are therefore contained in the Fitting subgroup of G. Therefore the next proposition applies in particular when $F(G) \unlhd N$:

4.5.3 Proposition. *Suppose that the group G is the product of its finite nilpotent subgroups A and B. Suppose that N is a normal subgroup of G such that G/N is nilpotent and that N contains $[A_p, B_p]$ for every prime p dividing $|G/N|$. Then the subgroups*

$A_\pi N/N$ and $B_\pi N/N$ are normal subgroups of G/N for every set of primes π. In particular, AN/N and BN/N are normal subgroups of G. If moreover N is factorized, then G is the direct product of AN/N and BN/N,

$$\begin{aligned} G/N &= AN/N \times BN/N \\ &= \mathop{\Large\times}_{p\in\pm\nvdash\mathbb{P}} A_pN/N \times \mathop{\Large\times}_{p\in\pm\nvdash\mathbb{P}} B_pN/N. \end{aligned}$$

Proof. Consider the Sylow p-subgroup A_pN/N of AN/N. Since G/N is nilpotent, A_pN/N is centralized by the normal p-complement of G/N; moreover $[A_p, B_p] \leq N$ by hypothesis; therefore also B_pN/N centralizes A_pN/N which shows that $A_pN/N \trianglelefteq G/N$. Therefore also $A_\pi N/N = \times_{p\pi\pi} A_pN/N$ is a normal subgroup of G.

If N is factorized, then $N = AN \cap BN$ and therefore $AN/N \cap BN/N = 1$, showing that $G/N = AN/N \times BN/N$. The rest of the statement follows. □

4.5.4 Corollary. *Let $G = AB$ where A and B are finite and nilpotent. If $M \trianglelefteq G$ and $N \trianglelefteq M$ such that M/N is nilpotent, if moreover N contains $[A_p, B_p]$ for all primes p dividing $|M/N|$, then $AN \cap M$ and $BN \cap M$ are normal subgroups of M. If M contains $AN \cap BN$, then $(AN \cap M)(AN \cap M) = (A \cap M)(B \cap M)$.*

Proof. Let $X = AM \cap BM$ be the factorizer of M. Then X/N is nilpotent by Corollary 3.5.2 and therefore $AN \cap X \trianglelefteq X$ and $BN \cap X \trianglelefteq X$. Hence $AN \cap M = AN \cap M \cap X$ and $BN \cap M$ are normal subgroups of M.

If M contains $AN \cap BN$, also $M^* = (AN \cap M)(AB \cap M)$ contains this intersection. Thus, M^*/N is a factorized subgroup of G/N, and M^* is a factorized subgroup of G. □

Since by Theorem 3.1.5 the Fitting subgroup and therefore every term of the Fitting series of a product of two nilpotent subgroups is factorized, we obtain

4.5.5 Corollary. *Let F_k be the k-th term of the Fitting series of $G = AB$ where A and B are finite nilpotent subgroups of G. If $k \geq 1$, then F_{k+1}/F_k is the direct product of $(A \cap F_{k+1})F_k/F_k$ and $(B \cap F_{k+1})F_k/F_k$.*

In this context, also the following Proposition is of interest:

4.5.6 Proposition (Heineken [29]). *Let the group G be the product of its finite nilpotent subgroups A and B. If $F_k = F_k(G)$ denotes the k-th term of the Fitting series of G, then for all $k \geq 1$, we have that $(A \cap F_{k+1})F_k/F_k$ and $(B \cap F_{k+1})F_k/F_k$ are normal subgroups of G.*

Proof. Since the setting is symmetrical in A and B, it clearly suffices to show that $(A \cap F_{k+1})F_k \trianglelefteq G$. Since $(A \cap F_{k+1})F_k = AF_k \cap F_{k+1}$ by the modular law and $G/F_k =$

$(AF_k)(BF_k)$, we may in addition pass to the quotient group G/F_{k-1}, hence it remains to show that $(F_2 \cap A)F = AF \cap F_2$ is normal in G where $F = F(G)$ $(= F_1)$.

Suppose by induction on $|G|$ that the proposition is true for all groups of smaller order; in view of $F/\Phi(G) = F\big(G/\Phi(G)\big)$, we may also assume that $\Phi(G) = 1$.

Next, consider the case when G contains two distinct minimal normal subgroups N^* and N^{**}. Now let $F^*/N^* = F(G/N^*)$, $F^{**}/N^{**} = F(G/N^{**})$ and $F_2^*/F^* = F(G/F^*)$, $F_2^{**}/F^{**} = F(G/F^{**})$ then we have $F = F^* \cap F^{**}$ and $F_2 = F_2^* \cap F_2^{**}$ by Lemma 1.3.5, observing that the classes of nilpotent and metanilpotent groups form Fitting formations. So we have $AF = A(F^* \cap F^{**}) = AF^* \cap AF^{**}$ by Lemma 1.1.9. Now by induction hypothesis, $AF^* \cap F_2^*$ and $AF^{**} \cap F_2^{**}$ are normal subgroups of G and thus $AF^* \cap F_2^* \cap AF^{**} \cap F_2^{**} = AF \cap F_2 = (A \cap F_2)F$ is a normal subgroup of G as required.

There remains the case when G has a unique minimal normal subgroup; let p denote its exponent. Since also $\Phi(G) = 1$, we conclude that G is primitive. But in this case, by Lemma 3.2.5, A is either a p-group and $A \cap F_2 = F$ whence $F = (A \cap F_2)F \trianglelefteq G$ or A is a Hall p'-group whence F_2 is contained in AF and therefore $AF \cap F_2 = F_2 \trianglelefteq G$. □

Next, we report a result of Heineken which will allow to relate products of finite nilpotent subgroups to other classes of groups.

4.5.7 Proposition (Heineken [28]). *Assume that the group G is the product of two finite nilpotent subgroups A and B. Then $G/F(G) \in \mathrm{R}_0\mathfrak{X}$ where $\mathfrak{X}$ is the class of finite groups whose order is divisible only by two primes one of which divides the order of A while the other divides the order of B.*

Indeed, from Heineken's proof of the above proposition, a stronger result can be obtained.

4.5.8 Proposition. *Let the group G be the product of its finite nilpotent subgroups A and B. If H/K is a principal factor of G such that $F(G) \leq K$, then $|G/C_G(H/K)|$ is divisible at most by one prime divisor of $|A|$ and one of $|B|$; moreover one of these prime divisors is the exponent of H/K.*

Proof. Suppose that the proposition is true for all groups of smaller order than G: this includes, of course, the case when G is nilpotent. Observe also that $C_{G/N}(H/K) = C_G(H/K)/N$ for every normal subgroup N of G with $N \leq K$.

As a first case, assume that G possesses two distinct minimal normal subgroups N^* and N^{**}. Let F^*/N^* and F^{**}/N^{**} denote the Fitting subgroups of G/N^* and G/N^{**} respectively, then $F^* \cap F^{**} = F = F(G)$ by Lemma 1.3.5. Refine the series

$$1 \trianglelefteq F^*/F \trianglelefteq G/F$$

to a chief series of G. Then by the Jordan-Hölder theorem, H/K is G-isomorphic with a factor of that chief series; therefore we may assume that H/K itself belongs to that chief series. Now if $F^* \leq K$, then H/K is already a chief factor of G/F^* and the result follows from the fact that $C_G(H/K)/N^* = C_{G/N^*}(H/K)$. If H/K is a chief factor of F^*/F, then since $F^*/F \cong_G F^{**}F^*/F^*$, the chief factor H/K is G-isomorphic with a chief factor of G/F^{**} and the result follows again.

Hence we may assume that G has a unique minimal normal subgroup N. If $\Phi(G) > 1$, we have $F/\Phi(G) = F(G)/\Phi(G)$ and the result follows by induction.

Therefore there remains the case when G is primitive and non-nilpotent. As before, we may assume that H/K is a principal factor of the chief series of G/F obtained by refining

$$1 \lhd F_2 \unlhd G/F,$$

where $F_2/F = F(G/F)$; by finite induction, we may also exclude the case when $F_2 \leq K < H \leq G$. Now by Lemma 3.2.5, we may w.l.o.g. assume that A is a Sylow p-group containing F and that B is a Hall p'-group of G, we also have $F_2/F \leq BF/F$. From the last statement, it follows that $H/K \leq B_q K/K$ where q is the exponent of H/K. Now since B is nilpotent, B_q is centralized by $B_{q'}$ and therefore $B_{q'} \leq C_G(H/K)$. Since $B_{q'}$ is a Hall $\{p,q\}'$-subgroup of G, the order of $G/C_G(H/K)$ can be divisible by p and q only. □

From this, Proposition 4.5.7 follows with the help of the characterization of the Fitting subgroup of a finite group as the intersection of the centralizers of all principal factors of G.

Following Huppert [30], a finite group G is called *a group with many Sylow bases* if set of Sylow subgroups of G containing exactly one Sylow subgroup for every prime p is a Sylow basis of G, or equivalently, if every Sylow p-subgroup of G permutes with every Sylow q-subgroup of G when $p \neq q$. [1]

Of course, the groups of order $p^\alpha q^\beta$ (where p and q are primes) are examples of groups with many Sylow bases. In the following, we will denote the class of all such groups by $\mathfrak{Q}$; clearly

$$\mathfrak{Q} = \bigcup_{p,q \in \pm\nvdash\mathbb{P}} \mathfrak{S}_{\{p,q\}}.$$

The class $\mathfrak{B}$ of all finite groups with many Sylow bases can be characterized as follows:

[1] These groups were first termed 'Gruppen mit vielen Sylowsystemen' by Huppert [30] who used the term 'Sylowsystem' for what we call a Sylow basis.

4.5.9 Theorem (Huppert [30]). *The following statements about the finite group G are equivalent:*

(i) *G is a group with many Sylow bases.*

(ii) *G is soluble; if H/K is a principal factor of G of exponent p, then there are a prime q and nonnegative integers a and B such that the order of $G/C_G(H/K)$ is $p^a q^b$.*

(iii) *G is contained in the smallest formation that contains all groups whose order is divisible by at most two primes.*

A proof of this can also be found in [31], VI, § 3.

In the following, we will denote the class of groups with many Sylow bases with $\mathfrak{B}$; we also recall that this class is s-closed. Furthermore, by (ii) and the characterization of primitive soluble groups in Lemma 3.2.3 as those groups in which the unique minimal normal subgroup is self-centralized, it is clear that the primitive $\mathfrak{B}$-groups are precisely the primitive groups whose order is divisible by at most two primes.

There follows that primitive $\mathfrak{B}$-groups are products of their nilpotent subgroups since in fact every $\mathfrak{Q}$-group is the product of its Sylow subgroups. As a consequence of this, $\mathfrak{B}$ is clearly contained in the smallest Schunck class that comprises all groups that are the product of two finite nilpotent subgroups. To simplify notation, we will for the rest of this section to denote the class of finite groups that are the product of two nilpotent subgroups by $\mathfrak{M}$.

Now proposition can be used to show that $\mathfrak{M} \subseteq \mathfrak{N}\mathfrak{B}$ in the following

4.5.10 Proposition. *Let $\mathfrak{N}\mathfrak{B}$ be the class of groups G that possess a normal subgroup N such that $N \in \mathfrak{N}$ and $G/N \in \mathfrak{B}$. Then*

(i) *$\mathfrak{N}\mathfrak{B}$ is a saturated formation;*

(ii) *$\mathfrak{N}\mathfrak{B} = \{G \mid G/F(G) \in \mathfrak{B}\}$ and*

(iii) *$\mathfrak{M} \subseteq \mathfrak{N}\mathfrak{B}$.*

Proof. Since $\mathfrak{N}$, the class of all finite nilpotent groups, is $\mathrm{s_n}$-closed, $\mathfrak{N}\mathfrak{F}$ equals the formation product of $\mathfrak{N}$ and $\mathfrak{F}$ and so it is a formation by [13], IV.1.8.

If $G \in \mathfrak{N}\mathfrak{B}$ and N is a nilpotent normal subgroup of G such that $G/N \in B$, we have $N \leq F(G) \in \mathfrak{N}$. On the other hand, $G/F(G)$ is an epimorphic image of G/N and therefore $G/F(G) \in \mathrm{Q}\mathfrak{B} = \mathfrak{B}$. This shows that

$$\mathfrak{N}\mathfrak{B} = \{G \mid G/F(G) \in \mathfrak{B}\}.$$

From this and the fact that $F(G)/\Phi(G) = F\big(G/\Phi(G)\big)$ for all finite groups G, it follows immediately that $\mathfrak{N}\mathfrak{B}$ is saturated. This proves (i) and (ii).

(iii) Let $G \in \mathfrak{M}$. Then by Proposition 4.5.8, the chief factors of $G/F(G)$ are of the form described in Theorem 4.5.9, (ii) and so $G/F(G) \in \mathfrak{B}$, hence $G \in \mathfrak{N}\mathfrak{B}$. □

4.5.11 Example. Let p be an odd prime that is not a Fermat prime. Suppose that N is a cyclic group of order p, and let G be the semidirect product of N with its automorphism group. Then $C_G(N) = N$, and since G is soluble, G is primitive by Lemma 3.2.3. Now the order of G is divisible by more than two primes (namely by p and by all prime divisors of $p-1$ which is divisible by 2 and some other prime $< p$ since p is not a Fermat prime). So by the remark after Theorem 4.5.9, G does not belong to the class of groups with many Sylow bases. But obviously, G is the product of N and a subgroup of order $p-1$ both of which are abelian.

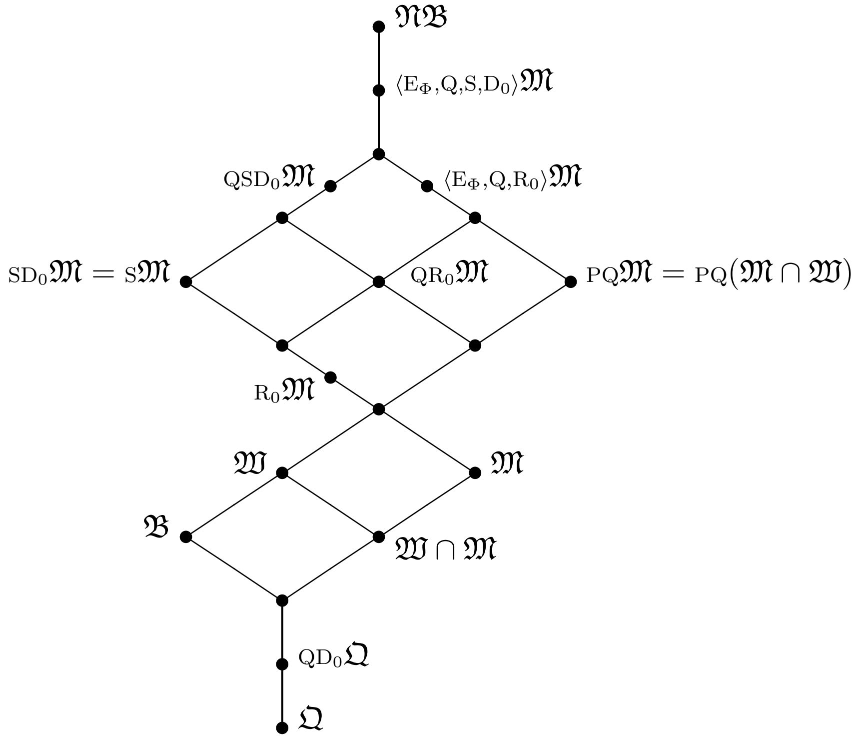
The next example shows that a group G such that $G/F(G)$ is a group with many Sylow bases is not necessarily the product of two nilpotent subgroups.

4.5.12 Example. Let G and p be as in Example 4.5.11. Now let $H = G \wr C$ where C is cyclic of order r where r is a prime $\neq p$ such that r does not divide $p-1$. Then H is primitive by [13], A.18.5 since H is non-abelian and primitive. If H were the product of two nilpotent subgroups, then one of them would have to be a Hall p'-subgroup of H by Lemma 3.2.5, and since the Sylow p-subgroup of H equals its Fitting subgroup F, the quotient group G/F is isomorphic with a Hall p'-subgroup of G. On the other hand, G/F is isomorphic with the regular wreath product of a group of order $p-1$ with a cyclic group of order r. But such a group is not nilpotent since its Sylow r-subgroup does not centralize the Hall r'-part. This shows that H is not the product of two nilpotent subgroups. But since the Sylow q-subgroups of G/F are normal for all primes $q \neq r$, the factor group G/F is clearly a $\mathfrak{B}$-group.

The last example also shows that $\mathrm{PQ}\mathfrak{M}$, the smallest Schunck class containing $\mathfrak{M}$, is properly contained in $\mathfrak{N}\mathfrak{B}$.

Question. Is $\mathfrak{N}\mathfrak{B}$ the smallest formation (saturated formation) containing all products of two finite nilpotent groups?

If the answer to the first question is negative, is it true that the class $\mathfrak{M}$ of finite groups which are factorized by two nilpotent subgroups closed with respect to subgroups? This would imply that $\mathfrak{M}$ is itself a formation since $\mathfrak{M}$ is D_0-closed and Q-closed. Moreover, every group with many Sylow bases would be the product of two nilpotent groups. (Observe that in order to prove that $\mathfrak{M}$ is S-closed, it would suffice to show that a maximal normal subgroup of a product of two finite nilpotent groups is the product of two finite nilpotent groups, for by Corollary 3.3.6, every nonnormal maximal subgroup



The ordering of the classes $\mathfrak{Q}$, $\mathfrak{W}$, $\mathfrak{M}$ and $\mathfrak{NB}$

of G has a conjugate which is factorized and thus every nonnormal maximal subgroup is the product of two nilpotent subgroups.)

In [30], Huppert also shows that if $G \in \mathfrak{B}$ and $p \neq q$ are prime divisors of $|G|$, then G has a normal $\{p, q\}$-subgroup N such that a Hall $\{p, q\}$-subgroup of G/N is nilpotent. It is easy to see that this last property of G is equivalent to the property that $G/O_\pi(G)$ has a nilpotent Hall π-subgroup for every set π of primes. Thus Huppert's statement is equivalent to the second statement of the following

4.5.13 Proposition. *Let $\mathfrak{W}$ be the class of finite groups G such that $G/O_\pi(G)$ has a nilpotent Hall π-subgroup for every set of primes π. Then*

(i) *$\mathfrak{W}$ is a subgroup-closed formation of finite soluble groups;*

(ii) *$\mathfrak{B} \subseteq \mathfrak{W}$;*

(iii) *$\mathfrak{W} \subseteq R_0\mathfrak{X}$ and*

(iv) *$\mathfrak{W}$ and $\mathfrak{M}$ contain the same primitive groups, hence the smallest Schunck classes containing $\mathfrak{W}$ and $\mathfrak{M}$ coincide.*

Proof. (i) Let $G \in \mathfrak{W}$ and let π be a set of primes. If $G_\pi/O_\pi(G)$ is a Hall subgroup of $G/O_\pi(G)$, then clearly G_π is a Hall subgroup of G. Therefore G possesses Hall π-

subgroups for all sets of primes π, hence is soluble. Since all Hall subgroups of the soluble group G are isomorphic, all Hall π-subgroups of $G/O_\pi(G)$ are nilpotent.

If $N \trianglelefteq G$ and π is a set of primes, then clearly $O_\pi(G)N/N \le O_\pi(G/N)$. Therefore $(G_\pi/N)/O_\pi(G/N)$ is an epimorphic image of G_π/N, hence is a nilpotent Hall π-subgroup of G/N, showing that $G/N \in \mathfrak{W}$. Therefore $\mathfrak{W}$ is Q-closed. To see that it is S-closed, let $S \le G$. Then $O_\pi(G) \cap S \le O_\pi(S)$, whence $S/O_\pi(S)$ is an epimorphic image of $S/S \cap O_\pi(G) \cong SO_\pi(G)/O_\pi(G)$. Now a Hall π-subgroup of $SO_\pi(G)/O_\pi(G)$ is contained in a Hall π-subgroup of $G/O_\pi(G)$, hence is nilpotent. Therefore $S \in \mathfrak{W}$ and $\mathfrak{W}$ is S-closed.

Now let M, $N \in \mathfrak{W}$ and put $G = M \times N$. If π is any set of primes and M_π and N_π are Hall π-subgroups of M and N respectively, then $G_\pi = M_\pi \times N_\pi$ is a Hall π-subgroup of G. Since $O_\pi(G)$ contains (even equals) $O_\pi(M) \times O_\pi(N)$, the group $G_\pi/O_\pi(G)$ is an epimorphic image of $M_\pi/O_\pi(M) \times N_\pi/O_\pi(N)$ which is nilpotent. Therefore $G \in \mathfrak{W}$ and thus $\mathfrak{W}$ is also D_0-closed, hence is a subgroup-closed formation.

(ii) Since clearly $\mathfrak{Q} \subseteq \mathfrak{W}$ and $\mathfrak{B}$ is the smallest formation containing $\mathfrak{Q}$, it follows that $\mathfrak{B} \subseteq \mathfrak{W}$.

(iii) Let $G \in \mathfrak{W}$. Then $G/O_{p'}(G)$ is the product of one of its Sylow p-subgroup and a nilpotent Hall p'-subgroup, hence $G/O_{p'}(G) \in \mathfrak{M}$. Since $\bigcap_{p\in\pm\nvdash\mathbb{P}} O_{p'}(G) = 1$, it follows that $G \in \mathrm{R}_0\mathfrak{M}$.

(iv) Let G be a primitive $\mathfrak{W}$-group. Since G is soluble, G has a unique minimal normal subgroup N of prime exponent p, say. Then $O_{p'}(G) = 1$ and therefore G has a nilpotent Hall p'-subgroup. Since G is the product of a Hall p'-subgroup and a Sylow p-subgroup, we have $G \in \mathfrak{M}$. Conversely, let G be a primitive M-group, let p be the exponent of the unique minimal normal subgroup N of G and let π be a set of primes. If $p \in \pi$, then $N \le O_\pi(G)$ and therefore $O_\pi(G/N) = O_\pi(G)/N$. Now $N = F(G)$ and thus $G/N \in \mathfrak{B} \subseteq \mathfrak{W}$ which shows that $(G/N)/O_\pi(G/N)$ has a nilpotent Hall π-subgroup and by an isomorphism theorem, the same is true for G. If $p \notin \pi$, then a Hall π-subgroup of G is contained in a Hall p'-subgroup of G which is nilpotent by Lemma 3.2.5. Hence $G/O_\pi(G)$ has a nilpotent Hall π-subgroup for all sets π of primes. $\square$

Remark. Since the class $\mathfrak{M}$ is D_0-closed by Lemma 1.3.2, it follows that $\mathrm{R}_0\mathfrak{M} \subseteq \mathrm{S}\mathrm{D}_0\mathfrak{M} = \mathrm{S}\mathfrak{M}$. Therefore every $\mathfrak{B}$-group and also every $\mathfrak{W}$-group is a subgroup of a group which is the product of two nilpotent subgroups. On the other hand, if $G \in \mathfrak{M}$, then $G/F(G) \in \mathfrak{B} \subseteq \mathfrak{W}$, and since $F(G) \le O_{\pi'\pi}(G)$ for every set of primes, it follows that $G/O_{\pi'\pi}(G)$ has a nilpotent Hall π-subgroup for every set of primes π.

List of symbols

In general, uppercase Latin letters denote groups (A, B, G, H, …) or sets, lowercase (Latin) letters symbolize elements of sets or groups. Uppercase Fraktur stands for classes of groups while script is used for sets of groups. Lowercase Greek letters usually denote homomorphisms of groups (α, β, …) or sets of primes (π, σ, τ …).

In the following, G and H will be groups, A and B are subgroups of G and g, $h \in G$. Ω will be a set acting on G via endomorphisms (the action is written exponentially). Integers are denoted by k, m and n, a prime by p.

$\pm\nvdash\mathbb{N}$	the set of positive integers
$\pm\nvdash\mathbb{N}_0$	the set of nonnegative integers
$GF(p^n)$	the finite field of order p^n
(m, n)	the greatest common divisor of the integers m and n
$\pm\nvdash\mathbb{P}$	the set of primes
π'	complement of the set π of primes in $\pm\nvdash\mathbb{P}$
p'	$= \{p\}' = \pm\nvdash\mathbb{P} \setminus \{p\}$
$G \cong H$	G is isomorphic with H
$G \times H$	the direct product of G and H
$G \wr H$	the regular wreath product of G and H
$\langle X \rangle$	the subgroup of G generated by the elements of $X \subseteq G$
$\langle x_1, x_2, \ldots \rangle$	the subgroup generated by the set $\{x_1, x_2, \ldots\}$
g^h	the action of h on G: $g^h = h^{-1}gh$
X^ω	the set $\{x^\omega \mid x \in X\}$
X^Ω	the subgroup of G generated by the set $\{x^\omega \mid x \in X,\ \omega \in \Omega\}$
X_Ω	$X_\Omega = \bigcap_{\omega\in\Omega} X^\omega$
$[g, \omega]$	the commutator of g and ω; $[g, \omega] = g^{-1}g^\omega$
$[A, B]$	the subgroup of G generated by all $[a, b]$ where $a \in A$ and $b \in B$
$N_\Omega(X)$	normalizer of the set X: $N_\Omega(X) = \{\omega \in \Omega \mid [x, \omega] \in X \text{ for all } x \in X\}$
$C_\Omega(X)$	centralizer of the set X: $C_\Omega(X) = \{\omega \in \Omega \mid [x, \omega] = 1 \text{ for all } x \in X\}$

$Z(G)$	centre of the group G; $Z(G) = C_G(G)$.
$G^{(n)}$	n-th derived subgroup of G defined recursively by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geq 0$
G', G''	$= G^{(1)}$, $G^{(2)}$
$F(G)$	Fitting subgroup of G, the subgroup generated by the normal nilpotent subgroups of G
$F_n(G)$	n-th term of the Fitting series of G defined recursively by $F_0(G) = 1$ and $F_{n+1}(G)/F_n(G) = F\big(G/F_n(G)\big)$ for every $n \geq 0$
$\Phi(G)$	Frattini-subgroup of G, the intersection of all maximal subgroups of G
$O^{\pi_1\pi_2\ldots\pi_k}(G)$	defined recursively; $O^{\pi_1}(G)$ is the π-residual of G, i.e. the intersection of all normal subgroups N of G with G/N a π-group and $O^{\pi_1\pi_2\ldots\pi_k}(G) = O^{\pi_k}\big(O^{\pi_1\pi_2\ldots\pi_{k-1}}(G)\big)$ if $k > 1$.
$O_{\pi_1\pi_2\ldots\pi_k}(G)$	defined recursively: $O_{\pi_1}(G)$ is the π-radical of G, i.e. the subgroup of G generated by all subnormal π-subgroups of G and $O_{\pi_1\pi_2\ldots\pi_k}(G)/O_{\pi_1\pi_2\ldots\pi_{k-1}}(G) = O_{\pi_k}\big(G/O_{\pi_1\pi_2\ldots\pi_{k-1}}(G)\big)$ if $k > 1$.
G_π	a Hall π-subgroup of the group G
$X_G(H)$	if G is the product of two subgroups, the factorizer $X_G(H)$ of H is the unique smallest factorized subgroup of G that contains H;
C_π	G satisfies C_π (conjugacy) if it satisfies C_π and all Hall π-subgroups of G are conjugate
D_π	G satisfies D_π (dominance) G satisfies C_π and every π-subgroup is contained in some Hall π-subgroup of G
E_π	G satisfies the property E_π (existence) if it possesses a Hall π-subgroup.
$\lvert G\rvert$	the cardinality of the set G
$\sigma(G)$	the set of primes dividing the order of some element of G; if G is finite, this equals the set of prime divisors of the group order $\lvert G\rvert$.
$c(G)$	the nilpotency class of the nilpotent group G
$d(G)$	the least integer k such that $G^{(k)} = 1$
$b_p(G)$	$p^{b_p(G)} = \lvert P\rvert$ where P is a Sylow p-subgroup of G
$c_p(G)$	the nilpotency class of a Sylow p-subgroup of G
$d_p(G)$	the derived length of a Sylow p-subgroup of G
$e_p(G)$	$p^{e_p(G)}$ is the exponent of a Sylow p-subgroup of G
$b_\pi(G)$	$= \max_{p\in\pi} b_p(G)$
$c_\pi(G)$	$= \max_{p\in\pi} c_p(G)$
$d_\pi(G)$	$= \max_{p\in\pi} d_p(G)$

$e_\pi(G)$	$= \max_{p\in\pi} e_p(G)$
$n(G)$	Fitting length of G; the least integer k such that $F_k(G) = G$
$l_\pi(G)$	π-length of G; the number of nontrivial π-factors in the series $1 \le O_{\pi'}(G) \le O_{\pi'\pi}(G) \le O_{\pi'\pi\pi'}(G) \le \ldots \le G$
$l_p(G)$	$= l_{\{p\}}(G)$

$\mathfrak{N}$	the class of all finite nilpotent groups
$\mathfrak{S}$	the class of all finite soluble groups
$\mathfrak{S}_\pi$	the class of all soluble π-groups

References

[1] Amberg, B.: *Factorizations of Infinite Groups.* Habilitationsschrift, Universität Mainz 1973.

[2] Amberg, B.: *Artinian and Noetherian Factorized Groups.* Rend. Sem. Mat. Univ. Padova 55 (1976), 105–122.

[3] Amberg, B., Franciosi, S. and de Giovanni, F.: *On normal subgroups of products of nilpotent groups.* J. Austral. Math. Soc. (Series A) 44 (1988), 275–286.

[4] Amberg, B., Franciosi, S. and de Giovanni, F.: *Products of Groups.* Oxford University Press, Oxford 1992.

[5] Amberg, B. and Fransman, A.: *On trifactorized groups.* unpublished.

[6] Baer, R.: *Classes of finite groups and their properties.* Illinois J. Math. 1 (1957), 115–187.

[7] Bryce, R. A., Cossey, J.: *Subgroup closed Fitting classes are formations.* Math. Proc. Cambridge Philos. Soc. 91 (1982), 225–258.

[8] Bryce, R. A., Cossey, J.: *Corrigenda: "Subgroup closed Fitting classes".* Math. Proc. Cambridge Philos. Soc. 91 (1982), 343.

[9] Berger, T. R. and Gross, F.: *2-Length and the Derived Length of a Sylow 2-subgroup.* Proc. London Math. Soc. (3) 34 (1977), 520–534.

[10] Blessenhohl, D. and Gaschütz, W.: *Über normale Schunck- und Fittingklassen.* Math. Z. 118 (1970), 1–8.

[11] Casolo, C.: *Subnormality in factorizable finite soluble groups.* Arch. Math. 57 (1991), 12–13.

[12] Dickenschied, O.: *Radikale Ringe und faktorisierte Gruppen.* Diplomarbeit, Mainz 1993.

[13] Doerk, K. and Hawkes, T.: *Finite Soluble Groups.* DeGruyter, Berlin 1992.

[14] Feit, W. and Thompson, J.: *Solvability of groups of odd order.* Pacific J. Math. 13 (1963), 775–1135.

[15] Fischer, B., Gaschütz, W. and Hartley, B.: *Injektoren endlicher auflösbarer Gruppen.* Math. Z. 102 (1967), 337–339.

[16] Franciosi, S., de Giovanni, F., Heineken, H. and Newell, M. L.: *On the Fitting length of a soluble product of nilpotent groups.* Arch. Math. 57 (1991), 313–318.

[17] Fransman, A.: *Factorizations of Groups.* Academisch Proefschrift, University of Amsterdam, 1991.

[18] Gaschütz, W.: *Zur Theorie der endlichen auflösbaren Gruppen.* Math. Z. 80 (1963), 300–315.

[19] Gillam, J. .D.: *A finite p-group* $P = AB$ *with* $\mathrm{Core}(A) = \mathrm{Core}(B) = 1$. Rocky Mountain J. Math. 3 (1973).

[20] Gross, F.: *The 2-length of Groups whose Sylow 2-Groups Are of Exponent 4.* J. Algebra 2 (1965), 312–314.

[21] Gross, F.: *The 2-length of a Finite Solvable Group.* Pacific Journal of Math. 15 (1965), 1221–1237.

[22] Gross, F.: *Finite groups which are the product of two nilpotent subgroups.* Bull. Austral. Math. Soc. 9 (1973), 267–274.

[23] Gross, F. and Scott, W. R.: *Solvable Products of Groups.* J. Algebra 9 (1968), 414–416.

[24] Hall, P.: *On the Sylow system of a soluble group.* Proc. London Math. Soc. (2) 43 (1937), 316–323.

[25] Hall, P.: *Theorems Like Sylow's.* Proc. London Math. Soc. (3) 6 (1956).

[26] Hall, P. and Higman, G.: *On the p-length of p-soluble groups and reduction theorems for Burnside's Problem.* Proc. London Math. Soc. (3) 6 (1956).

[27] Hawkes, Trevor O.: Finite Soluble Groups. In *Essays for Philip Hall eds* Gruenberg*, K. W. and* Rosenblade*, J. E.*. Academic Press, London (1984).

[28] Heineken, H.: *Products of finite nilpotent groups.* Math. Ann. 287 (1990), 643–652.

[29] Heineken, H.: *The product of two finite nilpotent groups and its Fitting series.* Arch. Math. 59 (1992), 209–214.

[30] Huppert, B.: *Zur Sylowstruktur endlicher auflösbarer Gruppen. II.* Arch. Math. XV (1964), 251–257.

[31] Huppert, B.: *Endliche Gruppen I.* Springer, Berlin 1967.

[32] Itô, N.: *Über das Produkt von zwei abelschen Gruppen.* Math. Z. 62 (1955), 400–401.

[33] Kegel, O.: *Produkte nilpotenter Gruppen.* Arch. Math. 12 (1961), 90–93.

[34] Kegel, O.: *Zur Struktur mehrfach faktorisierter endlicher Gruppen.* Math. Z. 87 (1965), 42–48.

[35] Maier, R.: *Endliche metanilpotente Gruppen.* Arch. Math. 23 (1972), 139–144.

[36] Maier, R.: *Um problema da teoria dos subgrupos subnormais.* Bol. Soc. Bras. Mat. 8 (1977), 127–130.

[37] Maier, R.: *A Completeness Property of Certain Formations.* Bull. London Math. Soc. 24 (1992), 540–544.

[38] Maier, R. and Sidki, S.: *A note on subnormality in factorizable finite group.* Arch. Math. 42 (1984), 97–101.

[39] Mann, A.: *System normalizers and subnormalizers.* Proc. London Math. Soc. (3), 20 (1970), 123–143.

[40] Pennington, E.: *On Products of Finite Nilpotent Groups.* Math. Z. 134 (1973), 81–83.

[41] Pennington, E.: *Trifactorisable groups.* Bull. Austral. Math. Soc. 8 (1973), 461–469.

[42] Peterson, F. G.: *Properties of multifactorizable groups.* unpublished.

[43] Robinson, D. J. S.: *A Course in the Theory of Groups.* Springer, Berlin 1982.

[44] Wielandt, H.: *Über das Produkt paarweise vertauschbarer nilpotenter Gruppen.* Math. Z. 55 (1951), 1–7.

[45] Wielandt, H.: *Über Produkte von nilpotenten Gruppen.* Illinois J. Math. 2 (1958), 611–618.

[46] Wielandt, H.: *Topics in the Theory of Composite Groups.* Dep. of Math., Univ. of Wisconsin, Madison, Wisconsin, 1967.

[47] Wielandt, H.: *Subnormalität in faktorisierbaren endlichen Gruppen.* J. Algebra 69 (1981), 305–311.

Erklärung

Hiermit erkläre ich, daß ich die vorliegende Arbeit selbständig verfaßt und keine anderen als die angegebenen Hilfsmittel verwendet habe.